Equations from Elasticity Theory ▲

Introduction

In this appendix, we will develop the basic equations of the theory of elasticity. These equations should be referred to frequently throughout the structural mechanics portions of this text.

There are three basic sets of equations included in theory of elasticity. These equations must be satisfied if an exact solution to a structural mechanics problem is to be obtained. These sets of equations are (1) the differential equations of equilibrium formulated here in terms of the stresses acting on a body, (2) the strain/displacement and compatibility differential equations, and (3) the stress/strain or material constitutive laws.

▲ C.1 Differential Equations of Equilibrium ▲

For simplicity, we initially consider the equilibrium of a plane element subjected to normal stresses σ_x and σ_y, in-plane shear stress τ_{xy} (in units of force per unit area), and body forces X_b and Y_b (in units of force per unit volume), as shown in Figure C–1. The stresses are assumed to be constant as they act on the width of each face. However, the stresses are assumed to vary from one face to the opposite. For example, we have σ_x acting on the left vertical face, whereas $\sigma_x + (\partial\sigma_x/\partial x)\,dx$ acts on the right vertical face. The element is assumed to have unit thickness.

Summing forces in the x direction, we have

$$\sum F_x = 0 = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\,dx\right) dy(1) - \sigma_x\,dy(1) + X_b\,dx\,dy(1)$$
$$+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\,dy\right) dx(1) - \tau_{yx}\,dx(1) = 0 \qquad \text{(C.1.1)}$$

After simplifying and canceling terms in Eq. (C.1.1), we obtain

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X_b = 0 \qquad \text{(C.1.2)}$$

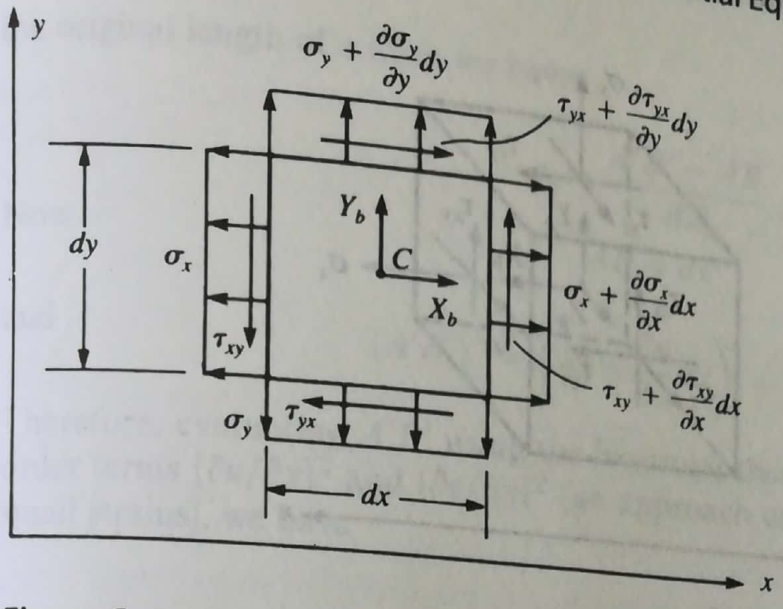


Figure C–1 Plane differential element subjected to stresses

Similarly, summing forces in the y direction, we obtain

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y_b = 0 \tag{C.1.3}$$

Because we are considering only the planar element, three equilibrium equations must be satisfied. The third equation is equilibrium of moments about an axis normal to the x-y plane; that is, taking moments about point C in Figure C–1, we have

$$\sum M_z = 0 = \tau_{xy}\, dy(1)\frac{dx}{2} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right)\frac{dx}{2}$$
$$- \tau_{yx}\, dx(1)\frac{dy}{2} - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right)\frac{dy}{2} = 0 \tag{C.1.4}$$

Simplifying Eq. (C.1.4) and neglecting higher-order terms yields

$$\tau_{xy} = \tau_{yx} \tag{C.1.5}$$

We now consider the three-dimensional state of stress shown in Figure C–2, which shows the additional stresses σ_z, τ_{xz}, and τ_{yz}. For clarity, we show only the stresses on three mutually perpendicular planes. With a straightforward procedure, we can extend the two-dimensional equations (C.1.2), (C.1.3), and (C.1.5) to three dimensions. The resulting total set of equilibrium equations is

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_b = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y_b = 0 \tag{C.1.6}$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z_b = 0$$

and

$$\tau_{xy} = \tau_{yx} \qquad \tau_{xz} = \tau_{zx} \qquad \tau_{yz} = \tau_{zy} \tag{C.1.7}$$

