CE8402-STRENGTH OF MATERIALS-II

UNIT 1

ENERGY PRINCIPLES

CONTENTS

TECHNICAL TERMS

1. **Determinate structure-** The structure in which the number of unknown reactions are equal to the number of available static equilibrium equations.
2. **Indeterminate structure-** The structure in which the number of unknown reactions are not equal to the number of available static equilibrium equations.
3. **Strain energy-** When an elastic material is deformed due to application of external force, internal resistance is developed in the material of body due to deformation. Some work is done by the internal resistance developed in the body which is stored in the form of energy. This energy is known as strain energy.
4. **Resilience-** The total strain energy stored in the body is known as resilience.
5. **ProofResilience-** The maximum strain energy that can be stored in a material within its elastic limit is known as proof resilience.
6. **Modulus of Resilience-** It is the proof resilience of the material per unit volume.
7. **Castigliano's I theorem-** The partial derivative of virtual strain energy with respect to virtual force or moment is equal to the deflection or rotation in that direction of virtual force or moment.
8. **Castiglaino's II theorem-** The partial derivative of virtual strain energy with respect to virtual deflection or rotation gives the virtual force or moment respectively, which induces the deflection or rotation.
9. **Maxwell's Reciprocal theorem-** In any beam or truss the deflection at any point D due to the load W at any other point C is the same as the deflection at C due to the same load W applied at D.

UNIT 1

ENERGY PRINCIPLES

1.1 Strain Energy and Strain energy Density

Strain Energy: Strain Energy in Gradual Load U = Average Load x Change in length = stress x strain x volume Substituting the value of stress, strain, and volume of the section

$$U = \frac{P\,\delta_L}{2} \qquad \delta_L = \frac{PL}{AE}$$

The stress ζ due to gradual load is P/A.

$$U = \frac{\sigma^2 V}{2E}$$

This is the strain energy stored in a body. – Equation (A)

Strain Energy in Sudden Load The stress due to sudden load is found by equating the equation (A) in the following equation. (B)

$$U = P \times \delta_L \text{ ---- Equation (B)}$$

$$\frac{\sigma^2 V}{2E} = P \times \delta_L$$

Therefore stress produced due to sudden load is

$$\sigma = \frac{2P}{A}$$

Strain energy due to sudden load is found by substituting the stress ζ due to sudden load in the following equation

$$U = \frac{\sigma^2 V}{2E}$$

Strain Energy in Impact Load U = Load x (height + Change in length)

The stress ζ due to impact load when δL is negligible

$$\sigma = \frac{\sqrt{2EPh}}{AL}$$

The stress ζ due to impact load when δL is not negligible

$$\sigma = \frac{P}{A} \quad 1 + \sqrt{1+2Eh(PL)}$$

Strain energy due to impact load is found by substituting the stress ζ due to impact load in the following equation.

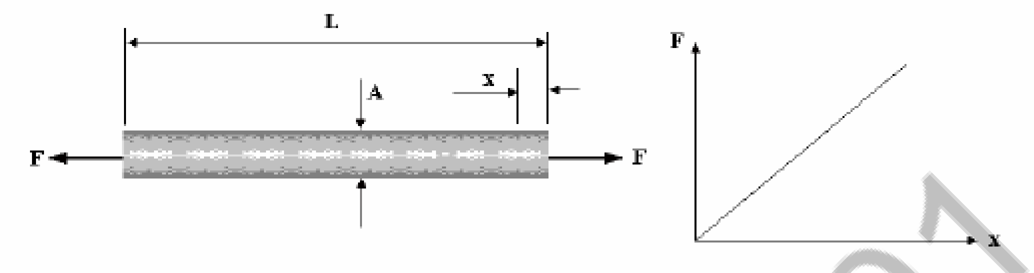
$$U = \frac{\sigma^2 V}{2E}$$

1.2 Strain energy in Traction

When an elastic body is deformed, work is done. The energy used up is stored in the body as strain energy and it may be regained by allowing the body to relax. The best example of this is clockwise device which stores strain energy and then gives it up.

We will examine strain energy associated with the most common forces of stress encountered in structure and use it to calculate the deflection of structures. Strain energy is usually given by the symbol U.

Consider a bar of length L and cross sectional area. If a tensile force is applied it stretches and the graph of force v extension is usually a straight line as shown. When the forces reaches a value of F and corresponding extension x, the work done (W) is the area under the graph. Hence W = Fx/2



Since the work done is the energy used up, this is now stored in the material as strain energy

$$\mathbf{U = Fx/2}$$

The stress in the bar is

$$\sigma = F/A$$

$$F = \sigma A$$

The strain in the bar as

$$\varepsilon = x/L$$

$$x = \varepsilon L$$

For an elastic material up to the limit of proportionality,

$$\sigma / \varepsilon = E$$

$$\varepsilon = \sigma / E$$

Substituting we find

$$U = \sigma A \varepsilon L/2 = \sigma^2 AL/2E$$

The volume of the bar is AL so

$$\mathbf{U = (\sigma^2/2E)\ x\ volume\ of\ the\ bar}$$

1.3 Strain energy in Flexure

Consider a rectangular element subjected to pure shear so that it deforms as shown. The height is h and plan area A. It is distorted a distance x by a shear force F. The graph of Force plotted against x is normally a straight line so long as the material remains elastic. The work done is the area under the F-x graph so

$$W = Fx/2$$

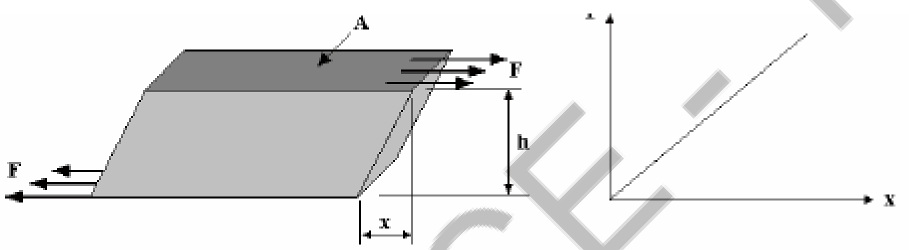


Fig 1.1 Strain energy due to shear

The work done is the strain energy stored hence

$$U = Fx/2$$

The shear stress is

$$\tau = F/A$$

Hence

$$F = \tau A$$

The shear strain is

$$\gamma = x/h$$

Hence

$$x = \gamma h$$

Note that since x is very small it is the same length as an arc of radius h and angle ϒ. It follows that the shear strain is the angle through which the element is distorted.

For an elastic material

$$\tau/\gamma = G$$

Hence

$$\gamma = \tau/G$$

Substituting we find

$$U = \bar{\tau}A\bar{\gamma}h/2 = \tau^2Ah/2G$$

The volume of the element is A h so

$$U = (\tau^2/2G)\ \text{x volume}$$

Pure shear does not often occur in structures and the numerical values are very small compared to that due to other forms of loading so it often ignored.

1.4 Strain energy due to torsion

Consider a round bar being twisted by a torque T. A line along the length rotates through angle ϒ and corresponding radial line on the face rotates angle ϴ. ϒ is the shear strain on the surface at radius R

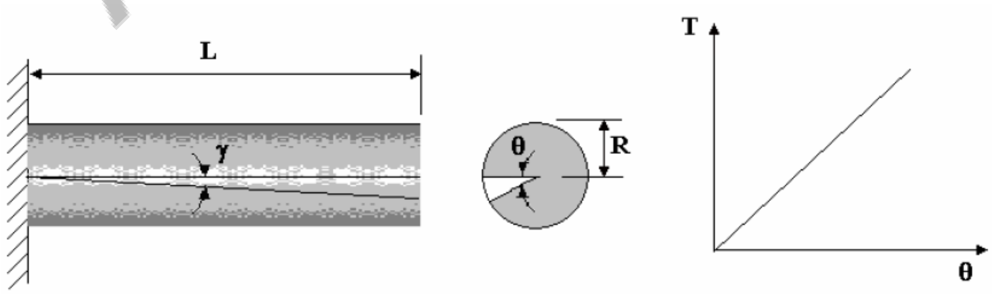


Fig 1.2 Strain energy due to torsion

The relationship between torque T and angle of twist Θ is normally a straight line. The work done is the area under the torque-angle graph. For a given pair of values

$$W = T\theta/2$$

The strain energy stored is equal to the work done hence

$$U = T\theta/2$$

From the theory of torsion

$$\theta = TL/GJ$$

G is the modulus of rigidity and J is the polar second moment of area.

$J = \Pi R^4/2$ for a solid circle.

Substitute $\Theta = TL/GJ$

And we get $U = T^2L/GJ$

Also from torsion theory

$$T = \tau J/R$$

Where ζ is maximum shear stress on the surface.

Substituting for T we get the following

$$U = (\tau J/R)^2/2GJ = \tau^2 JL/2GR^2 \qquad \text{Substitute } J = \pi R^4/2$$
$$U = \tau^2 \pi R^4 L/4GR^2 = \tau^2 \pi R^2 L/4G$$

The volume of the bar is $AL = \pi R^2 L$ so it follows that:

$$\mathbf{U = (\tau^2/4G) \text{ x volume of the bar.}}$$

1.4 Castigliano's Theorem

A linear force-displacement relationship between a force, *F*, and a collocateddisplacement, *D*, in statically determinate systems can be determined usingthe *principle of real work*,

$$W_E = U$$
$$\frac{1}{2} F \cdot D = \frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\} \, dV \, .$$

The force-displacement relationships for systems with multiple external forcesor distributed loads, or statically indeterminate systems, involve relationshipsbetween multiple forces and displacements. The external work is

$$W_E = \frac{1}{2} \sum_{i=1}^{n} F_i D_i$$

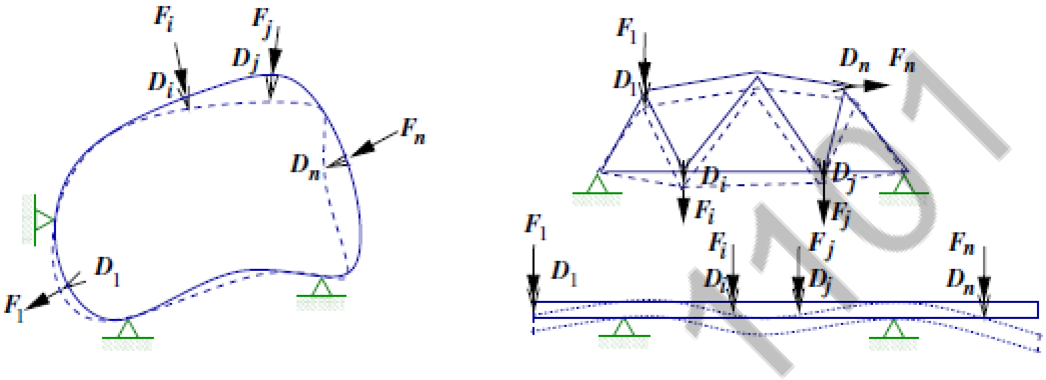


Fig 1.3 Castigliano's on beam and frame

A set of n force-displacement relationships cannot be found with the single principle of real work equation,$W_E = U$. Instead, a new method must be developed.

Castigliano's Theorem - Part I (Force Theorem)

The *strain energy* in any elastic solid subjected to n point forces Fi is afunction of the n collocated displacements, Di.

$$U(D_1, D_2, \cdots, D_n) = \sum_{i=1}^{n} \int_0^{D_i} F_i(\bar{D}_i) \, d\bar{D}_i$$

$$\Delta U \approx F_j \, \Delta D_j \, ,$$

$$F_j \approx \frac{\Delta U}{\Delta D_j} \, ,$$

$$\Rightarrow \quad F_j = \frac{\partial U(D)}{\partial D_j}$$

The force, Fj , on an elastic solid is equal to the partial derivative of the *strainenergy*, $U(D1,D2, \cdots ,Dn)$, with respect to the collocated displacement, Dj .

Castigliano's Theorem - Part II (Deflection Theorem)

The *complementary strain energy* in any elastic solid subjected to *n* pointforces *Fi* is a function of the *n* forces and is the complement of the strainenergy.

$$U^*(F_1, F_2, \cdots, F_n) = \sum_{i=1}^{n} F_i D_i - U(D_1, D_2, \cdots, D_n) = \sum_{i=1}^{n} F_i D_i - \sum_{i=1}^{n} \int_0^{D_i} F_i(\bar{D}_i)\, d\bar{D}_i$$

$$\Delta U^* \approx D_j\, \Delta F_j\,, \qquad D_j \approx \frac{\Delta U^*}{\Delta F_j}\,, \qquad \Rightarrow \qquad D_j = \frac{\partial U^*(F)}{\partial F_j}$$

The partial derivative of the *complementary strain energy* of an elastic system,*U*(*F*), with respect to a selected force acting on the system, *Fj* , gives thedisplacement of that force along its direction, *Dj* .

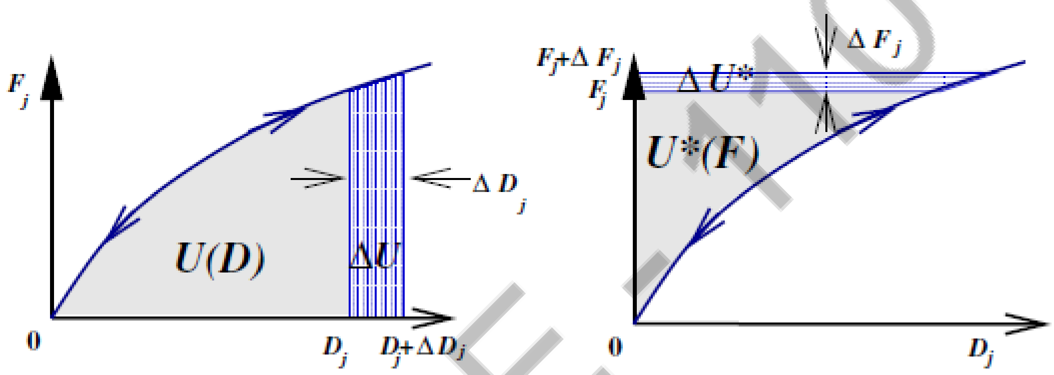
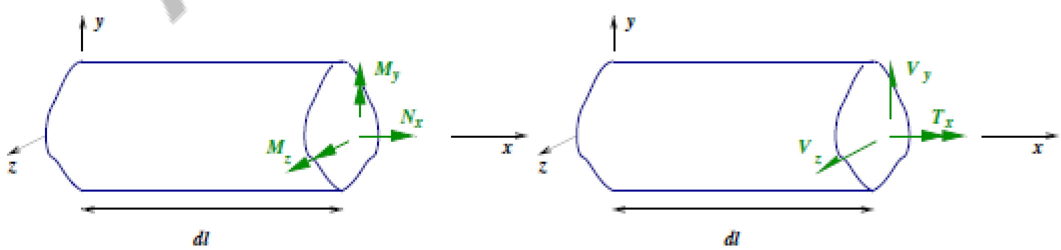


Fig 1.4 Complementary strain energy

If the solid is linear elastic, then *U*(*F*) = *U*(*D*).

For linear elastic *prismatic* solids in equilibrium,

$$U^*(F) = U(D) = \frac{1}{2}\int_l \frac{N^2}{EA}\,dl + \frac{1}{2}\int_l \frac{M_z^2}{EI_z}\,dl + \frac{1}{2}\int_l \frac{M_y^2}{EI_y}\,dl + \frac{1}{2}\int_l \frac{V_z^2}{G(A/\alpha_z)}\,dl + \frac{1}{2}\int_l \frac{V_y^2}{G(A/\alpha_y)}\,dl + \frac{1}{2}\int_l \frac{T^2}{GJ}\,dl\,.$$



So,

$$\frac{\partial U^*}{\partial F_j} = \frac{\partial U}{\partial F_j} = \int_l \frac{N\frac{\partial N}{\partial F_j}}{EA}\,dl + \int_l \frac{M_z\frac{\partial M_z}{\partial F_j}}{EI_z}\,dl + \int_l \frac{M_y\frac{\partial M_y}{\partial F_j}}{EI_y}\,dl + \int_l \frac{V_z\frac{\partial V_z}{\partial F_j}}{G(A/\alpha_z)}\,dl + \int_l \frac{V_y\frac{\partial V_y}{\partial F_j}}{G(A/\alpha_y)}\,dl + \int_l \frac{T\frac{\partial T}{\partial F_j}}{GJ}\,dl\ .$$

1.6 Principle of Virtual Work

The virtual work is defined as work done by the real forces due to hypothetical displacement or the work done by hypothetical forces during real displacement.

Principle

If the deformable body in equilibrium under a system of force is given virtual deformation. The virtual work done by the system of the forces is equal to the internal virtual work done by the stresses due to system of forces.

$$1 \times \Delta = \int \sigma\,\varepsilon\,dv$$

The other name is unit load method is used to solving the problems.

$\Delta \quad = \int Mm\,dx/EI$

Where M = Bending moment due to actual loads

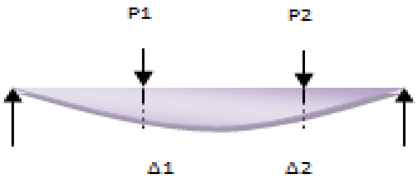
m = Bending moment due to unit load only at the particular location

1.7 Application of energy theorem for computing deflection in beams and Columns

Castigliano's I Theorem: In a linearly elastic structure partial derivative of the strain energy with respect to a load is equal to the deflection of the point where the loads acting the deflection being measured in the direction of the load.

$dV/dF_1 = \Delta_1$

$dV/dM = \Theta$



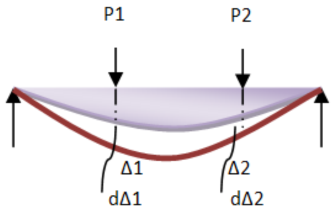


Fig 1.5 Deflection of beams by Castigliano's I and II theorem

Additional Displacement with respective points

If P_1 and P_2 are applied gradually

$V = ½ (P_1\Delta_1 + P_2\Delta_2)$

If additional loads dP, be added after the loads P_1 and P_2

Additional Strain energy is $½ (dp_1 .d\Delta_1 + P_1\Delta_1 + P_2\Delta_2)$

Total Strain energy = U + d U

$= ½ (dp_1 .d\Delta_1 + P_1\Delta_1 + P_2\Delta_2) + P_1 d\Delta_1 + P_2 d\Delta_2$ …………1

If $P_1 + dP_1$ are applied simultaneously

Strain energy stored = $½ (P+d\Delta_1).(\Delta_1 + d\Delta_1) + ½ P_2 (\Delta_2 + d\Delta_2)$……………...2

Load is same, sequence of loading is different

Equate 1 and 2

We get $dV/dp_1 = ½ dp_1 \Delta_1$

$dV/dp_1 = \Delta_1$

Castigliano's II theorem: It states that among all the statically possible states of stress in a structures subjected to a variation of stress during which the condition of equilibrium are maintained, the correct one is that which makes the strain energy of the system is minimum.