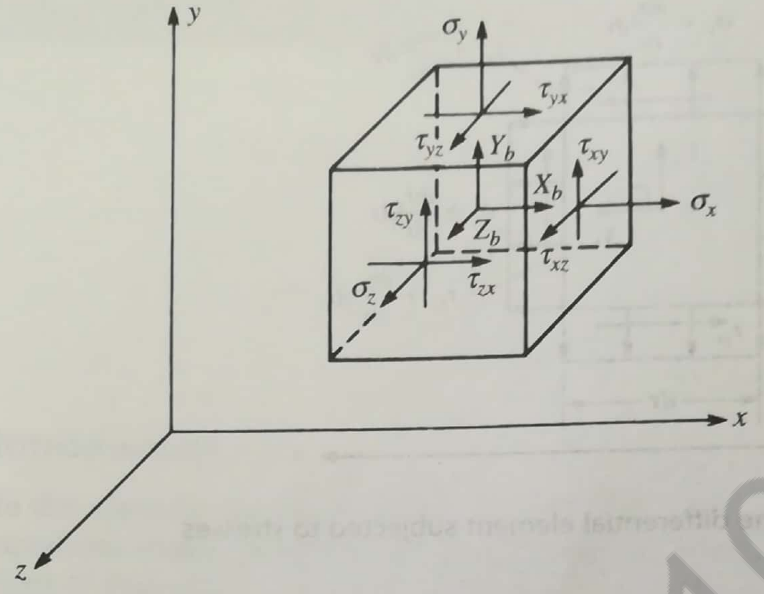


Figure C–2 Three-dimensional stress element

C.2 Strain/Displacement and Compatibility Equations

We first obtain the strain/displacement or kinematic differential relationships for the two-dimensional case. We begin by considering the differential element shown in Figure C–3, where the undeformed state is represented by the dashed lines and the deformed shape (after straining takes place) is represented by the solid lines.

Considering line element AB in the x direction, we can see that it becomes $A'B'$ after deformation, where u and v represent the displacements in the x and y directions. By the definition of engineering normal strain (that is, the change in length divided by

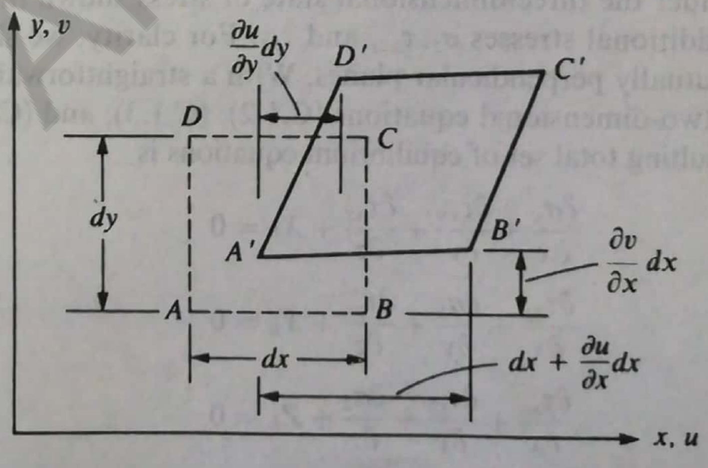


Figure C–3 Differential element before and after deformation the original length of a line), we have

$$\varepsilon_x = \frac{A'B' - AB}{AB} \tag{C.2.1}$$

Now

$$AB = dx \tag{C.2.2}$$

and

$$(A'B')^2 = \left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(\frac{\partial v}{\partial x} dx\right)^2 \tag{C.2.3}$$

Therefore, evaluating $A'B'$ using the binomial theorem and neglecting the higher-order terms $(\partial u/\partial x)^2$ and $(\partial v/\partial x)^2$ (an approach consistent with the assumption of small strains), we have

$$A'B' = dx + \frac{\partial u}{\partial x} dx \tag{C.2.4}$$

Using Eqs. (C.2.2) and (C.2.4) in Eq. (C.2.1), we obtain

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{C.2.5}$$

Similarly, considering line element *AD* in the *y* direction, we have

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{C.2.6}$$

The shear strain γ_{xy} is defined to be the change in the angle between two lines, such as *AB* and *AD*, that originally formed a right angle. Hence, from Figure C–3, we can see that γ_{xy} is the sum of two angles and is given by

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{C.2.7}$$

Equations (C.2.5) through (C.2.7) represent the strain/displacement relationships for in-plane behavior.

For three-dimensional situations, we have a displacement *w* in the *z* direction. It then becomes straightforward to extend the two-dimensional derivations to the three-dimensional case to obtain the additional strain/displacement equations as

$$\varepsilon_z = \frac{\partial w}{\partial z} \tag{C.2.8}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{C.2.9}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{C.2.10}$$

Along with the strain/displacement equations, we need compatibility equations to ensure that the displacement components *u*, *v*, and *w* are single-valued continuous functions so that tearing or overlap of elements does not occur. For the planar-elastic case, we obtain the compatibility equation by differentiating γ_{xy} with respect to both x and y and then using the definitions for ε_x and ε_y given by Eqs. (C.2.5) and (C.2.6). Hence,

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\frac{\partial u}{\partial y} + \frac{\partial^2}{\partial x \partial y}\frac{\partial v}{\partial x} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} \tag{C.2.11}$$

where the second equation in terms of the strains on the right side is obtained by noting that single-valued continuity of displacements requires that the partial differentiations with respect to x and y be interchangeable in order. Therefore, we have $\partial^2/\partial x \partial y = \partial^2/\partial y \partial x$. Equation (C.2.11) is called the *condition of compatibility*, and it must be satisfied by the strain components in order for us to obtain unique expressions for u and v. Equations (C.2.5), (C.2.6), (C.2.7), and (C.2.11) together are then sufficient to obtain unique single-valued functions for u and v.

In three dimensions, we obtain five additional compatibility equations by differentiating γ_{xz} and γ_{yz} in a manner similar to that described above for γ_{xy}. We need not list these equations here; details of their derivation can be found in Reference [1].

In addition to the compatibility conditions that ensure single-valued continuous functions within the body, we must also satisfy displacement or kinematic boundary conditions. This simply means that the displacement functions must also satisfy prescribed or given displacements on the surface of the body. These conditions often occur as support conditions from rollers and/or pins. In general, we might have

$$u = u_0 \qquad v = v_0 \qquad w = w_0 \tag{C.2.12}$$

at specified surface locations on the body. We may also have conditions other than displacements prescribed (for example, prescribed rotations).