$dV/dR_1 = 0$ and $dU/dR_2 = 0$

R_1 and R_2 are reduntant reaction

This theorem is useful to find the reaction in indeterminate beam.

1.8 Maxwell's Reciprocal theorems

Maxwell theorem states that for any linear elastic body (also called a Hookeanbody), that the movement at a d.o.f. A, caused by the application of a force/momentF at a d.o.f. B, is exactly the same as the movement at a d.o.f. B, caused by theapplication of a force/moment F at a d.o.f. A.

To illustrate, consider a cantilever with a location A and a location B. When weapply a force at B, the displacement at A is. When we apply a force at A, thedisplacement at B is. Maxwell said that δAB=δBA . Imagine that Maxwell basedthis on a general theory for elastic bodies called Castigliano's Theorem. Maxwell'stheorem is actually a corollary of Castigliano's.

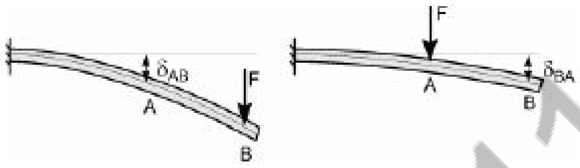


Fig 1.6 Displacement on beams

By considering elastic energy can show that Maxwell's theory works. We start by Assuming that for elastic bodies, the stored energy depends on the deformed shape, which depends on the total set of loads. The shape, and the stored energy do not depend on which load was applied first. (Elastic energy is 'path independent'). With this we next consider our beam with two equal forces at A and B.

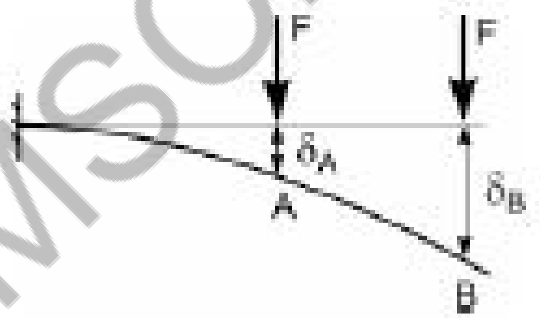


Fig 1.7Displacement on beams

The elastic energy stored in the beam comes from the work done by the forces as They were applied to the beam. We will apply the forces in two ways. In the first Way, we apply the force at A first, then a B. In the second way, we apply F to B first, then to A. We calculate the work done by the forces, and compare the two results.

QUESTION BANKS

2 Marks

1. Define Shear stress and shear strain.
2. Define stability.
3. Define strain energy density
4. State Castigliano's I theorem
5. State Castigliano's I theorem
6. Define Modulus of resilience
7. What is strain energy
8. What is Principle of Virtual Work
9. State Maxwell reciprocal theorem
10. What is proof resilience
11. What is the formulae to find the deflection from Castigliano's theorem

16 Marks

1. Derive the expression for traction while applying gradual applying load.
2. Derive the expression on strain energy for shear
3. Derive the expression on strain energy for flexure
4. Derive the expression on strain energy for flexure
5. A simply supported beam is 8m span. It is loaded with a concentrated load of 5 kN, 2m from left support. Find the deflection at center of the span by virtual work method.

STRENGTH OF MATERIALS-II

UNIT II

INDETERMINATE BEAMS

CONTENTS

TECHNICAL TERMS

1. **Continuous Beam-** A Beam which is supported on more than two supports is called a Continuous Beam.
2. **Propped Cantilever Beam-** A propped cantilever beam is a beam in which one end is fixed and is vertically supported with a prop at the free end or at any intermediate span.
3. **Fixed Beam-** A fixed beam also called 'Built in beam' or 'Encaster beam' is a beam, in which the ends are rigidly fixed.
4. **Conjugate Beam-** Conjugate beam is an imaginary beam of length equal to that of original beam but for which load diagram is M/EI diagram.
5. **Point of contra flexure-** It is a point where the bending moment changes its sign from +ve to –ve or –ve to +ve at that point bending moment is Zero.

UNIT II INDETERMINATE BEAMS

2.1 Propped Cantilever and Fixed Beams

When a cantilever or a beam carries some load, maximum deflection occurs at the free end in case of cantilever and at the middle point in case of simply supported beam. The deflection can be reduced by providing vertical support at these points or at any suitable point. Propped cantilevers means cantilevers supported on a vertical support at a suitable point.The vertical support is known as prop. The prop which does not yield under the loads is known as rigid.The prop which is of same height as the original position of the cantilever or a beam does not allow any deflection at the support when the cantilever or beam is loaded. The prop exerts an upward force on the cantilever or beam. As the deflection at the point of prop is zero, hence the upward force of the prop is such as magnitude as to give an upward deflection at the point of prop equal to the deflection due to the load on the beam.

Hence the reaction of the prop is calculated by equating the downward deflection due to load at the point of prop of the upward deflection due to prop reaction.

S.F and B.M Diagrams for a Propped Cantilever carrying a Point Load at the centre and Propped at free end

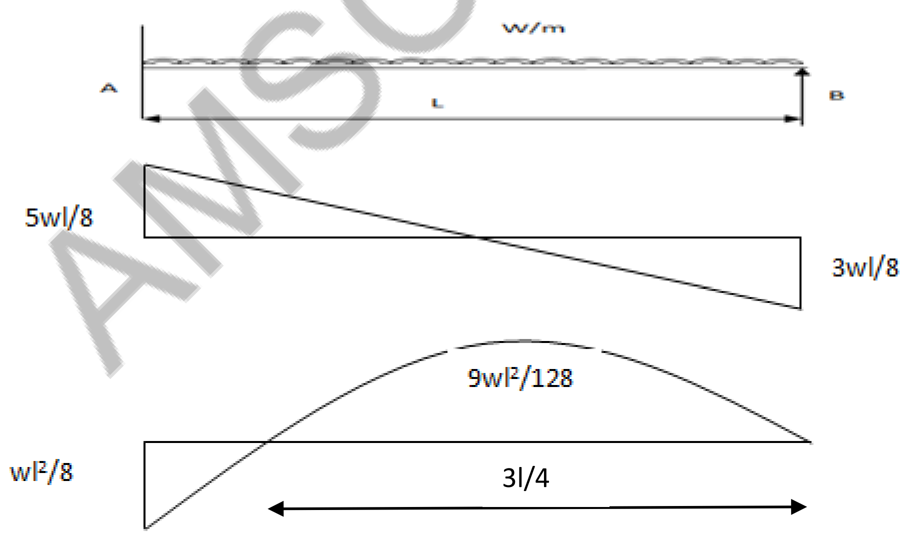


Fig 2.1 Cantilever beam with propped at one end subjected to UDL

Consider a cantilever AB fixed at A and propped at B and carrying a uniformly distributed load over its entire span as shown in Fig 2.1

Let l = span of the beam

w = UDL over the entire span

P = Reaction of prop

We know that the downward deflection of B due to uniformly distributed load (neglecting prop reaction)

$$y_B = wl^4/8EI \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

And the upward deflection of the cantilever due to force P (neglecting UDL)

$$y_B = Pl^3/3EI \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

Since both the deflections are equal, therefore equating equation (i) and (ii)

$$Pl^3/3EI = wl^4/8EI$$

$$P = 3W/8 \text{ where } W=wl$$

i. Shear Force Diagram

We know that the shear force at B,

F_B = -3wl/8 ….. (Minus sign due to right upwards)

F_A = wl/8 …..(Plus sign due to left upwards)

Let M be the point at a distance x from B, where shear force changes its sign,

Therefore $x = 3l/8$

ii. Bending Moment Diagram

We know that the bending moment at the propped end B,

$$M_B = 0$$

$$M_A = (3wl/8) \times l - wl^2/2 = -wl^2/8$$

And also we know that the bending moment will be maximum at M, where SF changes its sign

Therefore $M_M = 9wl^2/128$

And bending moment at a section X, at a distance x from the propped end B,

$$M_M = (3wl/8)\,x - wx^2/2$$

In order to find the out the point of contraflexure

$$x = 3l/4$$

S.F and B.M Diagrams for a Propped Cantilever carrying a Point Load at the centre and Propped at free end

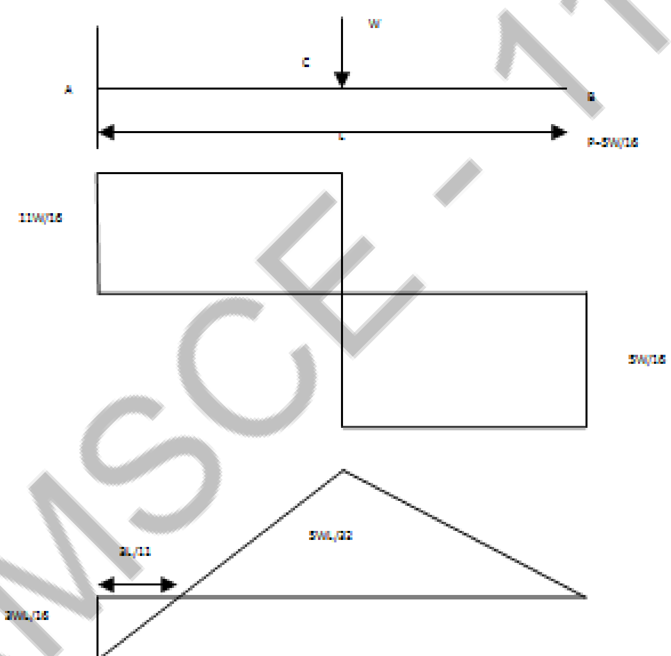


Fig 2.2 Cantilever beam with propped at one end subjected to Point Load

We know that the downward deflection of B due to load (neglecting prop reaction)

$y_B = 5WL^3/48EI$ …………………………… (i)

and the upward deflection of the cantilever due to force P (neglecting UDL)

$y_B = Pl^3/3EI$ ………………………………………. (ii)

Since both the deflections are equal, therefore equating equation (i) and (ii)

$Pl^3/3EI = 5WL^3/48EI$

$P = 5W/16$

i. Shear Force Diagram

We know that the shear force at B,

$F_B = -5WL/16$….. (Minus sign due to right upwards)

$F_C = -5WL/16 + W = 11\ W/16$

ii. Bending Moment Diagram

We know that the bending moment at the propped end B,

$M_B = 0$

$M_C = (5wl/16) \times l/2 = 5WL/16$

$M_A = (5wl/16) \times l/2 - wl/2 = -3WL / 16$

The BM at any section between C and A at a distance X from B

$M_M = (5WL/16)x - W(x - L/2) = 0$

In order to find the out the point of contraflexure

$x = 8L/11$

2.2 Fixed End Moments

A beam whose both ends are fixed is known as fixed beam. Fixed beam is also called a built-in or encaster beam. In case of a fixed beam, the slope and deflection at the fixed ends are zero. But the fixed ends are subjected to end moments. Hence end moments are not zero in case of a fixed beam.

Advantages of Fixed Beams

A fixed beam has the following advantages over simply supported beams

1. The beam is stiffer, stronger and more stable.

2. The slope at both the ends is zero
3. The fixing moments are developed at the two ends, whose effect is to reduce the maximum bending moment at the centre of the beam.
4. The deflection of a beam at its centre is very much reduced.

Fixed Beam carrying a Central Point Load

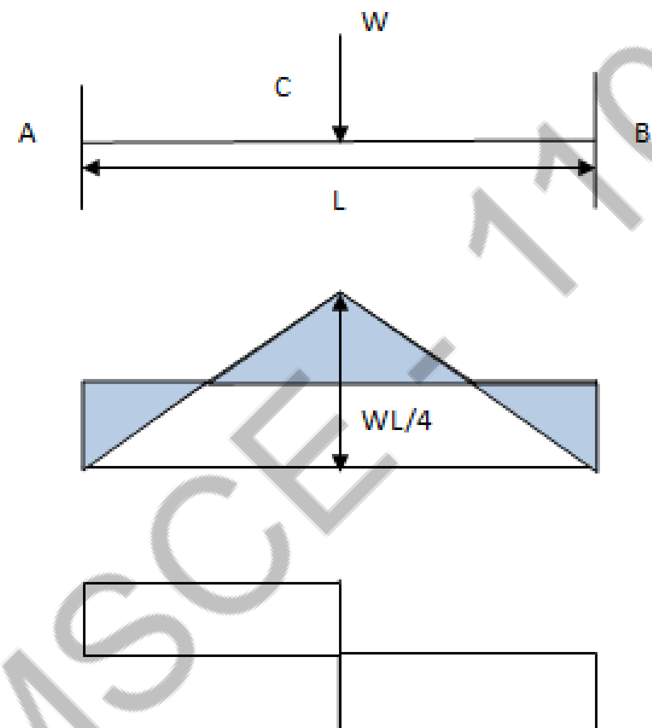


Fig 2.3 Fixed beams subjected to Central Point Load

Consider a beam AB of length l is fixed at both ends and carrying a central point load W as shown in Fig 2.3

i. BendingMoment Diagram

Let M_a = Fixing Moment at A

M_b = Fixing moment at B

Since the beam is symmetrical, therefore MA and MB will also be equal. Moreover the μ'-diagram will be rectangle as shown in figure. We know that μ-diagram i.e., BMD due to central point lead will be a triangle with the central ordinate equal to WL/4 as shown in fig2.3

Now equating the areas of the two diagrams

$$M_A.l = -(1/2)l \cdot (Wl)/4 = -Wl^2/8$$

$$M_A = -Wl/8$$

Similarly $M_B = -Wl/8$

ii. Shear Force Diagram

R_A = Reaction at A

R_B = Reaction at B

Equating clockwise moments and anticlockwise moments about A

$$R_B \times L + M_A = M_B + W \times (l/2)$$

$$R_A = R_B = W/2$$

iii. Deflection of beam

From the geometry of the fig, we find that the points of contraflexure will be at a distance of l/4 from both the ends of the beam.

We know that BM at any section X at a distance x from A

$$EI\,(d^2y/dx^2) = (Wx/2) - (Wl/8)$$

Intergrating the above equation

$$EI\,(dy/dx) = Wx^2/4 - Wlx/8 + C_1$$

Where C_1 is the first constant of integration, WKT when x = 0 then dy/dx= 0. Therefore $C_1=0$

$EI\,(dy/dx) \quad = Wx^2/4 - Wlx/8$

This is the equation for the slope of the beam at any section

Now integrating the equation once again

$EI.y \; = Wx^3/12 - Wlx^2/16 + C_2$

Where C_2 is the second constant of integration. WKT when x=0, then y=0. Therefore $C_2=0$

$EI.y \; = Wx^3/12 - Wlx^2/16$

This is the required equation for the deflection of the beam at any section. WKT the maximum deflection occurs at the centre of the beam. Therefore sub x = l/2 in the above equation.

$y_C = - WL^3/192\ EI$ …. (Minus signs means that the deflection is downwards)

2.3 Theorem of Three Moments

A beam which is supported on more than two supports is called continuous beam. Such a beam, when loaded will deflect with convexity upwards, over the intermediate supports and with concavity upwards over the mid of the spans. The intermediate supports of continuous beam are always subjected to some bending moment. The end supports, if simply supported will not be subjected to any bending moment. But the end supports, if fixed, will be subjected to fixing moments and the slope of the beam, at the fixed end will be zero.

Bending Moment Diagram for Continuous Beam

The analysis of continuous beam is similar to that of a fixed beam. The bending Moment diagram for a continuous beam under any system of loading may be drawn in the following two stages.

1. By considering the beam as a series of discontinuous beams, from support to support and drawing the usual μ-diagram due to vertical load
2. By superimposing the usual μ'- diagram, due to end moments over μ-diagram

2.4 Analysis of Continuous Beam

It states , " If a beam has n supports, the end ones being fixed, then the same number of equations required to determine the support moments may be obtained from the consecutive pairs of spans i.e., AB-BC-CD-DE and so on".

Consider a continuous beam ABC, fixed at A and C and supported at B as shown in figure 2.4

Let l1 = span of the beam

I1 = MOI of the beam of span AB

l2, I2 = corresponding values for the span AB

MA = Support moment at A

MB = Support moment at B

MC = Support moment at C

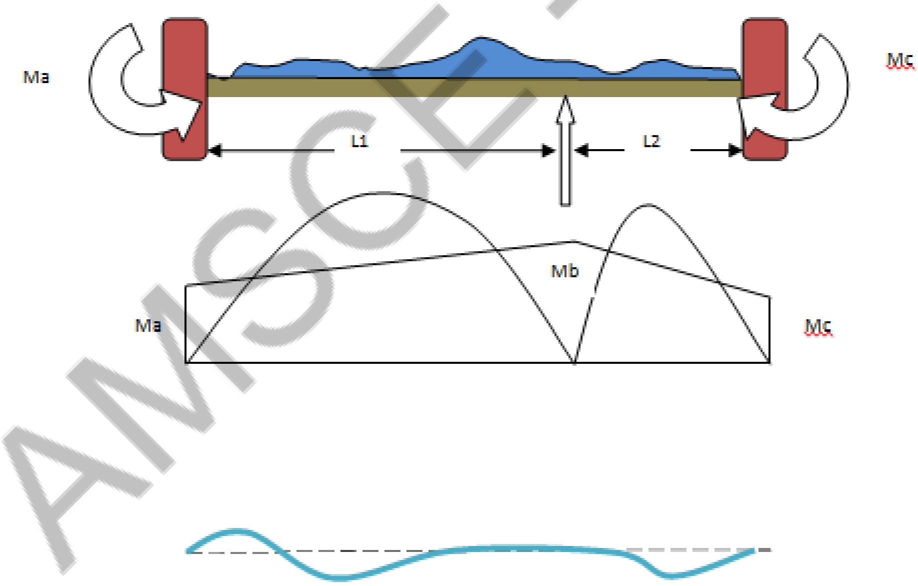


Fig 2.4 Continuous Beam with fixed end Moments

Let µx = Bending moment at any section X considering the beam between two supports as simply supported beam μx' = Fixing moment at any section X of the beam

$$M_A l_1 + 2\,M_B\,(l_1+l_2) + M_C l_2 = -\frac{6a1x1}{I1}+\frac{6a2x2}{I2}$$

Applications of Clapeyron's Theorem

1. Continuous beam with simply supported beam
2. Continuous beam with fixed end supports
3. Continuous beam with end span overhanging
4. Continuous beam with a sinking support

2.5 Slope and Deflection method

In this section we will use equation to find slope and deflection for some standard loading

cases of beams. The method adopted is called double integration method as the equation is to be integrated twice in order to develop a formula for deflection.