C.3 Stress-Strain Relationships

We will now develop the three-dimensional stress-strain relationships for an isotropic body only. This is done by considering the response of a body to imposed stresses. We subject the body to the stresses σ_x, σ_y, and σ_z independently as shown in Figure C–4.

We first consider the change in length of the element in the x direction due to the independent stresses σ_x, σ_y, and σ_z. We assume the principle of superposition to hold; that is, we assume that the resultant strain in a system due to several forces is the algebraic sum of their individual effects.

Considering Figure C–4(b), the stress in the x direction produces a positive strain

$$\varepsilon'_x = \frac{\sigma_x}{E} \tag{C.3.1}$$

where Hooke's law, $\sigma = E\varepsilon$, has been used in writing Eq. (C.3.1), and E is defined as the *modulus of elasticity*. Considering Figure C–4(c), the positive stress in the

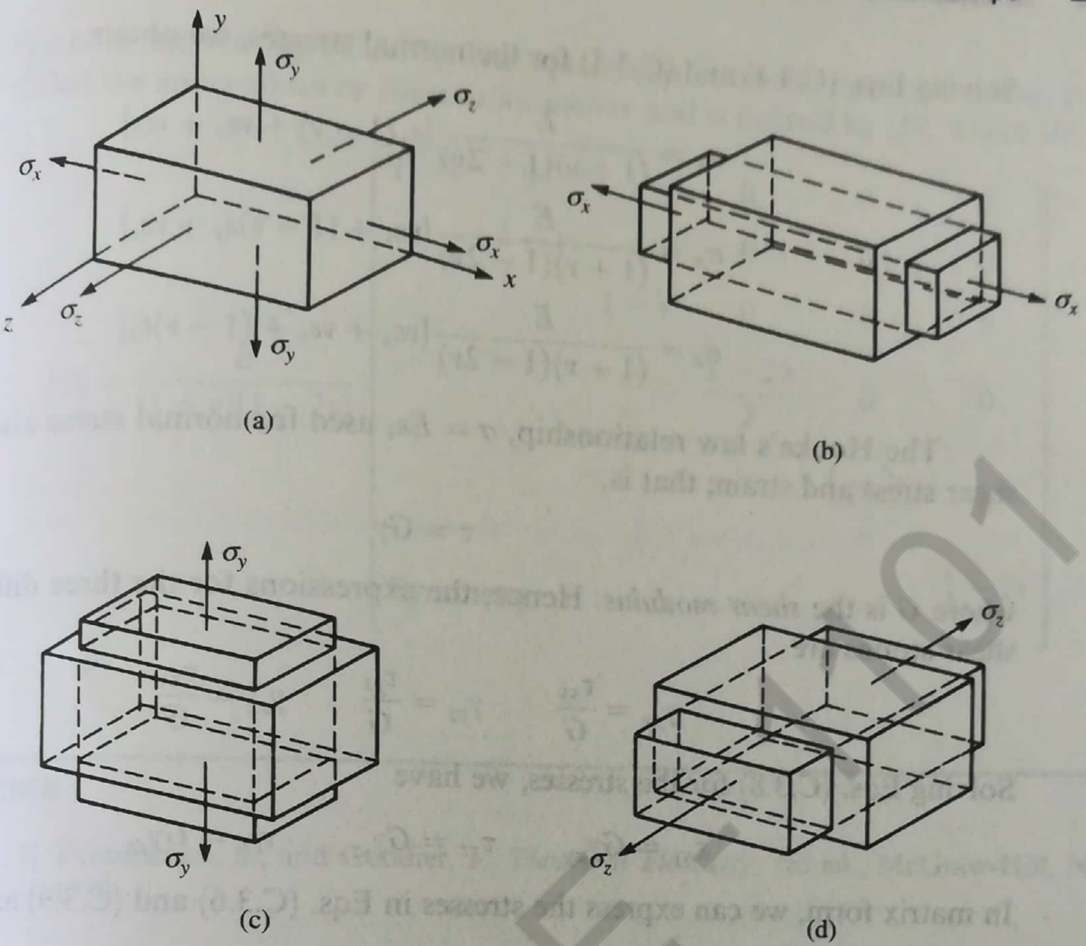


Figure C–4 Element subjected to normal stress acting in three mutually perpendicular directions

y direction produces a negative strain in the x direction as a result of Poisson's effect given by

$$\varepsilon_x'' = -\frac{\nu\sigma_y}{E} \tag{C.3.2}$$

where ν is Poisson's ratio. Similarly, considering Figure C–4(d), the stress in the z direction produces a negative strain in the x direction given by

$$\varepsilon_x''' = -\frac{\nu\sigma_z}{E} \tag{C.3.3}$$

Using superposition of Eqs. (C.3.1) through (C.3.3), we obtain

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \tag{C.3.4}$$

The strains in the y and z directions can be determined in a manner similar to that used to obtain Eq. (C.3.4) for the x direction. They are

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$
$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \tag{C.3.5}$$

Solving Eqs. (C.3.4) and (C.3.5) for the normal stresses, we obtain

$$\begin{aligned}
\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x(1-\nu)+\nu\varepsilon_y+\nu\varepsilon_z] \\
\sigma_y &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x+(1-\nu)\varepsilon_y+\nu\varepsilon_z] \\
\sigma_z &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x+\nu\varepsilon_y+(1-\nu)\varepsilon_z]
\end{aligned} \tag{C.3.6}$$

The Hooke's law relationship, $\sigma = E\varepsilon$, used for normal stress also applies for shear stress and strain; that is,

$$\tau = G\gamma \tag{C.3.7}$$

where G is the *shear modulus*. Hence, the expressions for the three different sets of shear strains are