2.5.1 Simply supported beam loaded with the central load

Consider a simply supported beam loaded with a concentrated load W as shown in fig

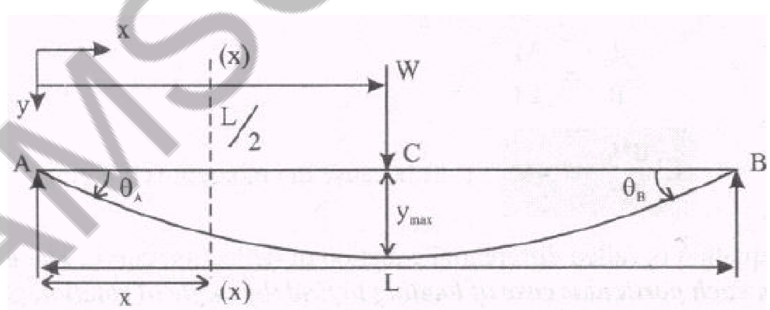


Fig 4.4

Now, consider a section x-x on the beam at a distance 'x' measured in positive x direction.

$$\text{Reaction at A} = \frac{W}{2} \text{ [Because of symmetry]}$$

$$\therefore \text{ B.M at section x-x} = M = \frac{W}{2}x$$

Now we press differential equation of elastic curve in to use, i.e.

$$EI\frac{d^2y}{dx^2} = -M$$

$$\therefore EI\frac{d^2y}{dx^2} = -\frac{W}{2}x \quad \text{.........(1)}$$

Integrating equation (1) once; we get

$$EI\frac{dy}{dx} = -\frac{W}{2}\frac{x^2}{2} + C_1 = -\frac{Wx^2}{4} + C_1$$

Where $C_1$is constant of integration. Its value can be obtained by using boundary condition. By virtue of symmetry of loading, the deflection at the centre of the beam was maximum. Then, a tangent drawn on the curve at this point will be horizontal

i.e., dx /dy = 0 thus at x = l/2 dy/dx =0

Putting the boundary condition for the above equation, we get

$$0 = -\frac{Wl^2}{16} + C_1$$

$$\therefore C_1 = \frac{Wl^2}{16}$$

Therefore, the general equation of slope is

$$EI\frac{dy}{dx} = -\frac{Wx^2}{4} + \frac{Wl^2}{16}$$

Now, in order to obtain slope at A, put x=0 in equ.2

$$\therefore \quad \text{at } x = 0, \frac{dy}{dx} = \theta_A$$

$$EI\theta_A = \frac{Wl^2}{16}$$

$$\therefore \quad \theta_A = \frac{Wl^2}{16EI}$$

$$\text{By symmetry } \theta_B = -\frac{Wl^2}{16EI}$$

Integrating equation (2) yet again, we get

$$EIy = -\frac{Wx^3}{12} + \frac{Wl^2}{16}x + C_2$$

Again, C2 is constant of integration, to be determined using boundary condition

At x =0 [at support point A], there can be no deflection, i.e., y = 0.

0 = 0 + 0 + C2

:. C2 = 0

Thus, the general equation of deflection is

$$EIy = -\frac{Wx^3}{12} + \frac{Wl^2}{16}x$$

As a special case, at $x = \frac{l}{2}$ (center of beam), $y = y_{max}$.

$$\therefore EIy_{max} = -\frac{Wl^3}{96} + \frac{Wl^3}{32} = \frac{2Wl^3}{96} = \frac{Wl^3}{48}$$

$$\therefore y_{max} = \frac{Wl^3}{48EI}$$

2.5.2 Simply supported beam loaded with Udl over entire span

Consider a simply supported beam of span 'l', loaded with Udl over the entire span as shown in figure 4.5.

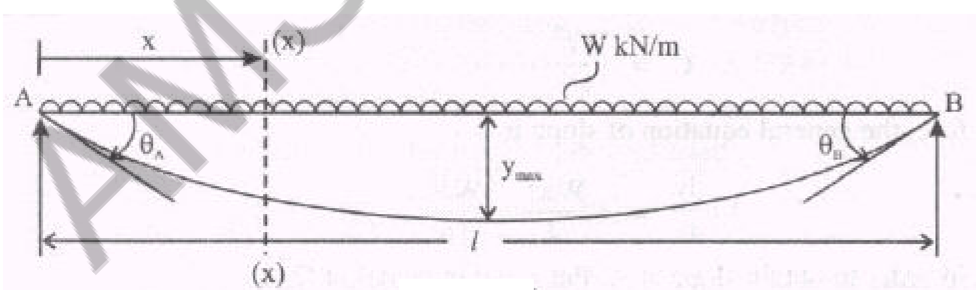


Fig 4.5

Now consider a section x-x on the beam at a distance x measured in positive 'x' direction

Reaction at A = WL / 2 [Because of symmetry]

BM at section x-x = M = $\frac{Wl}{2}x - \frac{Wx^2}{2}$

Substituting value of M in differential equation,

$$EI\frac{d^2y}{dx^2} = -\frac{Wlx}{2} + \frac{Wx^2}{2}$$

Integrating equation (1) once, we get.

$$EI\frac{dy}{dx} = -\frac{Wlx^2}{4} + \frac{Wx^3}{6} + C_1$$

Where C1 is constant of integration.

$$\text{At } x = \frac{l}{2}, \frac{dy}{dx} = 0$$

[As the deflection at centre happens to be maximum]

$$\therefore \quad 0 = -\frac{Wl^3}{16} + \frac{Wl^3}{48} + C_1$$

$$\therefore \quad C_1 = \frac{Wl^3}{24}$$

The general equation for slope is:

$$EI\frac{dy}{dx} = -\frac{Wlx^2}{4} + \frac{Wx^3}{6} + \frac{Wl^3}{24}$$

$$\text{At } x = 0, \frac{dy}{dx} = \theta_A$$

$$\therefore \quad EI\theta_A = \frac{Wl^3}{24}$$

$$\therefore \quad \theta_A = \frac{Wl^3}{24EI}$$

Similarly θ_B $= -\frac{Wl^3}{24EI}$

Integrating equation (2) again

$$EIy = -\frac{Wlx^3}{12} + \frac{Wx^4}{24} + \frac{Wl^3x}{24} + C_2$$

C_2 can be determined by applying boundary conditions At x = 0, y = 0

$$\therefore \quad 0 = 0 + 0 + 0 + C_2$$

$$\therefore \quad C_2 = 0$$

The general equation for deflection now becomes,

$$EIy = -\frac{Wlx^3}{12} + \frac{Wx^4}{24} + \frac{Wl^3x}{24}$$

As a special case, at $x = \frac{l}{2}$, $y = y_{max}$,

$$\therefore \quad EIy_{max} = -\frac{Wl^4}{96} + \frac{Wl^4}{384} + \frac{Wl^4}{48} = \frac{-4Wl^4 + Wl^4 + 8Wl^4}{384}$$

$$= \frac{5Wl^4}{384}$$

$$\therefore \quad y_{max} = \frac{5Wl^4}{384EI}$$

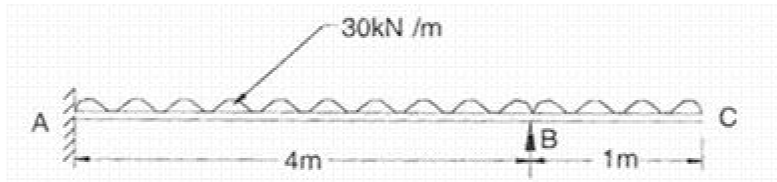
QUESTION BANKS

2 Marks

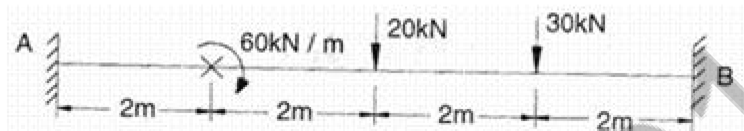
1. What is beam? What are the types? Differentiate three types of beams.
2. What is propped cantilever beam
3. State Clapeyron's theorem?
4. What is Modular ratio
5. Define shear force and bending moment?
6. Define Shear force and Bending Moment Diagram?
7. Define Prop
8. How will you draw a SF and BM diagrams for a beam which is subjected to inclined loads
9. State the relationship between BM, SF and load.
10. What are the different types of loads acting on a beam? Differentiate between a point load and a uniformly distributed load.
11. What do you mean by sagging bending moment and hogging bending moment?
12. What do you mean by shear centre?
13. What is a conjugate beam? Give the conjugate beam a simply supported beam and cantilever beam.
14. What is Bending Moment Diagram? Draw Bending Moment Diagram for a cantilever beam carries udl throughout the span.
15. What is the condition for maximum bending moment in a beam?

16 Marks

1. Draw the SFD and BMD of propped cantilever of span L having prop at end carrying a central load W. indicate the corresponding values on the diagram
2. . Draw the SFD and BMD for propped cantilever of span L carrying a UDL of w/m length through its length. Calculate the maximum bending moment and its position
3. Determine the prop reaction of the given diagram



4. Find the moment over the beam and draw the BM and SF diagram using three moment method



5. Find the bending moment diagram and sheaf force of continuous beam using slope deflection method

STRENGTH OF MATERIALS-II

UNIT III

COLUMNS AND CYLINDERS

CONTENTS

TECHNICAL TERMS

Column: A bar or a member of a structure inclined at 90° to the horizontal and carrying an axial compressive load is called a column.

Slenderness ratio. The ratio of the equivalent length of the column to the least radius of gyration is called the slenderness ratio.

Buckling load. The minimum axial load at which the column tends to have lateral

Displacement& buckle is called the buckling, crippling or critical load.

Equivalent length. It is the length of the column which gives the same buckling point, as given by a both ends hinged column.

Short Column. A column for which the slenderness ratio is less than12 called as short column.

Medium Column. A column for which the slenderness ratio lies between 32 and 120 is called a medium column.

Long Column. A column for which the slenderness ratio is more than 120 called a long column.

Safe load. It is the load under which the column will not buckle.

III COLUMNS

3.1 Eccentrically loaded short columns

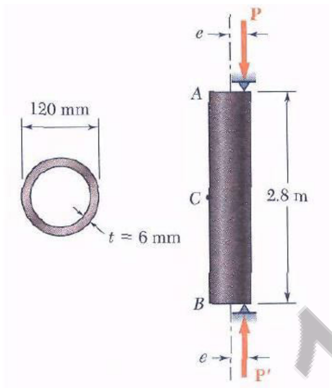


Fig 3.1 Eccentric loaded column

The load which acts apart from the axis is known as eccentrically loaded column

3.1.1 Load Acting Eccentric to One Axis

In order to study the effect of eccentric load more closely, let us consider a short axial member, loaded with load *P*, placed at a distance '*e*' from the centroidal vertical axis through the centroid of the section, as shown in Figure

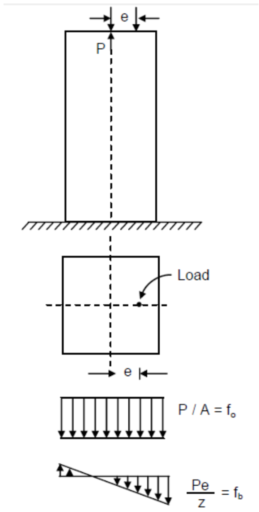


Fig 3.2 Load Acting Eccentric to One Axis

Along the vertical axis, introduce two equal and opposite forces, each equal to load *P*. Their introduction obviously makes no difference to the loading of the member, as they cancel out each other. However, if the upward force *P* is considered along with at a distance *e* from each other, from a clockwise couple of magnitude $P \times e$, the effect of which is to produce bending stress in the member. The remaining central downward force *P* produces a direct compressive stress f_0, of magnitude P/A as usual. Hence, we can conclude that an eccentric load produces direct compressive stress as well as the bending stress.

The bending couple $P \times e$ will cause longitudinal tensile and compressive stresses. The fibre stress due to bending f_0, at any distance '*y*' from the neutral axis is given by,

$$f_b = \frac{M}{I_{xx}} \times y = \frac{P \times e \times y}{I_{xx}} \qquad \text{(tensile or compressive)}$$

Hence, the total stress at any section is given by

$$f = f_0 \pm f_b = \frac{P}{A} \pm \frac{P \times e \times y}{I_{xx}}$$

$$f = \frac{P}{A} \pm \frac{M}{Z_{xx}}$$

$$\left[\text{where } P \times e = M \text{ and } \frac{I_{xx}}{y} = Z_{xx} \text{ (the section modulus)}\right.$$

The extreme fibre stresses are given by,

$$f_{\max} = f_0 + f_b = \frac{P}{A} + \frac{M}{Z_{xx}}$$

and

$$f_{\min} = f_0 - f_b = \frac{P}{A} - \frac{M}{Z_{xx}}$$

If f_0 is greater than f_b, the stress throughout the section will be of the same sign. If however, f_0 is less than f_b, the stress will change sign, being partly tensile and partly compressive across the section. Thus, there can be three possible stress distributions as shown in Figures

3.1.2 Load Acting Eccentric to Both Axis

If the axial load P is placed eccentric to both x-axis and y-axis as shown in Figure 7.4, then the system can be assumed to consist of

(a) A direct compressive force P acting at the centroid,

(b) A couple $P \times e_x$ about the x-axis, and

(c) A couple $P \times e_y$ about the y-axis.

As seen for the case of load acting eccentric to one axis, the stress at any point can be written as

$$\begin{aligned} f &= f_0 + f_{b1} \pm f_{b2} \\ &= \frac{P}{A} \pm \frac{P \times e_x}{I_{xx}} \times y \pm \frac{P \times e_y}{I_{xx}} \times x \\ &= \frac{P}{A} \pm \frac{M_{xx}}{Z_{xx}} \pm \frac{M_{yy}}{Z_{yy}} \end{aligned}$$

The maximum or minimum fibre stress will occur at the corner point A, B, C or D in Figure

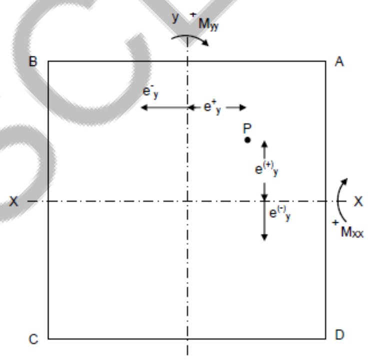


Fig 3.3 Maximum and Minimum stress

3.2 Middle Third Rule

In Figure $f_0 < f_b$ and therefore, stress changes sign, being partly tensile and partly compressive across the section. In masonry and concrete structures, the development of e-tensile stress in the section is not desirable, as they are weak in tension. This limits the eccentricity e to a certain value which will be investigated now for different sections.

In order that the stress may not change sign from compressive to tensile, we have

$$f_0 \geq f_b$$

$$\frac{P}{A} \geq \frac{Pe}{I} \times \frac{d}{2}$$

$$\frac{P}{A} \geq \frac{Ped}{2AK^2}$$

$$e \leq \frac{2k^2}{d}$$

Where, k = radius of gyration of the section with regard to N.A. and d is the depth of the section. Thus, for no tension in the section, the eccentrically must not exceed

$$\frac{2k^2}{d}.$$

Let us now take a rectangular section and find out the limiting value of e.

For a rectangular section of width b and depth d,

$$I = \frac{1}{12} bd^3 \text{ and } A = bd$$

$$k^2 = \frac{I}{A} = \frac{d^2}{12}$$

Substituting we get

$$e \leq \frac{2d^2}{d \times 12} \leq \frac{d}{6}$$

$$I_{\max} = \frac{d}{6}$$

The value of eccentricity can be on either side of the geometrical axis. Thus, the stress will be of the same sign throughout the section if the load line is within the middle third of the section.

In the case of rectangular section, the maximum intensities of extreme stresses are given by

$$f = \frac{P}{A} \pm \frac{Pl}{Z_{xx}} = \frac{P}{bd} \pm \frac{6pe}{bd^2}$$

$$= \frac{P}{bd}\left[1 \pm \frac{6e}{d}\right]$$

3.3 Core Section

If the line of action of the stress is on neither of the centre lines of the section, the bending is unsymmetrical. However, there is certain area within the line of action of the force *P* must cut the cross-section if the stress is not to become tensile. This area we call it as 'core' or 'kernel' of the section. Let us calculate this for a rectangular section.

3.3.1 Rectangular Section

Let the point of application of the load *P* have the coordinates (x, y), with reference to the axes shown in Figure in which x is positive when measured to right of O and y is positive when measured upwards.

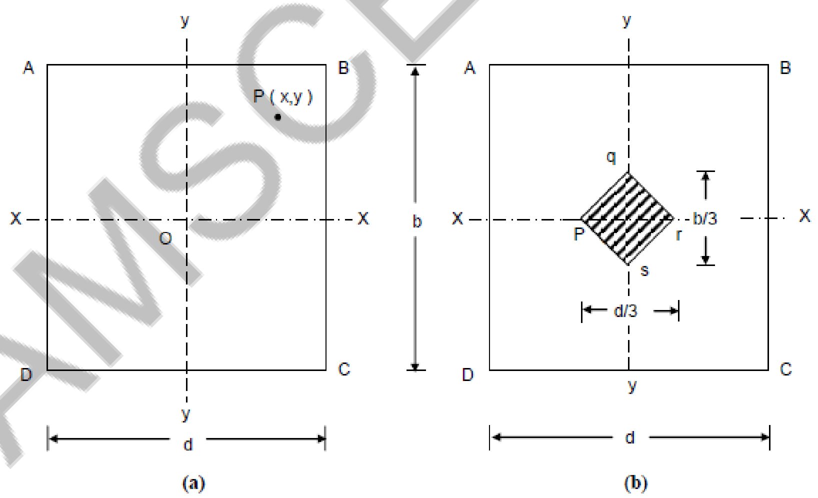


Fig 3. 4Rectangular Section

The stress at any point have coordinates (x', y') will be

$$f = \frac{P}{bd} + \frac{P \times yy'}{\frac{1}{12}db^3} + \frac{P \times xx'}{\frac{1}{12}bd^3}$$

$$= \frac{12P}{bd}\left(\frac{1}{12}\frac{yy'}{b^2} + \frac{xx'}{d^2}\right)$$

At *D*,

$x' = -\frac{d}{2}$ and $y' = -\frac{b}{2}$

And, therefore, *f* will be minimum. Thus, at *D*, we have,

$$f = \frac{6P}{bd}\left(\frac{1}{6} - \frac{y}{b} - \frac{x}{d}\right)$$

The value of *f* reaches zero when

$$\frac{y}{b} + \frac{x}{d} = \frac{1}{6} \text{ or } \frac{6y}{b} + \frac{6x}{d} = 1$$

Thus, the deviation of the load line is governed by the straight line of), whose intercepts on the axes are respectively b/6and d/6. This is true for the load line is the first quadrant. Similar limits will apply in other quadrants and the stress will be wholly compressive throughout the section, if the line of action of *P* will within the rhombus *pqrs* the diagonals of which are of length d/3and b/3, respectively. This rhombus is called the core of the rectangular section.

3.4 Column with Unsymmetrical section (angle channel section)

A short piece of ISA (200 × 100 × 15) angle carries a compressive load, the line of action of which coincides with the intersection of the middle planes of the legs. If the maximum compressive stress is not to exceed 112 N/mm^2, what is the safe axial load *P*? Given *A* = 4278 mm^2, r_{xx} = 64 mm, r_{yy} = 26.4 mm.

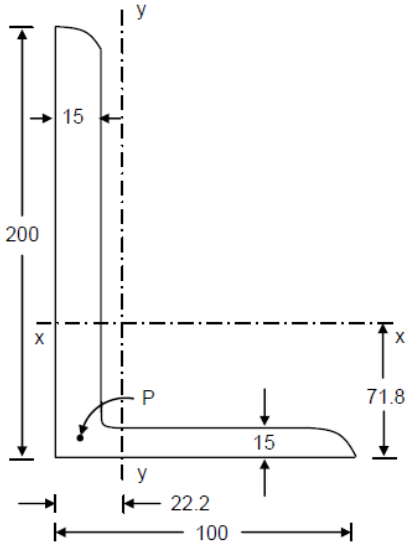


Fig 3.5 Angle Section

Solution

Area of cross-section $A = 4278 \text{ mm}^2$

Eccentricity of load with respect of xx-axis = (71.8 – 7.5) = 64.3 mm

Eccentricity of load with respect to yy-axis = 22.2 – 7.5 = 14.7 mm

Maximum compressive stress at any section

$$= \frac{P}{A} + \frac{M_{xx}}{I_{xx}} \times y + \frac{M_{yy}}{I_{yy}} \times x$$

or $$f_{\max} = \frac{P}{A}\left(1 + \frac{e_{xx}}{r_{xx}^2} \times y + \frac{e_{yy}}{r_{yy}^2} \times x\right)$$

Here, $r_{xx} = 64$ mm and $r_{yy} = 26.4$ mm

$f_{\max} = 112 \text{ N/mm}^2$

$$\therefore \qquad 112 = \frac{P}{A}\left(1 + \frac{64.3 \times 71.8}{(64)^2} + \frac{14.7 \times 22.2}{(26.4)^2}\right)$$

$$= \frac{P}{4278}[1 + 1.127 + 0.4684]$$

$$\therefore \qquad P = 184.6 \text{ kN.}$$

3.5 Euler's theory of long Columns

The following assumptions are made in this theory:

1. The column is initially straight and the applied load is truely axial.

2. The material of the column is homogeneous, linear and isotropic.

3. The length of the column is very large as compared to the cross-sectional dimensions of the column.

4. The cross-section of the column is uniform throughout.

5. The shortening of the column due to axial compression is negligible.

6. The self-weight of the column is neglected.

7. The ends of the column are frictionless.

1. **Fixed end.** For such an end, deflection (y) and slope $\left(\frac{dy}{dx}\right)$ are both zero. *i.e.*

$$y = 0$$

$$\frac{dy}{dx} = 0$$

2. Pinned end. For such an end, y =0

3. Free end, the column is neither fixed in position nor in direction.

Depending upon the end conditions, there are four types of columns.

• Both ends hinged

• Both ends fixed

• One end fixed and other end hinged

• One end fixed and other end free.

3.6 Critical Load of Prismatic Columns with different end conditions

3.6.1 Column hinged at Both Ends

Consider a column having both ends hinged and carrying an axial compressive load P as shown in Fig. Taking origin at A, the bending moment at a distance x is

$$M_x = Py$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = -M_x = -Py$$

or $$EI\frac{d^2y}{dx^2} + Py = 0$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

Let $k^2 = \frac{P}{EI}$, then

$$\frac{d^2y}{dx^2} + k^2y = 0$$

or $$(D^2 + k^2)\,y = 0$$

where $$D^2 = \frac{d^2}{dx^2}$$

The auxiliary equation is :

$$D^2 + k^2 = 0$$

or $$D = \pm ik, \; i = \sqrt{-1}$$

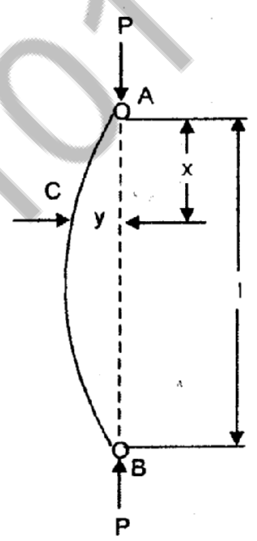


Fig 3.6 Hinged at both ends

The general solution of Eq. (*a*) is

$$y = A\cos kx + B\sin kx$$

where A and B are arbitrary constants.

The end conditions are :

At $x = 0, y = 0$ and at $x = l, y = 0$

$$\therefore \quad A = 0$$

$$B\sin kl = 0$$

Now $B \neq 0$, because if $B = 0$ then $y = 0$, and the column will remain straight, which is not true.

$$\therefore \quad \sin kl = 0$$

or $$kl = n\pi,\ n = 0, 1, 2, \ldots\ .$$

Taking the fundamental value, $n = 1$, we get

$$kl = \pi$$

or $$\sqrt{\frac{P}{EI}}\ l = \pi$$

$$P = \frac{\pi^2 EI}{l^2}$$

This load is called the Euler's critical load and is denoted by P_e.

$$P_e = \frac{\pi^2 EI}{l^2}$$

Now $I = AK^2$

where A = area of cross-section of the column

K = least radius of gyration

$$P_e = \frac{\pi^2 EA}{\left(\frac{l}{K}\right)^2}$$

Now $\frac{l}{K}$ = slenderness ratio of column = λ

$$P_e = \frac{\pi^2 EA}{(\lambda)^2}$$

Euler stress, $\sigma_e = \frac{P_e}{A} = \frac{\pi^2 E}{\lambda^2}$

3.6.2 Column fixed at both ends

Consider the column fixed at both ends as shown in Fig. Let M_A and M_B be the fixing moments at the ends. At a distance x from A,

$$M_x = Py - M_A$$

$$EI\frac{d^2y}{dx^2} = -M_x = M_A - P\underline{y}$$

or $$EI\frac{d^2y}{dx^2} + Py = M_A$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M_A}{EI} \qquad \ldots(a)$$

Let $$k^2 = \frac{P}{EI}$$

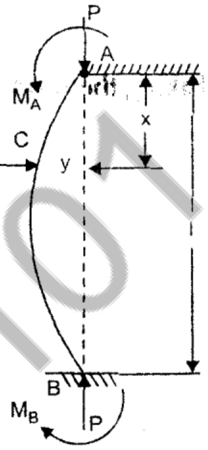


Fig 3.7 Column fixed at both ends

The general solution of Eq. (*a*) is :

$$y = A\cos kx + B\sin kx + \frac{M_A}{P}$$

The end conditions are

At $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$

$$\therefore \quad A + \frac{M_A}{P} = 0$$

$$A = -\frac{M_A}{P}$$

$$\frac{dy}{dx} = -Ak\sin kx + Bk\cos kx$$

$$0 = B$$

$$y = \frac{M_A}{P}(1 - \cos kx)$$

At $x = l$, $y = 0$ $$0 = \frac{M_A}{P}(1 - \cos kl)$$

Now $\frac{M_A}{P_E} \neq 0$

$$\therefore \quad 1 - \cos kl = 0$$
$$\cos kl = 1$$
$$kl = 2n\pi, n = 0, 1, 2, ...$$

Taking the fundamental value, $n = 1$,

$$kl = 2\pi$$
$$\sqrt{\frac{P}{EI}}\, l = 2\pi$$
$$P = \frac{4\pi^2 EI}{l^2} = \frac{\pi^2 EI}{\left(\frac{l}{2}\right)^2} = \frac{\pi^2 EI}{l_e^2}$$

where $l_e = \frac{l}{2}$ **is the equivalent length.**

3.6.3 Column fixed at one end and hinged at the other

Consider a column hinged at end A and fixed at B as shown in Fig. There will be a buckling moment$_{MB}$ and a horizontal force R_A will have to be applied to A to keep the column in equilibrium. At a distance x from A,

$$M_x = Py - R_A.x$$
$$EI\frac{d^2y}{dx^2} = -M_x = R_A x - Py$$
$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{R_A.x}{EI} \quad ....(a)$$

The general solution of Eq. (a) is :

$$y = A\cos kx + B\sin kx + \frac{R_A.x}{P}$$

where $k^2 = \frac{P}{EI}$

The end conditions are :

At $x = 0, y = 0$

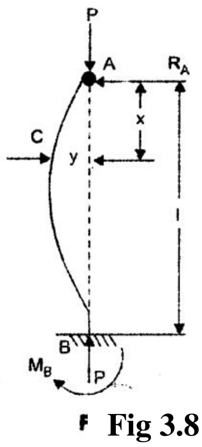


Fig 3.8 One end fixed other end Hinged

$\therefore \qquad A = 0$

At $x = l, y = 0$

$$0 = B \sin kl + \frac{R_A . l}{P}$$

At $x = l, \frac{dy}{dx} = 0$

Now $$\frac{dy}{dx} = Bk \cos kx + \frac{R_A}{P}$$

$$0 = Bk \cos kl + \frac{R_A}{P}$$

or $$B = -\frac{R_A}{Pk} \times \frac{1}{\cos kl}$$

Substituting in Eq. (*b*), we get

$$0 = -\frac{R_A}{Pk} . \tan kl + \frac{R_A . l}{P} = \frac{R_A}{P}\left[l - \frac{\tan kl}{k}\right]$$

Now $\frac{R_A}{P} \neq 0$

$\therefore \qquad \tan kl - kl = 0$

or $\qquad kl = 4.49$

$$\sqrt{\frac{P}{EI}}\, l = 4.49$$

$$P = \frac{20\,EI}{l^2} = \frac{2\pi^2 EI}{l^2} = \frac{\pi^2 EI}{\left(\frac{l}{\sqrt{2}}\right)^2} = \frac{\pi^2 EI}{l_e^2}$$

$\therefore$ Equivalent length, $l_e = \frac{l}{\sqrt{2}}$

3.6.4 Column fixed at one end and free at the other

Consider a column as shown in Fig. Let the horizontal deflection of end A be and fixing moment at end B be M_B.

At a distance x from A,

$$M_x = -P(e-y)$$

$$EI\frac{d^2y}{dx^2} = -M_x = P(e-y)$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{Pe}{EI} \quad ...(a)$$

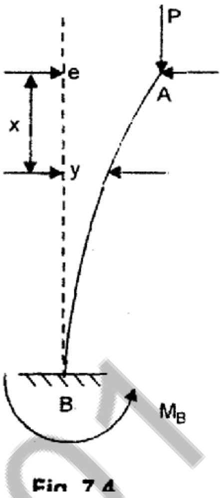


Fig 3.9 One end fixed and other end free

The general solution of Eq. (a) is :

$$y = A\cos kx + B\sin kx + e$$

where $k^2 = \frac{P}{EI}$

The end conditions are :

At $x = l$, $y = 0$ and $\frac{dy}{dx} = 0$

$$0 = A\cos kl + B\sin kl + e \quad ...(b)$$

$$\frac{dy}{dx} = -Ak\sin kx + Bk\cos kx$$

$$0 = -Ak\sin kl + Bk\cos kl \quad ...(c)$$

At $x = 0$, $y = e$

$$e = A + e$$

or $$A = 0 \quad ...(d)$$

From Eq. (b), we get

$$B = \frac{-e}{\sin kl}$$

Substituting in Eq. (c), we get

$$\frac{-ek}{\sin kl}.\cos kl = 0$$

or $$\cot kl = 0$$

$$kl = (2n+1)\frac{\pi}{2},\ n = 0, 1, 2, ...$$

For $n = 0$, $$kl = \frac{\pi}{2}$$

$$\sqrt{\frac{P}{EI}}\,l = \frac{\pi}{2}$$

$$P = \frac{\pi^2 EI}{4l^2} = \frac{\pi^2 EI}{(2l)^2} = \frac{\pi^2 EI}{l_e^2}$$

where equivalent length, $l_e = 2l$.

3.6. 5 Limitations of Euler's Theory

1. The Euler's theory is applicable to columns which are initially exactly straight and the load is truly axial. However, there is always some crookedness in the column and the load may not be exactly axial.
2. This theory is applicable to long columns only.
3. This theory does not take into account the axial compressive stress.

3.7 Rankine Gordon Formula for eccentrically loaded columns

A prediction of buckling loads by the Euler formula is only reasonable for very long and slenderstruts that have very small geometrical imperfections. In practice, however, most struts sufferplastic knockdown and the experimentally obtained buckling loads are much less than the Eulerpredictions. For struts in this category, a suitable formula is the Rankine4ordon formula whichis **a** semi-empirical formula, and takes into account the crushing strength of the material, its young's modulus and its slenderness ratio, namely *L/k,* where

L = length of the strut

k = least radius of gyration of the strut's cross-section

$P_c = \sigma_c A$

where

A = cross-sectional area

σ_c = crushing stress

Then

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

where

P_R = Rankine–Gordon buckling load
P_e = Euler buckling load

$$= \frac{\pi^2 EI}{L^2} \text{ for a pin-ended strut}$$

$$\therefore \frac{1}{P_R} = \frac{L_0^2}{\pi^2 EI} + \frac{1}{\sigma_{yc} \times A}$$

$$= \frac{L_0^2}{\pi^2 EAk^2} + \frac{1}{\sigma_{yc} A}$$

$$= \frac{L_0^2 \sigma_{yc} + \pi^2 Ek^2}{\pi^2 EAk^2 \sigma_{yc}}$$

or

$$P_R = \frac{\pi^2 EAk^2 \sigma_{yc}}{L_0^2 \sigma_{yc} + \pi^2 Ek^2}$$

$$= \frac{\sigma_{yc}}{L_0^2 \sigma_{yc} / \pi^2 EAk^2 + \pi^2 Ek^2 / \pi^2 EAk^2}$$

$$P_R = \frac{\sigma_{yc} \times A}{\left(\sigma_{yc} / \pi^2 E\right) \left(L_0 / k\right)^2 + 1}$$

Let

$$a = \frac{\sigma_{yc}}{\pi^2 E}$$

Then

$$P_R = \frac{\sigma_{yc} A}{1 + a(L_0 / K)^2}$$

Where*a* is the denominator constant in the Rankine-Gordon formula, which is dependent on theboundary conditions and material properties.

A comparison of the Rankine-Gordon and Euler formulae, for geometrically perfect struts, isgiven in Figure

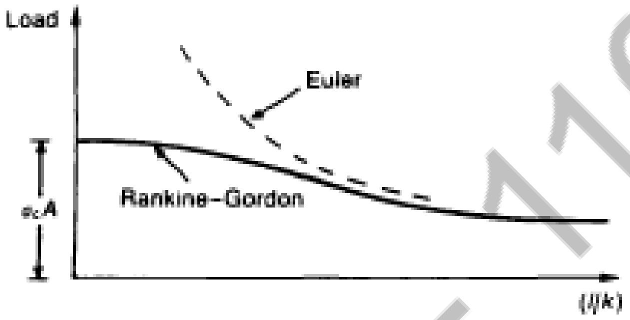


Fig 3.10 Graphical Representation

3.8 Thick Cylinders

3.8.1 Cylindrical Vessel with Hemispherical Ends:

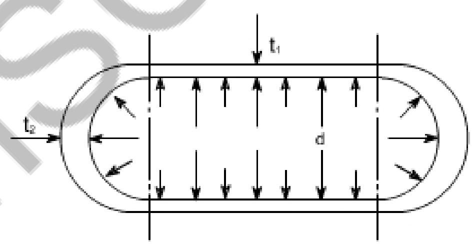


Fig 3.11Section of Cylindrical Vessel with Hemispherical Ends

* Let us now consider the vessel with hemispherical ends. The wall thickness of the cylindrical and hemispherical portion is different.While the internal diameter of both the portions is assumed to be equal

* Let the cylindrical vassal is subjected to an internal pressure p.

hoop or circumferential stress = σ_{HC} 'c' here synifies the cylindrical portion.

$$= \frac{pd}{2t_1}$$

longitudnal stress = σ_{LC}

$$= \frac{pd}{4t_1}$$

hoop or circumferential strain $\epsilon_2 = \frac{\sigma_{HC}}{E} - \nu\frac{\sigma_{LC}}{E} = \frac{pd}{4t_1E}[2-\nu]$

or $\boxed{\epsilon_2 = \frac{pd}{4t_1E}[2-\nu]}$

3.8.2 For the Hemispherical Ends

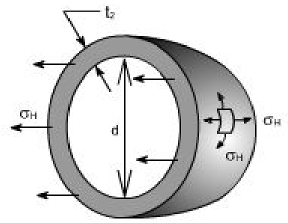


Fig 3.10 Section of Hemispherical ends

Because of the symmetry of the sphere the stresses set up owing tointernal pressure will be two mutually perpendicular hoops orcircumferential stresses of equal values.

Again the radial stresses are neglected in comparison to the hoopstresses as with this cylinder having thickness to diameter lessthan1:20.