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G} \tag{C.3.8}$$

Solving Eqs. (C.3.8) for the stresses, we have

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx} \tag{C.3.9}$$

In matrix form, we can express the stresses in Eqs. (C.3.6) and (C.3.9) as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}$$

$$\times \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & & & & \frac{1-2\nu}{2} & 0 \\ \text{Symmetry} & & & & & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{C.3.10}$$

where we note that the relationship

$$G = \frac{E}{2(1+\nu)}$$

has been used in Eq. (C.3.10). The square matrix on the right side of Eq. (C.3.10) is called the *stress-strain* or *constitutive matrix* and is defined by $[D]$, where $[D]$ is

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & & & & \frac{1-2\nu}{2} & 0 \\ \text{Symmetry} & & & & & \frac{1-2\nu}{2} \end{bmatrix} \tag{C.3.11}$$

5.4 2-d Elements for Structural Mechanics

5.4.1 Generic Relations

We will now summarise certain generic relations which will be useful for all the 2-d finite elements for structural mechanics. For a two-dimensional structural problem, each point on the structure may have two independent displacements, viz., u and v, along the two Cartesian coordinates X and Y, respectively. The strain-displacement relations are given by

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{5.83}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{5.84}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{5.85}$$

For axisymmetric problems, we can consider the r-z plane (analogous to x-y plane elasticity) as shown in Figure 5.17. We observe that, even though the deformation

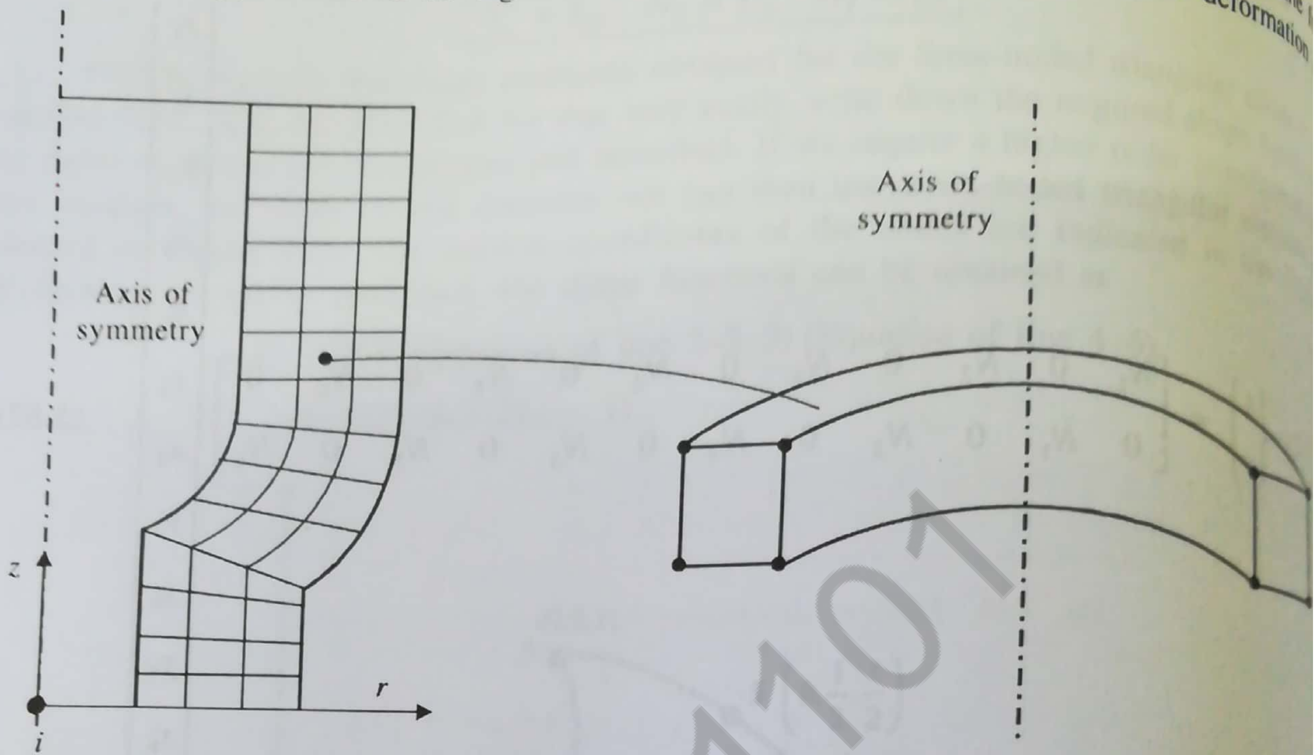


Fig. 5.17 An axisymmetric ring element.

axisymmetric, there can be four independent, nonzero strains (as a change in rad automatically leads to circumferential strain ε_θ):

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad (5.$$

$$\varepsilon_\theta = \frac{u}{r} \quad (5.$$

$$\varepsilon_z = \frac{\partial v}{\partial z} \quad (5.$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \quad (5.$$

In our standard finite element notation, we write the strain-displacement relations as

$$\{\varepsilon\} = [B]\{\delta\}^e \quad (5.9$$

where the size of $[B]$ and $\{\delta\}$ will be dictated by the element and $[B]$ will contain the derivatives of the shape functions. The stress-strain relations for plane stress specialised from the 3-d Hooke's law, are given by

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & 1-\nu^2/2\end{bmatrix}\left(\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} - \{\varepsilon\}^0\right) + \{\sigma\}^0 \quad (5.9$$

where $\{\varepsilon\}^0$ and $\{\sigma\}^0$ refer to the initial strain and stress, respectively.

The stress-strain relations for plane strain are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu/2 \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \{\varepsilon\}^0 \right) + \{\sigma\}^0 \quad (5.92)$$

For axisymmetric problems, we have four independent nonzero stresses given by

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \nu/1-\nu & \nu/1-\nu & 0 \\ & 1 & \nu/1-\nu & 0 \\ & & 1 & 0 \\ \text{Symmetric} & & & 1-2\nu/2(1-\nu) \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix} - \{\varepsilon\}^0 \right) + \{\sigma\}^0$$

(5.93)

In our standard finite element notation, we write the generic stress-strain relation as

$$\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon\}^0) + \{\sigma\}^0 \quad (5.94)$$

where the elements of the $[D]$ matrix depend on plane stress/strain/axisymmetric situation and are taken from Eqs. (5.91)–(5.93).

A structure can, in general, be subjected to distributed loading in the form of body (volume) forces $\{q_v\}$ such as gravity, forces distributed over a surface $\{q_s\}$ such as a pressure or concentrated forces lumped at certain points $\{P_i\}$. It may also be subjected to initial strains and stresses (such as those due to preloading and thermal loading). Under the action of all these loads, the structure, when properly supported (so as to prevent rigid body motion), undergoes deformation and stores internal strain energy. The strain energy expression can be written as

$$U = \frac{1}{2}\int_v \{\varepsilon\}^T \{\sigma\}\, dv = \frac{1}{2}\int_v \{\varepsilon\}^T \left([D](\{\varepsilon\} - \{\varepsilon\}^0) + \{\sigma^0\}\right) dv$$

$$= \int_v \left(\frac{1}{2}\{\varepsilon\}^T [D]\{\varepsilon\} - \{\varepsilon\}^T [D]\{\varepsilon\}^0 + \{\varepsilon\}^T \{\sigma^0\}\right) dv \quad (5.95)$$

The derivation of the above expression follows closely our discussion in Section 4.1 (Eq. (4.3)) except that we introduce the matrix notation to account for two-dimensional state of stress/strain.

From Eqs. (5.90) and (5.95), element level strain energy is given as

$$U^e = \int \frac{1}{2}\{\delta\}^{e^T} [B]^T [D][B]\{\delta\}^e\, dv - \int \{\delta\}^{e^T} [B]^T [D]\{\varepsilon\}^0\, dv$$
$$+ \int \{\delta\}^{e^T} [B]^T \{\sigma\}^0\, dv \quad (5.96)$$

The potential of external forces is given by

$$V^e = -\int_v \{\delta\}^T \{q_v\}\, dv - \int_s \{\delta\}^T \{q_s\}\, ds - \sum \{\delta_i\}^T \{P_i\} \quad (5.97)$$

where $\{\delta\}_i$ is the displacement of the point i on which a concentrated force P_i is acting and the summation is taken over all such points. It is observed that these displacements $\{\delta\}$ and forces $\{q\}$ have both X- and Y-components. Hence, typically at a point,

$$\{\delta\} = \begin{Bmatrix} u \\ v \end{Bmatrix} \quad (5.98)$$

$$\{q\} = \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} \quad (5.99)$$

From Eq. (5.97), using the notation of Eq. (5.14) and (5.21), we get

$$V^e = -\int_v \{\delta\}^{e^T} [N]^T \{q_v\} dv - \int_s \{\delta\}^{e^T} [N]^T \{Q_S\}\, ds - \sum \{\delta\}^{e^T} [N]_i^T \{P_i\} \quad (5.100)$$

where $[N]_i$ is the value of shape functions at the location i. Thus the total potential of an element can be written as

$$\Pi_p^e = \int \frac{1}{2} \{\delta\}^{e^T} [B]^T [D][B] \{\delta\}^e\, dv - \int \{\delta\}^{e^T} [B]^T [D] \{\varepsilon\}^0 dv + \int \{\delta\}^{e^T} [B]^T \{\sigma\}^0\, dv - \int_v \{\delta\}^{e^T} [N]^T \{q_v\}\, dv - \int_S \{\delta\}^{e^T} [N]^T \{Q_S\}\, ds - \sum \{\delta\}^{e^T} [N]_i^T \{P_i\} \quad (5.101)$$

Since $\{\delta\}^e$ is a vector of nodal d.o.f., it can be taken outside the integration. Thus we get

$$\Pi_p^e = \frac{1}{2} \{\delta\}^{e^T} \int_v \left([B]^T [D][B]\, dv \right) \{\delta\}^e - \{\delta\}^{e^T} \int_v [B]^T [D] \{\varepsilon\}^0 dv + \{\delta\}^{e^T} \int_v [B]^T \{\sigma\}^0 dv - \{\delta\}^{e^T} \int_v [N]^T \{q_v\}\, dv - \{\delta\}^{e^T} \int_s [N]^T \{q_s\} ds - \{\delta\}^{e^T} \sum [N]_i^T \{P_i\} \quad (5.102)$$

Defining the element stiffness matrix $[k]^e$ and the load vector $\{f\}^e$ as

$$[k]^e = \int_v [B]^T [D][B]\, dv \quad (5.103)$$

$$\{f\}^e = \int_v [B]^T [D] \{\varepsilon\}^0 dv - \int_v [B]^T \{\sigma\}^0 dv + \int_v [N]^T \{q_v\} dv + \int_s [N]^T \{q_s\} ds + \sum [N]_i^T \{P_i\} \quad (5.104)$$

we can rewrite Eq. (5.102) as

$$\Pi_p^e = \frac{1}{2} \{\delta\}^{eT} [k]^e \{\delta\}^e - \{\delta\}^{eT} \{f\}^e \quad (5.105)$$