* Consider the equilibrium of the half – sphere

* Force on half-sphere owing to internal pressure = pressure x projected Area

= p. pd^2 /4

Resisting force = $\sigma_H . \pi.d.t_2$

$$\therefore \quad p.\frac{\pi.d^2}{4} = \sigma_H . \pi d.t_2$$

$$\Rightarrow \sigma_H \text{(for sphere)} = \frac{pd}{4t_2}$$

similarly the hoop strain $= \frac{1}{E}\left[\sigma_H - \nu.\sigma_H\right] = \frac{\sigma_H}{E}[1-\nu] = \frac{pd}{4t_2E}[1-\nu]$ or $\boxed{\epsilon_{2s} = \frac{pd}{4t_2E}[1-\nu]}$

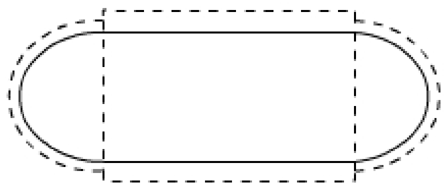


Fig 3.11 Strain of Cylinder

Fig – shown the (by way of dotted lines) the tendency, for the cylindrical portion and the spherical ends to expand by a different amount under the action of internal pressure.

* So owing to difference in stress, the two portions (i.e.cylindrical and spherical ends) expand by a different amount.

* This incompatibly of deformations causes a local bending and sheering stresses in the neighborhood of the joint.

* Since there must be physical continuity between the ends and the cylindrical portion, for this reason, properly curved ends must be used for pressure vessels.

* Thus equating the two strains in order that there shall be no distortion of the junction

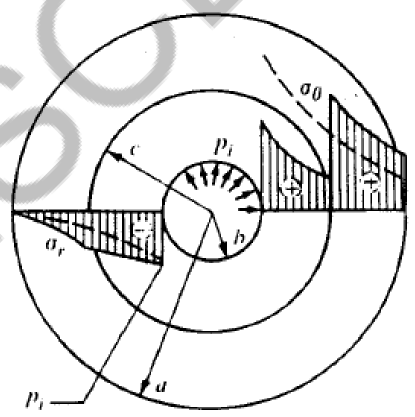
$$\frac{pd}{4t_1E}[2-\nu] = \frac{pd}{4t_2E}[1-\nu] \quad \text{or} \quad \frac{t_2}{t_1} = \frac{1-\nu}{2-\nu}$$

* But for general steel works ν = 0.3, therefore, the thickness ratiosbecomes

$$t_2 / t_1 = 0.7/1.7 \text{ or}$$

$$\frac{t_1}{t_2} = 2.4$$

3.9 Compound Cylinders



(b)

Fig 3.12 Compound Cylinder

- Once Pcis known, the residual stress in the 2 cylinders can be calculated by Lamesolution
- Superposing the stresses of internal pressure Pi → more efficient use of material

QUESTION BANKS

2 Marks

1. Define column
2. Define crippling load
3. Differentiate Long & Short Column?
4. Explain Buckling of Column?
5. Explain Neutral Axis?
6. State Euler's formula for crippling load
7. State Rankine formula for the crippling
8. State slenderness ratio
9. State two assumptions made in the Euler's column's theory.
10. What are the assumption in Euler's theory
11. Define thick cylinder and thin cylinder
12. What is hoop stress
13. What are the limitations of Euler's formula
14. Draw the hoop stress distribution diagram across the thickness of cylinder
15. How will you differentiate a thin cylinder from thick cylinder
16. What are the different methods of reducing hoop stresses?
17. What do you mean by thick compound cylinder?
18. What are auxiliary equations
19. Define Middle Third Rule
20. What is Kernel Section

16 Marks

1. A thick cylinder 120 mm internal diameter and 180 mm external diameter is used for a working pressure of 15 MN/m^2. Because of external corrosion the outer diameter of the cylinder is machined to 178 mm. Determine by how much the internal pressure is to be reduced so that the maximum hoop stress remains the same as before.
2. A compound cylinder is formed by shrinking one cylinder on to another, the final dimensions being internal diameter 120mm, external diameter 240 mm and diameter

at junction 200mm. After shrinking on the radial pressure at the common surface is 10 MN/m^2

i. Calculate the necessary difference in diameter of the two cylinders at the common surface.

ii. What is the minimum temperature through which outer cylinder should be heated before it is slipped on?

3. Derive and expression for critical load of prismatic columns with both end hinged in conditions
4. Derive and expression for critical load of prismatic columns with both end fixed in conditions
5. Derive and expression for critical load of prismatic columns with one end hinged and other end fixed in conditions
6. Derive and expression for critical load of prismatic columns with one end fixed and other end free in conditions

STRENGTH OF MATERIALS-II

UNIT IV

STATE OF STRESS IN THREE DIMENSIONS

CONTENTS

TECHNICAL TERMS

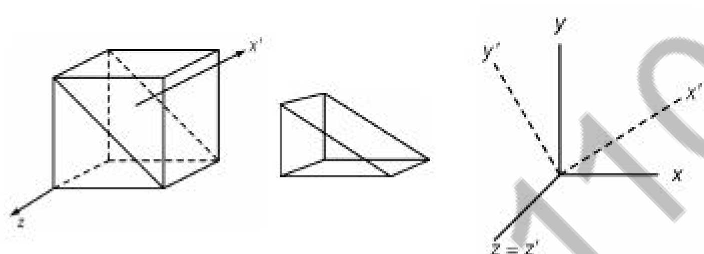
1. **Stress Tensor-** The nine stress components given by the group of square matrix of stresses or the components of a mathematical entity is called stress tensor.
2. **Stress Invariants-** The combinations of stresses at a point which do not change with the orientation of the co-ordinate axes are called Stress Invariants.
3. **Principal Plane-** It is a plane where shear force is zero is called principal plane
4. **PrincipalStress**- The normal stress on the principal plane is called principal stress
5. **Strain energy-** Whenever a body is strained, the energy is absorbed in the body. The energy is absorbed in the body due to straining effect is known as strain energy.
6. **Virtual work-** It is the work done by the forces acting on the particle during a virtual displacement.

IV STATE OF STRESS IN THREE DIMENSIONS

4.1 Spherical and deviatory components of stress tensor

Stress tensor

The state of stress at a point can be defined by three components on each of the three mutually perpendicular axes in mathematical terminology



Basically we have done so far for this type of coordinate system

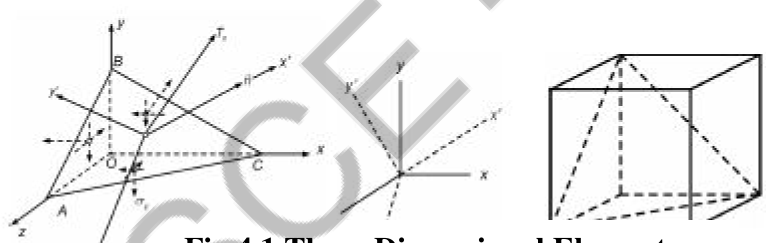


Fig 4.1 Three Dimensional Elements

$$n_{x'x} \quad n_{x'y} \quad n_{x'z} \; - \; Dir.cosines \; of \; x'$$

$$\hat{i}' = n_{x'x}\hat{i} + n_{x'y}\hat{j} + n_{x'z}\hat{k}$$

$$n_{y'x} \quad n_{y'y} \quad n_{y'z}$$

$$\hat{j}' = n_{y'x}\hat{i} + n_{y'y}\hat{j} + n_{y'z}\hat{k}$$

$$n_{z'x} \quad n_{z'y} \quad n_{z'z}$$

$$\hat{k}' = n_{z'x}\hat{i} + n_{z'y}\hat{j} + n_{z'z}\hat{k}$$

$$\bar{T}_n = \bar{T}_{x'x}\hat{i} + T_{x'y}\hat{j} + T_{x'z}\hat{k}s$$

$$\bar{T}_n = \sigma_{x'x'}\hat{i}' + \tau_{x'y'}\hat{j}' + \tau_{x'z'}\hat{k}'$$

$$ABC - dA$$
$$PAB - dA\, n_{x'x}$$
$$PAC - dA\, n_{x'x}$$
$$PBC - dA\, n_{x'z}$$

$$[\Sigma F_x \rightarrow + = 0]$$

$$T_{x'x} da = \sigma_x dA n_{x'x} + \tau_{yx} dA n_{x'y} + \tau_{zx} dA n_{x'z}$$

$$\left.\begin{aligned} T_{x'x} &= \sigma_x n_{x'x} + \tau_{yx} n_{x'y} + \tau_{zx} n_{x'z} \\ T_{x'y} &= \tau_{xy} n_{x'x} + \sigma_y n_{x'y} + \tau_{zy} n_{x'z} \\ T_{x'z} &= \tau_{xz} n_{x'x} + \tau_{yz} n_{x'y} + \sigma_z n_{x'z} \end{aligned}\right\} \begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{y'z'} \\ \tau_{x'y'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{z'x'} & \tau_{y'z'} & \sigma_{z'} \end{bmatrix}$$

$\sigma_{x'}, \tau_{x'y'}, \tau_{x'z'}$

$$\sigma_{x'} = \bar{T}_n \hat{i}' = \left(T_{x'x}\hat{i} + T_{x'y}\hat{j} + T_{x'z}\hat{k}\right).\left(n_{x'x}\hat{i} + n_{x'y}\hat{j} + n_{x'z}\hat{k}\right) \qquad (1)$$

$$\tau_{x'y'} = \bar{T}_n \hat{j}' = \left(T_{x'x}\hat{i} + T_{x'y}\hat{j} + T_{x'z}\hat{k}\right).\left(n_{y'x}\hat{i} + n_{y'y}\hat{j} + n_{y'z}\hat{k}\right) \qquad (2)$$

$$\tau_{x'z'} = \bar{T}_n \hat{k}' = \left(T_{x'x}\hat{i} + T_{x'y}\hat{j} + T_{x'z}\hat{k}\right).\left(n_{z'x}\hat{i} + n_{z'y}\hat{j} + n_{z'z}\hat{k}\right) \qquad (3)$$

$$T_{y'x} = \sigma_x n_{y'x} + \tau_{yx} n_{y'y} + \tau_{zx} n_{y'z}$$
$$T_{y'y} = \tau_{xy} n_{y'y} + \sigma_y n_{y'y} + \tau_{zy} n_{y'z}$$
$$T_{y'z} = \tau_{xz} n_{y'y} + \tau_{yz} n_{y'y} + \sigma_z n_{y'z}$$

$$\sigma_{y'} = \left(T_{y'x}\hat{i} + T_{y'y}\hat{j} + T_{y'z}\hat{k}\right)\left(n_{y'x}\hat{i} + n_{y'y}\hat{j} + n_{y'z}\hat{k}\right) \qquad (4)$$

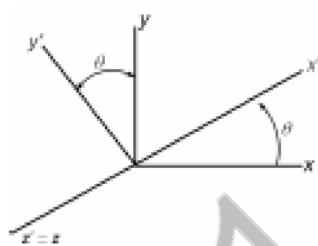
$$\sigma_{z'} = \left(T_{z'x}\hat{i} + T_{z'y}\hat{j} + T_{z'z}\hat{k}\right)\left(n_{z'x}\hat{i} + n_{z'y}\hat{j} + n_{z'z}\hat{k}\right) \qquad (5)$$

$$\tau_{y'z'} = \left(T_{y'x}\hat{i} + T_{y'y}\hat{j} + T_{y'z}\hat{k}\right)\left(n_{z'x}\hat{i} + n_{z'y}\hat{j} + n_{z'z}\hat{k}\right) \qquad (6)$$

$$n_{x'x} = Cos\theta \quad n_{x'y} = Sin\theta \quad n_{x'z} = 0$$

$$n_{y'x} = -Sin\theta \quad n_{y'y} = Cos\theta \quad n_{y'z} = 0$$

$$n_{z'x} = 0 \quad n_{z'y} = 0 \quad n_{z'z} = 1$$



$$\sigma_{z'} = 0 : \tau_{x'z'} = 0 : \tau_{y'z'} = 0$$

$$= \sigma_z$$

$$\sigma_{x'} = \sigma_x Cos^2\theta + \sigma_y Sin^2\theta + 2\tau_{xy} Sin\theta Cos\theta$$

$$\sigma_{y'} = \sigma_x Sin^2\theta + \sigma_y Cos^2\theta - 2\tau_{xy} Sin\theta Cos\theta$$

$$\tau_{x'y'} = -\left(\sigma_x - \sigma_y\right) Sin\theta Cos\theta + \tau_{xy}\left(Cos^2\theta - Sin^2\theta\right)$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4.2 Principal stress and Principal Plane

Principal stresses

$$n_x, n_y, n_z$$

$$\vec{T}_n = \sigma\hat{n} = \sigma\left(n_x\hat{i} + n_y\hat{j} + n_z\hat{k}\right)$$

$$\vec{T}_n = T_{nx}\hat{i} + T_{ny}\hat{j} + T_{nz}\hat{k}$$

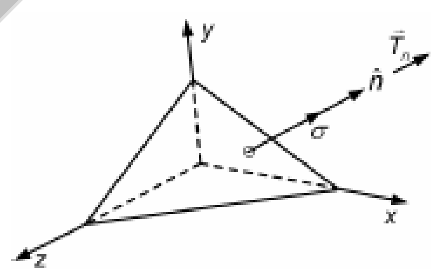


Fig 4.2Stress Elements

Where

$$T_{nx} = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z$$

$$T_{ny} = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z$$

$$T_{nz} = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$

$$Tn_x = \sigma n_x \,\Big|\, Tn_y = \sigma n_y \,\Big|\, Tn_z = \sigma n_z$$

$$\left.\begin{aligned}(\sigma_x-\sigma)n_x+\tau_{yx}n_y+\tau_{zx}n_z=0\\ \tau_{yx}n_x+(\sigma_y-\sigma)n_y+\tau_{zy}n_z=0\\ \tau_{xz}n_x+\tau_{yz}n_y+(\sigma_z-\sigma)n_z=0\end{aligned}\right\}\ Syst.\,of\ linear\ hom\,og.\,eqns.$$

$$n_x=n_y=n_z=0:\ \ n_x^2+n_y^2+n_z^2=1$$

$$\begin{bmatrix}\sigma_x-\sigma & \tau_{xy} & \tau_{zx}\\ \tau_{xy} & \sigma_y-\sigma & \tau_{zy}\\ \tau_{zx} & \tau_{yz} & \sigma_z-\sigma\end{bmatrix}\begin{Bmatrix}n_x\\ n_y\\ n_z\end{Bmatrix}=(0)$$

For non trivial solution || must be zero.

$$\sigma^3-(\sigma_x+\sigma_y+\sigma_z)\sigma^2+(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2)\sigma$$
$$-(\sigma_x\sigma_y\sigma_z+2\tau_{xy}\tau_{yz}\tau_{zx}-\sigma_x\tau_{yz}^2-\sigma_y\tau_{zx}^2-\sigma_z\tau_{xy}^2)=0$$

This has 3- real roots $\sigma_1,\sigma_2,\sigma_3$

$$\left.\begin{aligned}(\sigma_x-\sigma_1)n_x+\tau_{yx}n_y+\tau_{zx}n_z=0\\ \tau_{yx}n_x+(\sigma_y-\sigma_1)n_y+\tau_{zy}n_z=0\\ and\qquad n_x^2+n_y^2+n_z^2=1\end{aligned}\right\}$$

$$\Rightarrow\quad n_x,n_y,n_z\to\sigma_1$$
$$\sigma_1>\sigma_2>\sigma_3$$

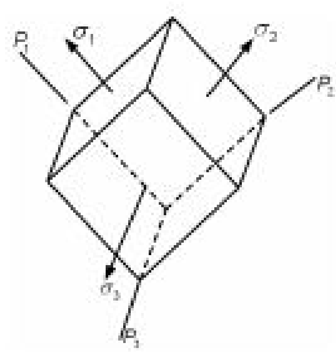


Fig 4.2Square Elements

Stress invariants

$$\sigma^3-I_1\sigma^2+I_2\sigma-I_3=0\qquad(1)$$

$$\left.\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z \\ I_2 &= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \\ I_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 \end{aligned}\right\} stress\ invariants$$

$$\sigma^3 - I_1'\sigma^2 + I_3' = 0$$

$$I_1' = \sigma_{x'} + \sigma_{y'} + \sigma_{z'} \Big| I_2' = \sigma_{x'}\sigma_{y'} + \sigma_{x'}\sigma_{z'} + \sigma_{y'z'} - \tau_{x'y'}^2 - \tau_{y'z'}^2 - \tau_{x'z'}^2$$

$$I_1 = I_1'; \quad I_2 = I_2'; \; I_3 = I_3'$$

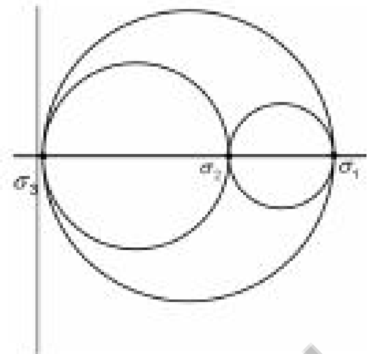
3D	2D
$I_1 = \sigma_1 + \sigma_2 + \sigma_3$ $I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$ $I_3 = \sigma_1\sigma_{2\sigma_3}$	$I_1 = \sigma_1 + \sigma_2$ $I_2 = \sigma_1\sigma_2$ $I_3 = 0$

To get an idea about the plane on which the resultant stresses are wholly normal and the plane on which the normal stress is maximum and shear stress is zero

Plane stress

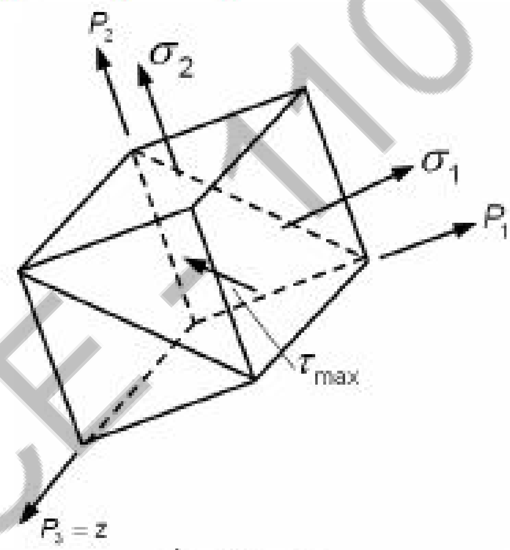
$\sigma_1 > \sigma$

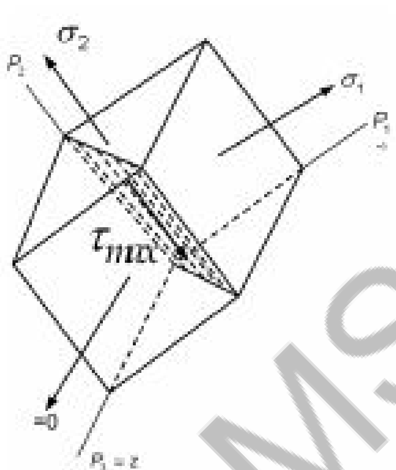
$\sigma_3 = \sigma_z = 0$

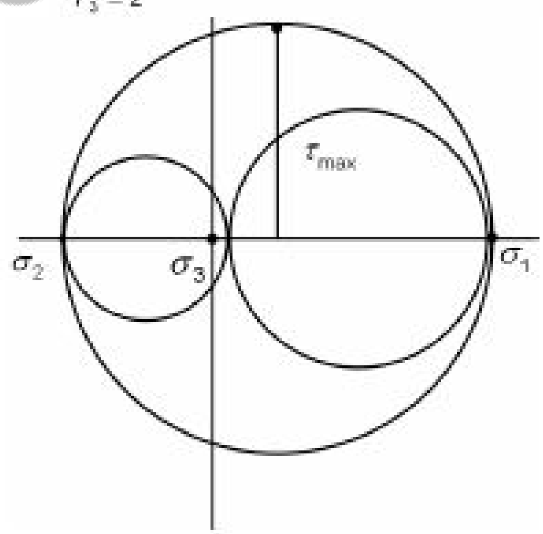


$$\tau = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$ ---- in plane principal shear stresses.

$$\tau_{max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = \left|\frac{\sigma_1}{2}\right|$$







4.2.1 Octahedral Planes and stresses

If $n_x = n_y = n_z$ w.r.t to ***the principal planes***, then these planes are known as octahedral planes. The corresponding stresses are known as octahedral stresses

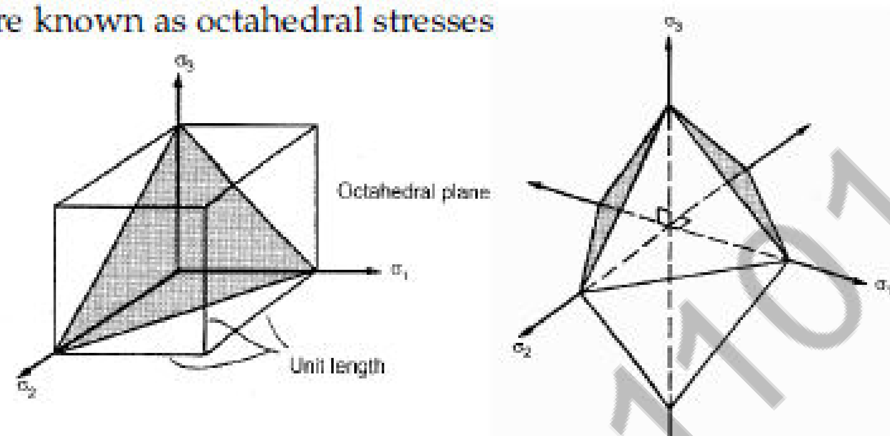


Fig 4.3Tetrahedral and Octahedral

Eight number of such planes can be identified at a given point --- Octahedron

$$\sigma = \sigma_1 n_x^2 + \sigma_2 n_y^2 + \sigma_3 n_z^2$$

$$|T_n|^2 = \sigma_1^2 n_x^2 + \sigma_2^2 n_y^2 + \sigma_3^2 n_z^2$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$n_x = n_y = n_z = \pm \frac{1}{\sqrt{3}} = 54.73^0$$

$$\sigma_{oct} = \sigma_1 \left(\frac{1}{\sqrt{3}}\right)^2 + \sigma_1 \left(\frac{1}{\sqrt{3}}\right)^2 + \sigma_1 \left(\frac{1}{\sqrt{3}}\right)^2$$

$$= \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

$$\frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{I_1}{3} = mean\,stress$$

σ_{oct} = can be interpreted -- mean normal stress at a pt.

$$\tau_{oct} = \sqrt{|T_n|^2 - \sigma_{oct}^2}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

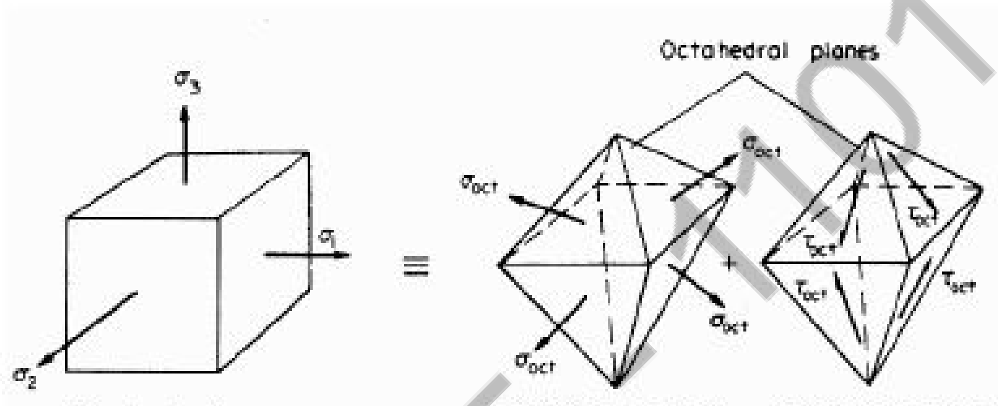


Fig 4.4shear stress and normal stress for Octahedral

Therefore, the state of stress at a point can be represented with reference to

(i) stress components of x,y,z coordinate system

(ii) stress components of x′,y′z′ coordinate system

(iii) using principal stresses

(iv) using octahedral shear and normal stresses

4.3 Volumetric Strain

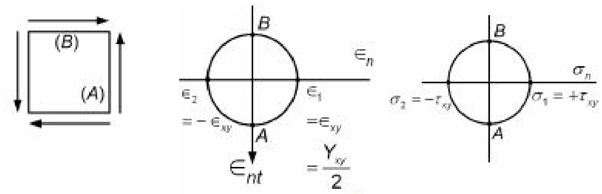


Fig 4.5shear strain on a plane

$$\sigma_1 = \tau_{xy} \qquad \in_1 = \frac{1}{E}(\sigma_1 - v\sigma_2)$$

$$\sigma_2 = -\tau_{xy} \qquad \in_2 = \frac{1}{E}(\sigma_2 - v\sigma_1)$$

$$\in_1 = \frac{1}{E}\left(\tau_{xy} + v\tau_{xy}\right) = \frac{\tau_{xy}(1+v)}{E}$$

$$\in_2 = \frac{-\tau_{xy}(1+v)}{E}$$

$$\in_1 = \in_{xy} = \frac{\gamma_{xy}}{2} = \frac{\tau_{xy}}{2G}$$

$$\in_2 = \frac{-\tau_{xy}}{2G}$$

$$\frac{\tau_{xy}(1+v)}{E} = \frac{\tau_{xy}}{2G} \quad \Rightarrow$$

$$G = \frac{E}{2(1+v)}$$

Only two elastic constants are independent.

4.4 Volumetric strain dilatation and distortion

Consider a stress element size dx, dy, dz

dv = dxdydz

After deformation

$$dx^* = (1+\epsilon_x)dx$$
$$dy^* = (1+\epsilon_y)dy$$
$$dz^* = (1+\epsilon_z)dz$$

In addition to the changes of length of the sides, the element also distorts so that right angles no longer remain sight angles. For simplicity consider only ϒxy.

The volume dv* of the deformed element is then given by

$$dv^* = Area(OA^*B^*C^*) \times dz^*$$
$$Area(OA^*B^*C^*) = dx^*(dy^* CosY_{xy})$$
$$= dx^* dy^* CosY_{xy}$$
$$\therefore \quad dv^* = dx^* dy^* dz^* CosY_{xy}$$

For small Y_{xy} $CosY_{xy} = 1$

$$\therefore \quad dv^* = dx^* dy^* dz^* - Volume\ change\ doesn't\ depend\ on\ Y$$
$$= (1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z)dxdydz$$

dropping all second order infinitesimal terms

$$dv^* = (1+\epsilon_x+\epsilon_y+\epsilon_z)dxdydz$$

Now, analogous to normal strain, we define the measure ofvolumetric strain as

$$Volumetric\ strain = \frac{final\ volume\text{-}initial\ volume}{initialvolume}$$
$$e = \frac{dv^* - dv}{dv}$$
$$e = \epsilon_x + \epsilon_y + \epsilon_z$$

e =volumetric strain = dilatation. This expression isvalid only for infinitesimal strains and rotations

$e = \epsilon_x + \epsilon_y + \epsilon_z = J_1 =$ *first in variance of strain.*

Volumetric strain is scalar quantity and does notdepend on orientation of coordinate system.

Dilatation is zero for state of pure shear.

4.5 Theories of Failures

To explain the cause of failure maximum principal stress theory and rankine's theory is adopted.

The failure will occur when maximum principal tensile stressσ_1 in the complex system reaches the value of maximum stress at elastic limit σ_{el} in simple tension or the mini principal stress reaches the elastic limit stress σ_{ec} in simple compression.

If $\sigma_1 > \sigma_2 > \sigma_3$

$\sigma_1 = \sigma_{el}$ in simple tension

$\sigma_s = \sigma_{ec}$ in simple compression

So the maximum principal stress must not be exceed the working stress of the material hence $\sigma_1 < \sigma_2$

The failure of the material reaches the total strain energy at the elastic limit of simple tension

$$U = ½ E (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 – 2/m (\sigma_1\sigma_2 + \sigma_2 \sigma_3)$$

Whereσ_1, σ_2, $\sigma_3$are same sign

Principal stress dilatation

Now we are in position to compute the direction andmagnitude of the stress components on any inclined plane atany point, provided if we know the state of stress (Planestress) at that point. We also know that any engineeringcomponent fails when the internal forces or stresses reach a particular value of all the stress components on all of theinfinite number of planes only stress components on someparticular planes are important for solving our basicquestion i.e. under the action of given loading whether thecomponent will fail or not. Therefore our objective of this

Class is to determine these plane and their correspondingstresses.

$$(1)\ \sigma_n = \sigma_n(\theta) = \frac{\sigma_n + \sigma_y}{2} + \frac{\sigma_n - \sigma_y}{2} Cos2\theta + \tau_{xy} Sin2\theta$$

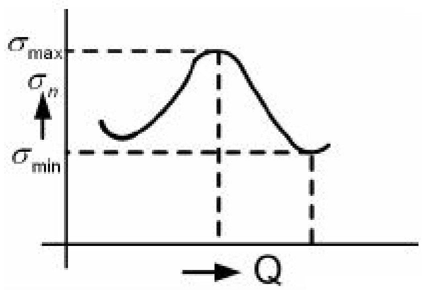


Fig 4.5Graphical representation stress strain

(2) Of all the infinite number of normal stresses at a point, what is the maximum normal stress value, what is the minimum normal stress value and what are theircorresponding planes i.e. how the planes are oriented? Thus mathematically we are looking for maxima and minima of n (Q)

Function...

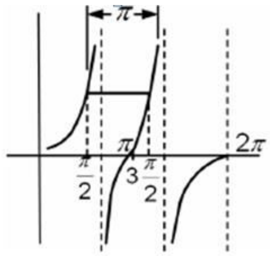
$$(3)\ \sigma_n = \frac{\sigma_n + \sigma_y}{2} + \frac{\sigma_n - \sigma_y}{2} Cos2\theta + \tau_{xy} Sin2\theta$$

For maxima or minima, we know that

$$\frac{d\sigma_n}{d\theta} = 0 = -\left(\sigma_x - \sigma_y\right) Sin2\theta + 2\tau_{xy} Cos2\theta$$

$$tan\,2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

(4) The above equations have two roots, because tan repeatsitself after p. Let us call the first root as Ɵ P1



$$tan 2\theta_{P_1} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$tan 2\theta_{P_2} = tan\left(2\theta_{P_1} + \pi\right) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\theta_{P_2} = \theta_{P_1} + \frac{\pi}{2} s$$

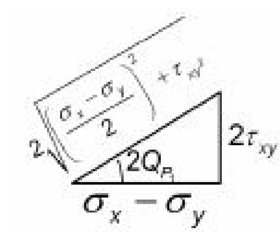
(5) Let us verify now whether we have minima or minima at θ_{P_1} and θ_{P_2}

$$\frac{d^2\sigma_n}{d\theta^2} = -2\left(\sigma_x - \sigma_y\right)Cos 2\theta - 4\tau_{xy} Sin 2\theta$$

$$\therefore \left.\frac{d^2\sigma_n}{d\theta^2}\right|_{\theta=\theta_{P_1}} = -2\left(\sigma_x - \sigma_y\right)Cos 2\theta_{P_1} - 4\tau_{xy} Sin 2\theta_{P_1}$$

We can find $Cos 2\theta_{P_1} s$ and $Sin 2\theta_{P_1} s$ as

$$Cos 2\theta_{P_1} = \frac{\sigma_x - \sigma_y}{2\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$



$$Sin 2\theta_{P_1} = \frac{2\tau_{xy}}{2\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} = \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

Substituting $Cos 2\theta_{P_1}$ and $Sin 2\theta_{P_1}$

$$\left.\frac{d^2\sigma_n}{d\theta^2}\right|_{\theta=\theta_{P_1}} = \frac{-2(\sigma_x-\sigma_y)(\sigma_x-\sigma_y)}{2\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}} - \frac{4\tau_{xy}\tau_{xy}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}$$

$$= \frac{-(\sigma_x-\sigma_y)^2}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}} - \frac{4\tau_{xy}^2}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}$$

$$= \frac{-4}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}\left[\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\right]$$

$$\therefore \frac{d^2\sigma_n}{d\theta^2} = -4\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} \qquad \text{(-ve)}$$

$$\left.\frac{d^2\sigma_n}{d\theta^2}\right|_{\theta=\theta_{P_2}=\theta_{P_1}+\frac{\pi}{2}} = 2(\sigma_x-\sigma_y)Cos(2\theta_{P_1}+\pi) - 4\tau_{xy}Sin(2\theta_{P_1}+\pi)$$

$$= 2(\sigma_x-\sigma_y)Cos2\theta_{P_1} + 4\tau_{xy}Sin2\theta_{P_1}$$

Substituting $Cos2\theta_{P_1}$ & $Sin2\theta_{P_1}$ m we can show that

$$\therefore \left.\frac{d^2\sigma_n}{d\theta^2}\right|_{\theta=\theta_{P_2}} = -4\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2 s} \qquad \text{(+ve)}$$

Thus the angles q P s 1 and q P s 2 define planes of eithermaximum normal stress or minimum normal stress.

(6) Now, we need to compute magnitudes of these stresses

We know that,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} Cos2\theta + \tau_{xy} Sin2\theta$$

$$\sigma_n \Big|_{\theta=\theta_{P_1}} = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} Cos2\theta_{P_1} + \tau_{xy} Sin2\theta_{P_1}$$

Substituting $Cos2\theta_{P_1} s$ and $Sin2\theta_{P_1}$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Max. Normal stress because of +sign

Similarly,

$$\sigma_n \Big|_{\theta=\theta_{P_2}=\theta_{P_1}=\frac{\pi}{2}} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} Cos\left(2\theta_{P_1} + \pi\right) + \tau_{xy} Sin\left(2\theta_{P_1} + \pi\right)$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} Cos2\theta_{P_1} - \tau_{xy} Sin2\theta_{P_1}$$

Substituting $Cos2\theta_{P_1}$ and $Sin2\theta_{P_1}$

$$\sigma = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Min. normal sress because of − ve sign

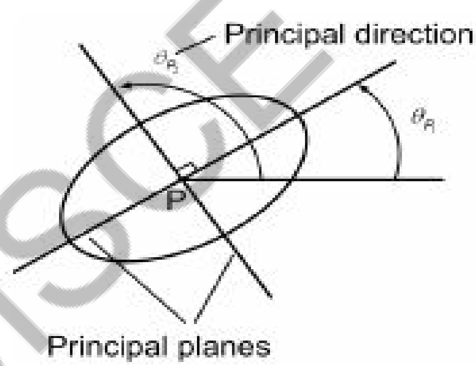
We can write

$$\sigma_1 \, or \, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

(7) Let us se the properties of above stress.

(1) $\theta_{P_2} = \theta_{P_1} + \frac{\pi}{2}s$ - planes on which maximum normal stress and minimum normal stress act are $\perp$ to each other.

(2) Generally maximum normal stress is designated by σ_1 and minimum stress by σ_2. Also $\theta_{P_1} \rightarrow \sigma_1 ; \theta_{P_2} \rightarrow \sigma_2$



$\sigma_1 > \sigma_2$ *algebraically i.e.,*

$0 - \quad \sigma_1$

$-1000 - \sigma_2$

(3) Maximum and minimum normal stresses are collectivelycalled as principal stresses.

(4) Planes on which maximum and minimum normal stressact are known as principal planes.

(5) ϴ P1 and ϴ P2 that define the principal planes are known asprincipal directions.

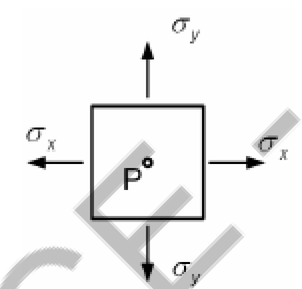
(6) Let us find the planes on which shearing stresses are zero.

$$\tau_{nt} = 0 = -\left(\sigma_x - \sigma_y\right) Sin2\theta + \tau_{xy} Cos2\theta$$

$$tan2\theta = \frac{2\tau_{xy}}{\sigma_x = \sigma_y}$$

$$= directions\,of\ principal\ plans$$

Thus on the principal planes no shearing stresses act.Conversely, the planes on which no shearing stress acts areknown as principal planes and the corresponding normalstresses are principal stresses. For example the state of stressat a point is as shown.



Then σ x and σ y areprincipal stresses becauseno shearing stresses areacting on these planes.