Once all the element matrices are 'assembled' together, we obtain the total potential of the system as

$$\Pi_p = \Sigma\Pi_p^e = \frac{1}{2}\{\delta\}^T[K]\{\delta\} - \{\delta\}^T\{F\} \tag{5.106}$$

...re the global stiffness matrix of the structure $[K]$ and the global load vector $\{F\}$ are given

$$[K] = \sum_{n=1}^{\text{NOELEM}} [k]^e \tag{5.107}$$

$$\{F\} = \sum_{1}^{\text{NOELEM}} \{f\}^e \tag{5.108}$$

...ELEM refers to the number of elements and $\{\delta\}$ contains all the nodal d.o.f. variables for ...entire finite element mesh. The summations indicating assembly imply that the individual ...ent matrices have been appropriately placed in the global matrices following the standard ...edure of assembly.

Using the Principle of stationary total potential we set the total potential stationary with ...pect to small variations in the nodal d.o.f., i.e.*

$$\frac{\partial\Pi_p}{\partial\{\delta\}^T} = 0 \tag{5.109}$$

...us we have the system level equations given by

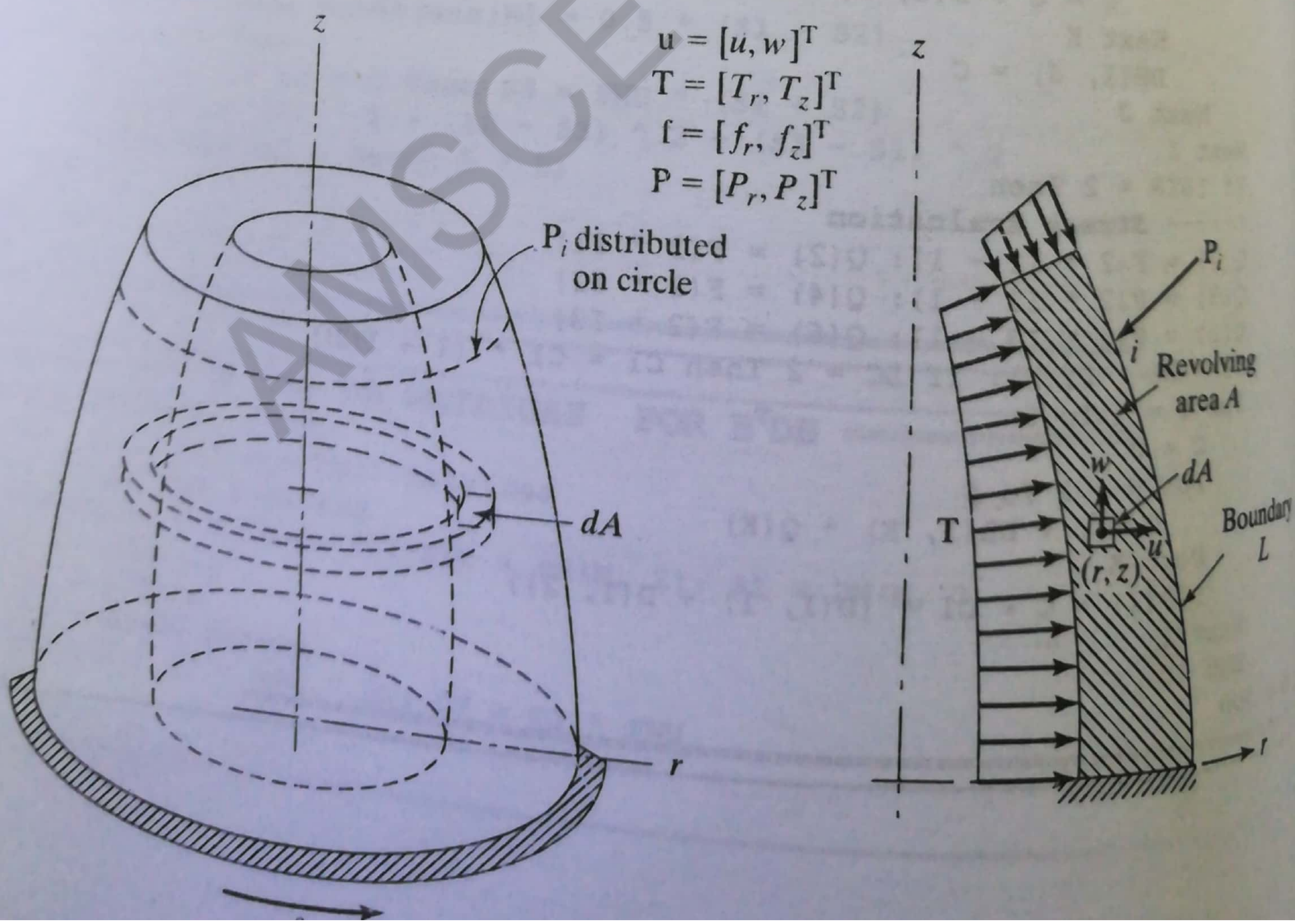
$$[K]\{\delta\} = \{F\} \tag{5.110}$$

We will now discuss each individual 2-d element and describe how the element stiffness ...atrix and nodal load vectors can be obtained.

Axisymmetric Solids Subjected to Axisymmetric Loading

INTRODUCTION

Problems involving three-dimensional axisymmetric solids or solids of revolution su- jected to axisymmetric loading, reduce to simple two-dimensional problems. Becau of total symmetry about the z-axis, as seen in Fig. 6.1, all deformations and stresses a independent of the rotational angle θ. Thus, the problem needs to be looked at as a tw dimensional problem in rz, defined on the revolving area (Fig. 6.1b). Gravity forces ca be considered if acting in the z direction. Revolving bodies like flywheels can be an lyzed by introducing centrifugal forces in the body force term. We now discuss t axisymmetric problem formulation.



6.2 AXISYMMETRIC FORMULATION

Considering the elemental volume shown in Fig. 6.2, the potential energy can be written in the form

$$\Pi = \frac{1}{2}\int_0^{2\pi}\int_A \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}r\,dA\,d\theta - \int_0^{2\pi}\int_A \mathbf{u}^{\mathrm{T}}\mathbf{f}r\,dA\,d\theta - \int_0^{2\pi}\int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}r\,d\ell\,d\theta - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{6.1}$$

where $r\,d\ell\,d\theta$ is the elemental surface area and the point load $\mathbf{P}_i$ represents a line load distributed around a circle, as shown in Fig. 6.1.

All variables in the integrals are independent of θ. Thus, Eq. 6.1 can be written as

$$\Pi = 2\pi\left(\frac{1}{2}\int_A \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}r\,dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f}r\,dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}r\,d\ell\right) - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{6.2}$$

where

$$\mathbf{u} = [u, w]^{\mathrm{T}} \tag{6.3}$$

$$\mathbf{f} = [f_r, f_z]^{\mathrm{T}} \tag{6.4}$$

$$\mathbf{T} = [T_r, T_z]^{\mathrm{T}} \tag{6.5}$$

From Fig. 6.3, we can write the relationship between strains $\boldsymbol{\epsilon}$ and displacements $\mathbf{u}$ as

$$\begin{aligned}\boldsymbol{\epsilon} &= [\epsilon_r, \epsilon_z, \gamma_{rz}, \epsilon_\theta]^{\mathrm{T}} \\ &= \left[\frac{\partial u}{\partial r}, \frac{\partial w}{\partial z}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \frac{u}{r}\right]^{\mathrm{T}}\end{aligned} \tag{6.6}$$

The stress vector is correspondingly defined as

$$\boldsymbol{\sigma} = [\sigma_r, \sigma_z, \tau_{rz}, \sigma_\theta]^{\mathrm{T}} \tag{6.7}$$

The stress–strain relations are given in the usual form, viz.,

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon} \tag{6.8}$$

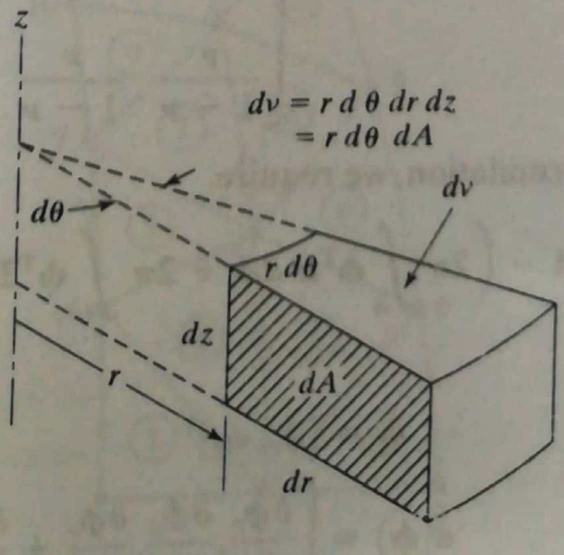


FIGURE 6.2 Elemental volume.

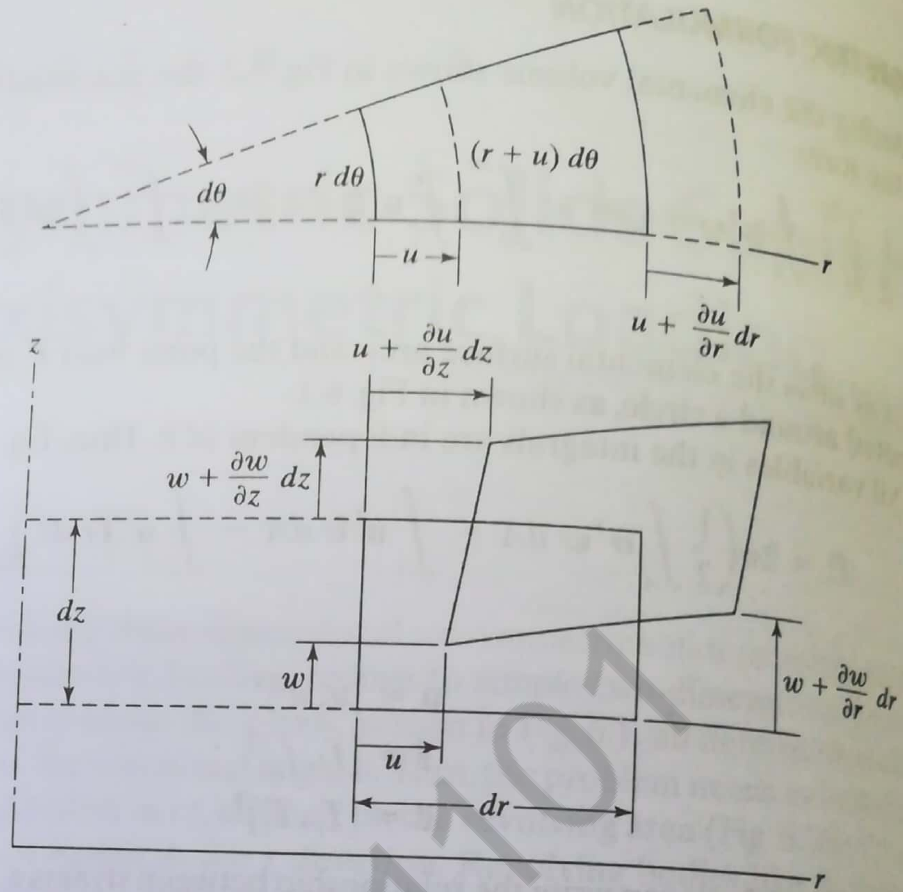


FIGURE 6.3 Deformation of elemental volume.