4.6 Principal strain

$$\left.\begin{aligned} (\epsilon_x - \epsilon) n_x + \epsilon_{xy}\, n_y + \epsilon_{xz}\, n_z = 0 \\ \epsilon_{xy}\, n_x + (\epsilon_y - \epsilon) n_y + \epsilon_{yz}\, n_z = 0 \\ \epsilon_{xz}\, n_x + \epsilon_{yz}\, n_y + (\epsilon_z - \epsilon) n_z = 0 \end{aligned}\right\}$$

System of linear homogeneous equations

$$\begin{vmatrix} (\in_x - \in) & \in_{xy} & \in_{xz} \\ \in_{xy} & (\in_y - \in) & \in_{yz} \\ \in_{xz} & \in_{yz} & (\in_z - \in) \end{vmatrix} = 0$$

$$\in^3 - J_1 \in_1^2 + J_2 \in - J_3 = 0$$

$$J_1 = \in_x + \in_y + \in_z$$

-

$$J_2 = \in_x \in_y + \in_x \in_z + \in_y \in_z - \in_{xy}^2 - \in_{yz}^2 - \in_{zx}^2 \quad \begin{vmatrix} \in_x & \in_{xy} \\ \in_{xy} & \in_y \end{vmatrix} +$$

$$\begin{vmatrix} \in_y & \in_{yz} \\ \in_{yz} & \in_z \end{vmatrix} + \begin{vmatrix} \in_x & \in_{xz} \\ \in_{xz} & \in_z \end{vmatrix}$$

$$J_3 = \in_x \in_y \in_z + \in_{xy} \in_{yz} \in_{zx} - \in_x \in_{yz}^2 - \in_y \in_{xz}^2$$

$$- \in_z \in_{xy}^2 \begin{vmatrix} \in_x & \in_{xy} & \in_{xz} \\ \in_{yx} & \in_y & \in_{yz} \\ \in_{zx} & \in_{zy} & \in_z \end{vmatrix}$$

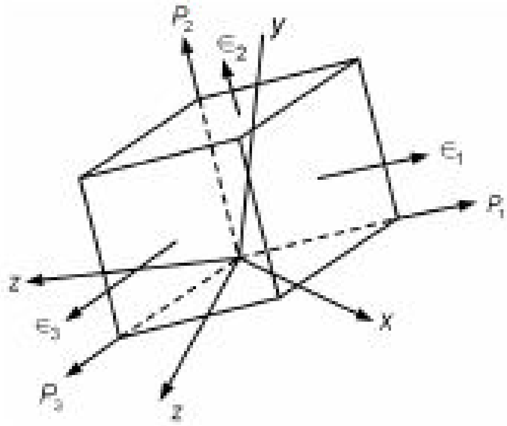
$$\in_1 > \in_2 > \in_3$$

$$(\in_x - \in_1) n_x + \in_{xy} n_y + \in_{zx} n_z = 0$$

$$\in_{xy} n_x + (\in_y - \in_1) n_y + \in_{zy} n_z = 0$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

$\Rightarrow n_x, n_y \,\&\, n_z$ unique



$$J_1 = \in_1 + \in_2 + \in_3$$

$$J_2 = \in_1\in_2 + \in_2\in_3 + \in_3\in_1$$

$$J_3 = \in_1\in_2\in_3$$

4.7 Strain energy and distortion energy theories

Strain energy:

The elastic strain energy U is the energy expended by the action of externalforces in deforming an elastic body. All the work performed during elasticdeformation is stored as elastic energy, and this energy is recovered on therelease of the applied forces. In the deformation of an elastic body, the average strain energy is

$$U = \frac{1}{2} P\delta$$

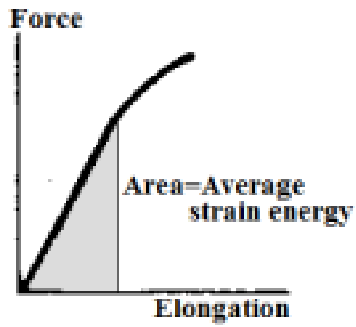


Fig 4.6Force - Elongation curve diagram

For an elemental cube subjected to only a tensile stress along x-axis, theelastic strain energy is:

$$dU = \tfrac{1}{2}(\sigma_x A)(\varepsilon_x\, dx) = \tfrac{1}{2}(\sigma_x \varepsilon_x)(A\, dx)$$

But (Adx) is the volume of the element, so the strain energy per unit volume or strain energy density Uo is:

$$U_0 = \frac{1}{2}\sigma_x \varepsilon_x = \frac{1}{2}\frac{\sigma_x^2}{E} = \frac{1}{2}\varepsilon_x^2 E$$

Note that the lateral strains which accompany deformation in simple tension do not enter into the expression for strain energy because forces do not exist in the direction of the lateral strains.

By the same type of reasoning, the strain energy per unit volume of an element subjected to pure shear is given by

$$U_0 = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{1}{2}\frac{\tau_{xy}^2}{G} = \frac{1}{2}\gamma_{xy}^2 G$$

The elastic strain energy for a general three dimensional stress

$$U_0 = \tfrac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})$$

But

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right];\ \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right]$$

and $$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$ also:

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{xz} = G\gamma_{xz}$$

So total strain energy per unit volume:

$$U_0 = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) - \frac{\nu}{E}\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z\right) + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)$$

$$\frac{\nu E}{(1+\nu)(1-2\nu)} = \lambda \quad \text{(called Lamé's constant)}$$

and:

$$\Delta = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

The strain energy can be expressed in terms of strains and the elasticconstants as:

$$U_0 = \tfrac{1}{2}\lambda\Delta^2 + G\left(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2\right) + \tfrac{1}{2}G\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)$$

The derivative of U0 with respect to any strain (or stress) component gives

The corresponding stress (or strain) component, so:

$$\frac{\partial U_0}{\partial \varepsilon_x} = \lambda\Delta + 2G\varepsilon_x = \sigma_x \quad \text{and} \quad \partial U_0/\partial\sigma_x = \varepsilon_x$$

The distortion energy theory (Von.Mises theory):

Also called shear energy theory.

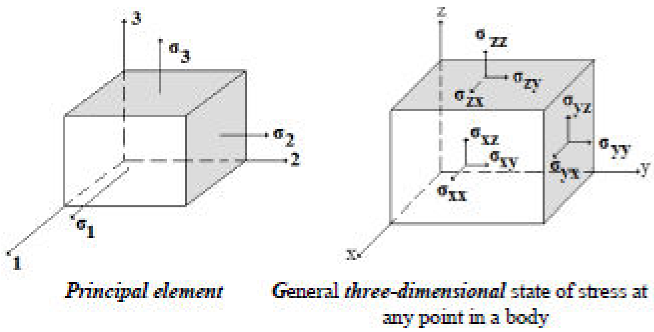


Fig 4.7 Three Dimensional state of stress at any point in a body

Total strain energy per unit volume for general ***three-dimensional*** state of stress:

$$U_0=\frac{1}{2E}\left(\sigma_x^2+\sigma_y^2+\sigma_z^2\right)-\frac{\nu}{E}\left(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_x\sigma_z\right)+\frac{1}{2G}\left(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2\right)$$

Total strain energy per unit volume for ***Principal element in*** three ***dimensional*** state of stress:

$$U_0=\frac{1}{2E}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3\right)\right)$$

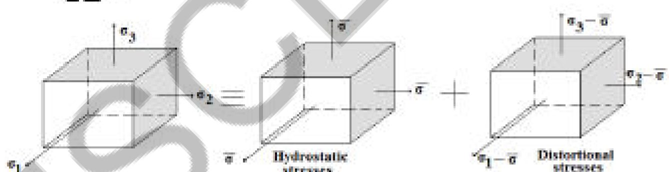


Fig 4.8 Hydrostatic and Distortional Stress

1- Volumetric strain energy (associated with the hydrostatic stresses

$$\sigma_1=\sigma_2=\sigma_3=\frac{1}{3}\left(\sigma_1+\sigma_2+\sigma_3\right)$$

By substituting the above equation in the equation of total strainenergy per unit volume for ***Principal element in*** three-***dimensional*** state ofstress:

$$U_{vol} = \frac{(1-2\upsilon)}{6E}\left[(\sigma_1+\sigma_2+\sigma_3)^2\right]$$

2- Shear strain energy per unit volume (Us)/distortion of the shape due to shear stress:

$$U_s = \frac{(1+\upsilon)}{6E}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \quad ...(1)$$

or:

$$U_s = \frac{1}{6G}\left[(\sigma_1^2+\sigma_2^2+\sigma_3^2)-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right]$$

Yielding will occur when the strain energy of the distortion per unit volume equals that or a specimen in uniaxial tension or compression (strained to the yield point).

4.8 Residual Stresses

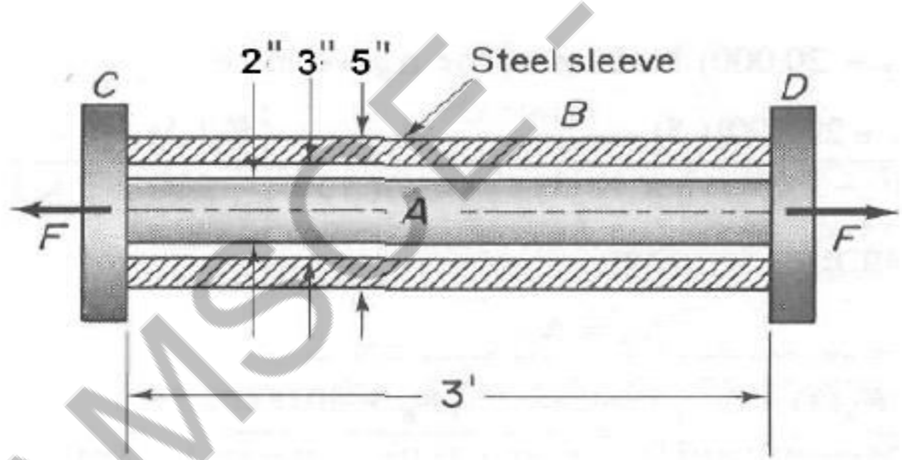


Fig 4.9 Composite material

Shown in Figure are an aluminum rod A (Eal = 15^06 psi) and a steel sleeve (EST = 30 x 10^6

psi) of equal length welded to stiff end plates C and D. An axial tensile force F is applied to the system.If we wish to cause an elongation equal to 0.030 ft for the system, what minimum value, Fmin, is neededto accomplish this? Also, compute the stresses in the members when this external load is released.Assume that the materials behave in an elastic, perfectly plastic manner with (ƒĐY)al = 60000 psi and(ƒĐY)st = 200000 psi.

Solution:

Maximum elastic elongations in the aluminum shaft and in the steel sleeve are

$$\begin{aligned} (\delta_{al})_{el} &= \left(\frac{\sigma}{E} l\right)_{al} = \frac{60000}{15 \times 10^6} \times 3 = 0.012 \text{ ft } < \Delta l = 0.030 \text{ ft} \\ (\delta_{st})_{el} &= \left(\frac{\sigma}{E} l\right)_{st} = \frac{200000}{30 \times 10^6} \times 3 = 0.020 \text{ ft } < \Delta l = 0.030 \text{ ft} \end{aligned}$$

We have plastic deformation in both steel sleeve and aluminum rod. The minimum force Fmin required as follows

$$F_{\min} = (\sigma_Y)_{st} \; A_{st} + (\sigma_Y)_{al} \; A_{al}$$

the cross-sectional areas are

$$\begin{aligned} A_{st} &= \frac{\pi}{4}\left(5^2 - 3^2\right) = 12.566 \text{ mm}^2 \\ A_{al} &= \frac{\pi}{4} \times 2^2 = 3.142 \text{ mm}^2 \end{aligned}$$

$$\begin{aligned} F_{\min} &= 200000 \times \; 12.566 + 60000 \times 3.142 \\ F_{\min} &= 2\,701\,720 \text{ lb} \end{aligned}$$

The yield strains for steel and aluminum are

$$
\begin{aligned}
(\varepsilon_Y)_{st} &= \left(\frac{\sigma_Y}{E}\right)_{st} = \frac{200000}{30 \times 10^6} = 6.6667 \times 10^{-3} \\
(\varepsilon_Y)_{al} &= \left(\frac{\sigma_Y}{E}\right)_{al} = \frac{60000}{15 \times 10^6} = 4 \times 10^{-3}
\end{aligned}
$$

Total strain for this assembly

$$\varepsilon = \frac{\Delta l}{l} = \frac{0.03}{3} = 0.01$$

For the final configuration, the residual stresses satisfy the equilibrium equation

$$
\begin{aligned}
(\sigma_R)_{st}\ A_{st} + (\sigma_R)_{al}\ A_{al} &= 0 \\
12.566 \times (\sigma_R)_{st} + 3.142 \times (\sigma_R)_{al} &= 0
\end{aligned}
$$

The compatibility demands that the decrease in length of the members during unloading must be the same. Thus, the strains "released" on unloading must be the same for the steel and aluminum members:

$$
\begin{aligned}
(\varepsilon_Y - \varepsilon_R)_{st} &= (\varepsilon_Y - \varepsilon_R)_{al} \\
\left(\frac{\sigma_Y - \sigma_R}{E}\right)_{st} &= \left(\frac{\sigma_Y - \sigma_R}{E}\right)_{al} \\
\left(\frac{200000 - \sigma_R}{30 \times 10^6}\right)_{st} &= \left(\frac{60000 - \sigma_R}{15 \times 10^6}\right)_{al} \\
80000 &= (\sigma_R)_{st} - 2\,(\sigma_R)_{al}
\end{aligned}
$$

Solving Equations (1) and (2) for (σR) st and (σR) al simultaneously

$$12.566 \times (\sigma_R)_{st} + 3.142 \times (\sigma_R)_{al} = 0$$
$$(\sigma_R)_{st} - 2(\sigma_R)_{al} = 80000$$

The residual stress in the aluminum shaft is

$$(3.142 + 2 \times 12.566)(\sigma_R)_{al} = -80000 \times 12.566$$
$$(\sigma_R)_{al} = -35555.\text{psi}$$

The residual stress in the steel sleeve is

$$(\sigma_R)_{st} - 2(-35555) = 80000$$
$$(\sigma_R)_{st} = 8890 \text{ psi}$$

QUESTION BANKS

2 Marks

1. Define principle plane.
2. Define principle stress.
3. Define Volumetric Strain.
4. What is stress tensor?
5. What are the types of Stresses?
6. What are the stresses that exist across an Inclined plane?
7. Write Stress Tensor in 3D?
8. Write the Torsion Equation?
9. What is residual stress?
10. Differentiate normal stress and shear stress?
11. What are Stress invariants?
12. State stress phenomena?
13. What are energy distortion theories?
14. State Von.Mises Theory
15. What is strain energy and strain energy density?
16. State Rankine theory of elastic failure?

16 Marks

1. The state of stress (N/mm^2) at a point is given by

$$\begin{matrix} 40 & 20 & 30 \\ 20 & 60 & 10 \\ 30 & 10 & 50 \end{matrix}$$

Determine the principal stress and the orientation of any one of the principal plane

2. The stress tensor (MPa) at a point is given by

$$\begin{matrix} 18 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & 32 \end{matrix}$$

Calculate the maximum shear stress at this point.

3. At the central point in a strained material the principal stresses (MPa) are 60 (tensile), 40(tensile) and 40(compression) respectively. Calculate i. The total strain energy per unit volume. ii. Volumetric strain energy per unit volume iii. Shear strain energy per unit volume. Assume the modulus of elasticity and Poisson's ratio for the material as 120 kN/m^2 and 0.3.
4. Explain any three theories of Failures?
5. In a material, the principal stresses are +40 MN/m^2, +48MN/m^2 and -30MN/m^2. Calculate (i) Total strain energy per unit volume (ii) Shear strain energy per unit volume. (iii) Volumetric strain energy per unit volume (iv) Factor of Safety on the total strain energy criterion, if the materials at 110 MN/m^2. Poisson's ratio – 0.3, E – 200 x 10^9 N/m^2

STRENGTH OF MATERIALS

UNIT V

ADVANCED TOPICS

CONTENTS

TECHNICAL TERMS

1. **Unsymmetrical bending of beams-** If the plane of loading or that of bending does not lie in a plane that contains the principal centroidal axis of the cross section, this bending is called unsymmetrical bending.
2. **Stress concentration-** Stresses of much higher magnitudes occur or produce as a result of discontinuities in the geometry (shape) of the structure are called stress concentration or localized stresses.
3. **Fatigue-** It is the progressive failure of a structure or component under repeated loading of relatively low magnitude.
4. **Notch sensitivity-** The effect of a notch on the fatigue strength of a part varies considerably with material and notch geometry is called notch sensitivity.

V ADVANCED TOPICS IN BENDING OF BEAMS

5.1 Unsymmetrical bending of beams of symmetrical and unsymmetrical sections

Another of the limitations of the usual development of beam bending equations is thatbeams are assumed to have at least one longitudinal plane of symmetry and that the load is applied in the plane of symmetry. The beam bending equations can be extended to coverpure bending (i.e., bent with bending moments only and no transverse forces) of 1) beamswith a plane of symmetry but with the load (couple) applied not in or parallel to the plane ofsymmetry or 2) beams with no plane of symmetry

Fig depicts a beam of unsymmetrical cross section loaded with a couple, M, in aplane making an angle, α, with the xy plane, where the origin of coordinates is at the centroid ofthe cross section. The neutral axis, which passes through the centroid for linearly elastic actionmakes an unknown angle, β, with the z axis. The beam is straight and of uniform cress sectionand a plane cross section is assumed to remain plane after bending. Note that the followingdevelopment is restricted to elastic action.Since the orientation of the neutral axis is unknown, the usual flexural stress distributionfunction

$$\sigma = E(\varepsilon_c / c)y = (\sigma_c / c)y$$

Cannot be expressed in terms of one variable.

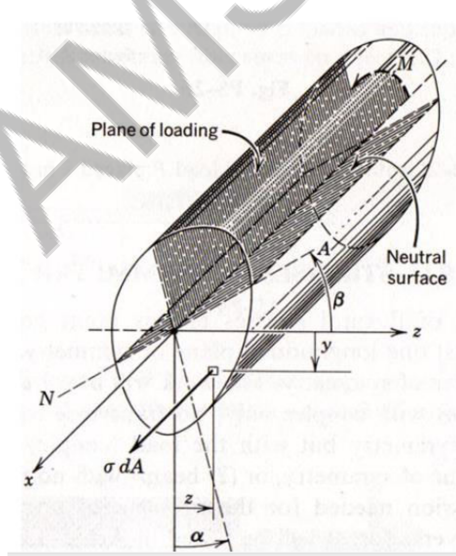


Fig 5.1 Unsymmetrical section

However, since the plane section remains plane, the stress variation can be written as:

$$\sigma = k_1 y + k_2 z$$

The resisting moments with respect to the z and y axes can be written as

$$M_{rz} = \int_A \sigma dAy = \int_A k_1 y^2 dA + \int_A k_2 zydA = k_1 I_z + k_2 I_{yz}$$
$$M_{ry} = \int_A \sigma dAz = \int_A k_1 yzdA + \int_A k_2 z^2 dA = k_1 I_{yz} + k_2 I_y$$

Where Iy and Iz are the moment of inertia of the cross sectional area with respect to the z and y axes, respectively, and Iyz is the product of inertia with respect to these two axes. It will be convenient to let the y and z axes are principal axes, Y and Z; then Iyz is zero. Equating the applied moment to the resisting moment and solving for k1 and K2 gives:

$$M_{rZ} = k_1 I_Z = M\cos\alpha \qquad k_1 = \frac{M\cos\alpha}{I_Z}$$
$$M_{rY} = k_2 I_Y = M\sin\alpha \qquad k_2 = \frac{M\sin\alpha}{I_Y}$$

Substituting the expressions for k given in Eq. gives the elastic flexure formula for unsymmetrical bending.

$$\sigma = \frac{M\cos\alpha}{I_Z} Y + \frac{M\sin\alpha}{I_Y} Z$$

Since σ is zero at the neutral axis, the orientation of the neutral axis is found by setting

$$\frac{\cos\alpha}{I_Z} Y = -\frac{\sin\alpha}{I_Y} Z$$

$$Y = -\tan\alpha \frac{I_Z}{I_Y} Z$$

Where Y is the equation of the neutral axis in the YZ plane. The slope of the line is the dY/dZ and since dY/dZ= tanβ , the orientation of the neutral axis is given by the expression

$$\tan\beta = -\frac{I_Z}{I_Y}\tan\alpha$$

The negative sign indicates that the angles, αand βare in adjacent quadrants.

Note that the neutral axis is not perpendicular to the plane of loading unless 1) the angle, α , is zero, in which case the plane of loading is (or is parallel to) a principal angle, or 2) the two principal moments of inertia are equal. This reduces to the special kind of symmetry where all centroidal moments of inertia are equal (e.g., square, rectangle, etc.)

5.2 Curved Beams

One of the assumptions of the development of the beam bending relations is that alllongitudinal elements of the bean have the same length, thus restricting the theory to initiallystraight beams of constant cross section. Although considerable deviations from this restrictioncan be tolerated in real problems, when the initial curvature of the beams becomes significant, the linear variations of strain over the cross section are no longer valid, even though theassumption of plane cross sections remaining plane is valid. A theory for a beam subjected topure bending having a constant cross section and a constant or slowly varying initial radius ofcurvature in the plane of bending is developed as follows. Typical examples of curvedbeams include hooks and chain links. In these cases the members are not slender but ratherhave a sharp curve and their cross sectional dimensions are large compared with their radius ofcurvature.

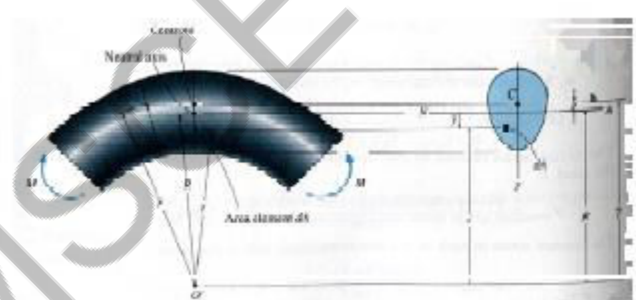


Fig 5.2 Curved beams

Fig is the cross section of part of an initially curved beam. The x-y plane is the plane ofbending and a plane of symmetry. Assumptions for the analysis are: cross sectional area isconstant; an axis of symmetry is perpendicular to the applied moment; M, the material ishomogeneous, isotropic and linear elastic; plane sections remain plane, and any distortions ofthe cross section within its own plane are neglected. Since a plane section before bendingremains a plane after bending, the longitudinal deformation of any element will be proportionalto the distance of the element from the neutral surface.

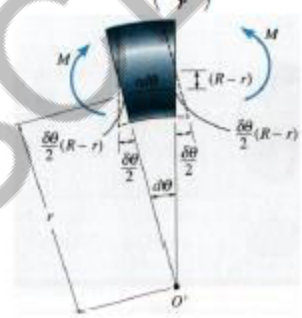
In developing the analysis, three radii, extending from the center of curvature, O', of themember are shown in Fig. The radii are: *r* that references the location of the centroid of thecross sectional area; R that references the location of the neutral axis; and r references somearbitrary point of area element dA on the cross section. Note that the neutral axis lies within thecross section since the moment M creates compression in the beams top fibers and tension inits bottom fibers. By definition, the neutral axis is al line of zero stress and strain.

If a differential segment is isolated in the beam the stress deforms thematerial in such a way that the cross section rotates through an angle ofδΘ /2. The normal strainin an arbitrary strip at location r can be determined from the resulting deformation. This strip hasan original length of Lo=r dΘ. The strip's total change in length,

$$\varepsilon = \frac{\Delta L}{L_o} = \frac{(R-r)\delta\theta}{r\,d\theta}$$

To simplify the relation, a constant k is defined as $k = \delta\Theta / d\Theta$ such that the normal strain can be rewritten as:

$$\varepsilon = k\left(\frac{R-r}{r}\right)$$



Note that Equation shows that the normal strain is a nonlinear function of r varying in ahyperbolic fashion. This is in contrast to the linear variation of strain in the case of the straight. The nonlinear strain distribution for the curved beam occurs even thoughthe cross section of the beam remains plane after deformation. The moment, M, causes thematerial to deform elastically and therefore Hooke's law applies resulting the following relationfor stress:

$$\sigma = E\varepsilon = Ek\left(\frac{R-r}{r}\right)$$

Because of the linear relation between stress and strain, the stress relation is alsohyperbolic. However, with the relation for stress determined, it is possible to determine thelocation of the neutral axis and thereby relate the applied moment, M to this resulting stress.First a relation for the unknown radius of the neutral axis from the center of curvature, R, isdetermined. Then the relation between the stress, σ , and the applied moment, M isdetermined.

Force equilibrium equations can apply to obtain the location of R (radius of the neutralaxis). Specifically, the internal forces caused by the stress distribution acting over the crosssection must be balanced such that the resultant internal force is zero:

$$\sum F_x = 0$$

Now since $\sigma = \frac{dF}{dA}$ then $dF = \sigma\, dA$ and

$$F = \sum \sigma\, dA = \int \sigma\, dA = 0$$

or $\int Ek\left(\frac{R-r}{r}\right) dA = 0$

Because Ek and R are constants Eq 4.4 can be rearranged such that:

$$Ek\left[\int\left(\frac{R}{r}\right)dA - \int\left(\frac{r}{r}\right)dA\right] = 0 \Rightarrow R\int\frac{dA}{r} - \int dA = 0$$

Solving Equation 4.5 for R results in:

$$R = \frac{\int dA}{\int \frac{dA}{r}} = \frac{A}{\int \frac{dA}{r}}$$

Moment equilibrium equations can be applied to relate the applied moment, M, to theresulting stress, σ. Specifically, the internal moments caused by the stress distribution actingover the cross section about the neutral must be balanced such that the resultant internalmoment balances the applied moment:

Recall that since $\sigma = \frac{dF}{dA}$ then $dF = \sigma\, dA$

if y is the distance from the neutral axis such that $y = R - r$

then $dM = y\, dF$ or $dM = y\,(\sigma\, dA)$

Applying moment equilibrium such that $\sum M = 0$ gives

$M - \sum y\,(\sigma\, dA) = 0$ or $M = \sum y\,(\sigma\, dA)$

Substituting the derived relations for y and σ gives

$$M = \sum y\,(\sigma\, dA) = \int (R - r) E k \left(\frac{R - r}{r}\right) dA$$

5.3 Winkler Bach Formula

5.3.1 Assumptions

1. Transverse sections which are plane before bending remain plane even after bending.
2. Longitudinal fibres of the bar, parallel to the central axis exert no pressure on each other.
3. All cross-sections possess a vertical axis of symmetry lying in the plane of the centroidal axis passing through *C*.
4. The beam is subjected to end couples *M*. The bending moment vector is normal throughout the plane of symmetry of the beam.

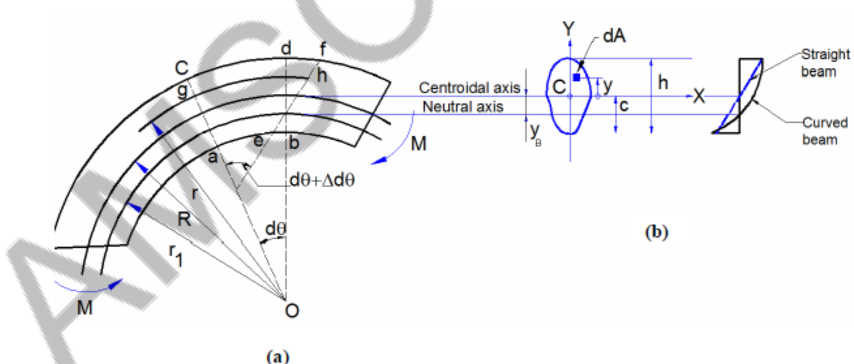


Fig 5.3 Beam with Initial Curve

Consider a curved beam of constant cross-section, subjected to pure bending produced bycouples *M* applied at the ends. On the basis of plane sections remaining plane, we can statethat the total deformation of a beam fiber obeys a linear law, as the beam element rotatesthrough small angle $\Delta d\Theta$. But the tangential strain $e\Theta$does not follow a linear relationship.The deformation of an arbitrary fiber, $gh = \varepsilon_c R d\Theta + y \Delta d\Theta$

where ε_c denotes the strain of the centroidal fiber

But the original length of the fiber

$$gh = (R + y)\, d\theta$$

Therefore, the tangential strain in the fiber

$$gh = \varepsilon_\theta = \frac{[\varepsilon_c R d\theta + y\Delta d\theta]}{(R+y)d\theta}$$

Using Hooke's Law, the tangential stress acting on area dA is given by

$$\sigma_\theta = \frac{\varepsilon_c R + y(\Delta d\theta / d\theta)}{(R+y)} E \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 1$$

Let angular strain

$$\frac{\Delta d\theta}{d\theta} = \lambda$$

Hence Eqn 1 becomes

$$\sigma_\theta = \frac{\varepsilon_c R + y\lambda}{(R+y)} E \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 2$$

Adding and subtracting $\varepsilon_c y$ in the numerator of Equation 2 we get

$$\sigma_\theta = \frac{\varepsilon_c R + y\lambda + \varepsilon_c y - \varepsilon_c y}{(R+y)} E \qquad \ldots\ldots\ldots\ldots\ldots \quad 3$$

Simplifying, we get

$$\sigma_\theta = \left[\varepsilon_c + (\lambda - \varepsilon_c)\frac{y}{(R+y)}\right] E \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 4$$

The beam section must satisfy the conditions of static equilibrium,

$F_z = 0$ and $M_x = 0$, respectively.

$$\therefore \int \sigma_\theta dA = 0 \quad \text{and} \quad \int \sigma_\theta y dA = M \qquad \ldots\ldots\ldots\ldots \quad 5$$

Substituting the above boundary condition 5 in 4 we get

$$0 = \int \left[\varepsilon_c + (\lambda - \varepsilon_c) \frac{y}{(R+y)} \right] dA$$

$$\text{or } \int \varepsilon_c dA = -(\lambda - \varepsilon_c) \int \frac{y}{(R+y)} dA$$

$$\text{or } \varepsilon_c \int dA = -(\lambda - \varepsilon_c) \int \frac{y}{(R+y)} dA$$

Also,

$$\text{M} = \left[\varepsilon_c \int y dA + (\lambda - \varepsilon_c) \int \frac{y^2}{(R+y)} dA \right] E$$

Also,

$$\text{M} = \left[\varepsilon_c \int y dA + (\lambda - \varepsilon_c) \int \frac{y^2}{(R+y)} dA \right] E$$

Here $\int dA = A$, and since y is measured from the centroidal axis, $\int y dA = 0$.

$$\text{Let } \int \frac{y}{(R+y)} dA = -mA$$

$$\text{Or } m = -\frac{1}{A} \int \frac{y}{(R+y)} dA$$

$$\text{Therefore, } \int \frac{y^2}{(R+y)} dA = \int \left(y - \frac{Ry}{(R+y)} \right) dA$$

$$= \int y dA - \int \frac{Ry}{(R+y)} dA$$

$$= 0 - R\,[-mA]$$

$$\therefore \int \frac{y^2}{(R+y)} dA = mRA$$

Substituting the above values

$\varepsilon_c = (\lambda - \varepsilon_c)m$

and $M = E\,(\lambda - \varepsilon_c)\,mAR$

From the above, we get

$$\varepsilon_c = \frac{M}{AER} \text{ and } \lambda = \frac{1}{AE}\left(\frac{M}{R} + \frac{M}{mR} \right)$$

Therefore,

$$, \sigma_\theta = \frac{M}{AR}\left[1 + \frac{y}{m(R+y)} \right]$$

The above expression for σ_Θ is generally known as the "Winkler-Bach formula". Thedistribution of stress $\sigma_\Theta$is given by the hyperbolic (and not linear as in the case of straightbeams) as shown in the Figure

5.4 Stress Concentration

- High stresses are known as stress concentrations
- Sources of stress concentrations- stress raisers

Stress concentrations are due to:

(1)Concentrated loads

(2)Geometric discontinuities

5.4.1 Stress concentration due to concentrated loads

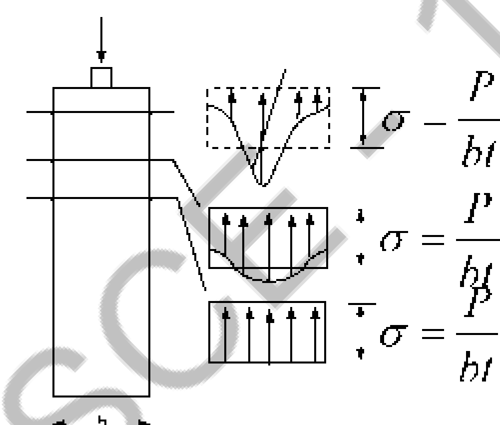


Fig 5.4 Stress concentration loads

$$\text{Stress concentration factor} = K = \frac{\sigma_{max}}{\sigma_{ave}}$$

$$\sigma_{nom} = \frac{P}{bt}$$

5.4.2 Stress concentration due to hole

Discontinuities of cross section may result in high localizedor concentrated stresses.

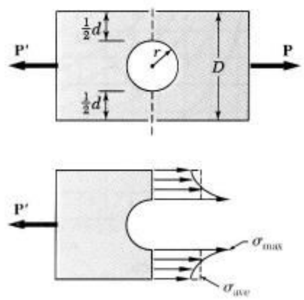
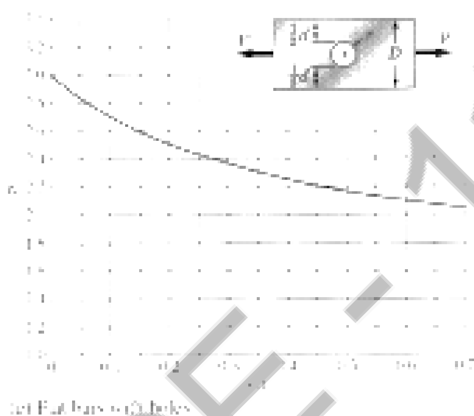


Fig 5.5 Stress concentration on hole



$$K = \frac{\sigma_{max}}{\sigma_{nom}} \qquad \sigma_{nom} = \frac{P}{dt}$$

$$K = Stress\ concentration\ factor$$

5.5 Fatigue and Fracture

Whilst a wide range of failure modes might occur under flexural loading,dependent on the particular test method used and the type or layup ofmaterial under test, these are broadly very similar for three- or four-pointflexure tests.The types of failure likely to be observed are shown in Fig.not all of which might be considered as acceptable flexural failures, thoseincluding evidence of interlinear shear being particularly suspect. Certainlyfor specimens with axially aligned fibres one would not expect to seeinterlinear shear accompanying failure, as this would suggest that thespan-to-thickness ratio used in the test was too low.

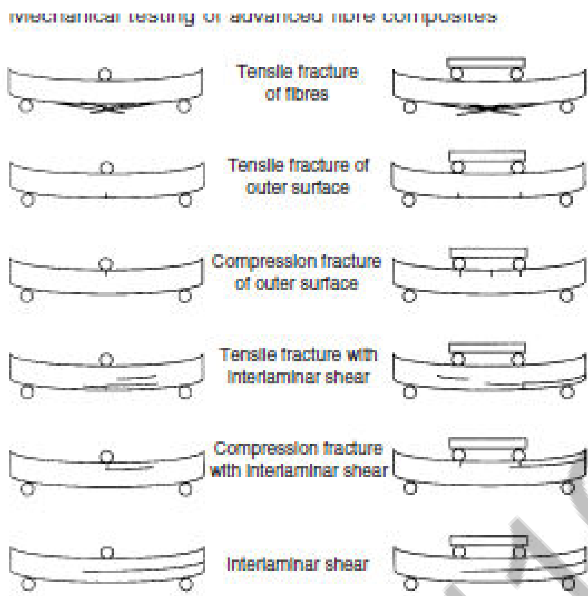


Fig 5.6 Fracture pattern

QUESTION BANKS

2 Marks

1. Explain Neutral Axis?
2. What is Winkler Bach theory?
3. Explain the assumptions followed in Winkler Bach formula?
4. What do you mean by bending?
5. Differentiate symmetrical and unsymmetrical section
6. What is stress?
7. What is stress concentration?
8. Define Stress interference?
9. What is fatigue?
10. What do you mean by Fracture?

16 Marks

1. Derive an expression for finding the pure bending of unsymmetrical section
2. Derive an expression for Winkler Bach formula
3. State stress concentration and explain in detail
4. What are Fatigue and fracture .Explain