where the (4×4) matrix $\mathbf{D}$ can be written by dropping the appropriate terms from the three-dimensional matrix in Chapter 1, as

$$\mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix} \qquad (6.9)$$

In the Galerkin formulation, we require

$$2\pi \int_A \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\epsilon}(\boldsymbol{\phi}) r\, dA - \left(2\pi \int_A \boldsymbol{\phi}^{\mathrm{T}} \mathbf{f} r\, dA + 2\pi \int_L \boldsymbol{\phi}^{\mathrm{T}} \mathbf{T} r\, d\ell + \sum \boldsymbol{\phi}_i^{\mathrm{T}} \mathbf{P}_i \right) = 0 \qquad (6.10)$$

where

$$\boldsymbol{\phi} = [\phi_r, \phi_z]^{\mathrm{T}} \qquad (6.11)$$

$$\boldsymbol{\epsilon}(\boldsymbol{\phi}) = \left[\frac{\partial \phi_r}{\partial r}, \frac{\partial \phi_z}{\partial z}, \frac{\partial \phi_r}{\partial z} + \frac{\partial \phi_z}{\partial r}, \frac{\phi_r}{r} \right]^{\mathrm{T}} \qquad (6.12)$$

6.3 FINITE ELEMENT MODELING: TRIANGULAR ELEMENT

The two-dimensional region defined by the revolving area is divided into triangular elements, as shown in Fig. 6.4. Though each element is completely represented by the area in the rz plane, in reality, it is a ring-shaped solid of revolution obtained by revolving the triangle about the z-axis. A typical element is shown in Fig. 6.5.

The definition of connectivity of elements and the nodal coordinates follow the steps involved in the CST element discussed in Section 5.3. We note here that the r- and z-coordinates, respectively, replace x and y.

Using the three shape functions N_1, N_2, and N_3, we define

$$\mathbf{u} = \mathbf{N}\mathbf{q} \tag{6.13}$$

where $\mathbf{u}$ is defined in (6.3) and

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \tag{6.14}$$

$$\mathbf{q} = [q_1, \ q_2, \ q_3, \ q_4, \ q_5, \ q_6]^{\mathrm{T}} \tag{6.15}$$

If we denote $N_1 = \xi$ and $N_2 = \eta$, and note that $N_3 = 1 - \xi - \eta$, then Eq. 6.13 gives

$$\begin{aligned} u &= \xi q_1 + \eta q_3 + (1 - \xi - \eta) q_5 \\ w &= \xi q_2 + \eta q_4 + (1 - \xi - \eta) q_6 \end{aligned} \tag{6.16}$$

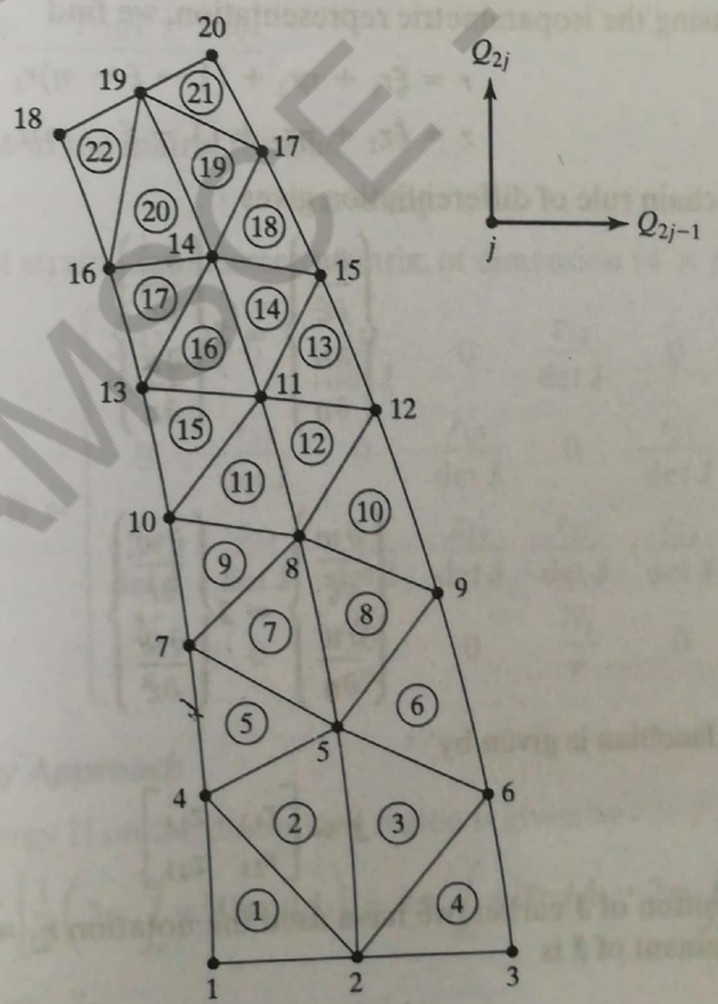


FIGURE 6.4 Triangulation.

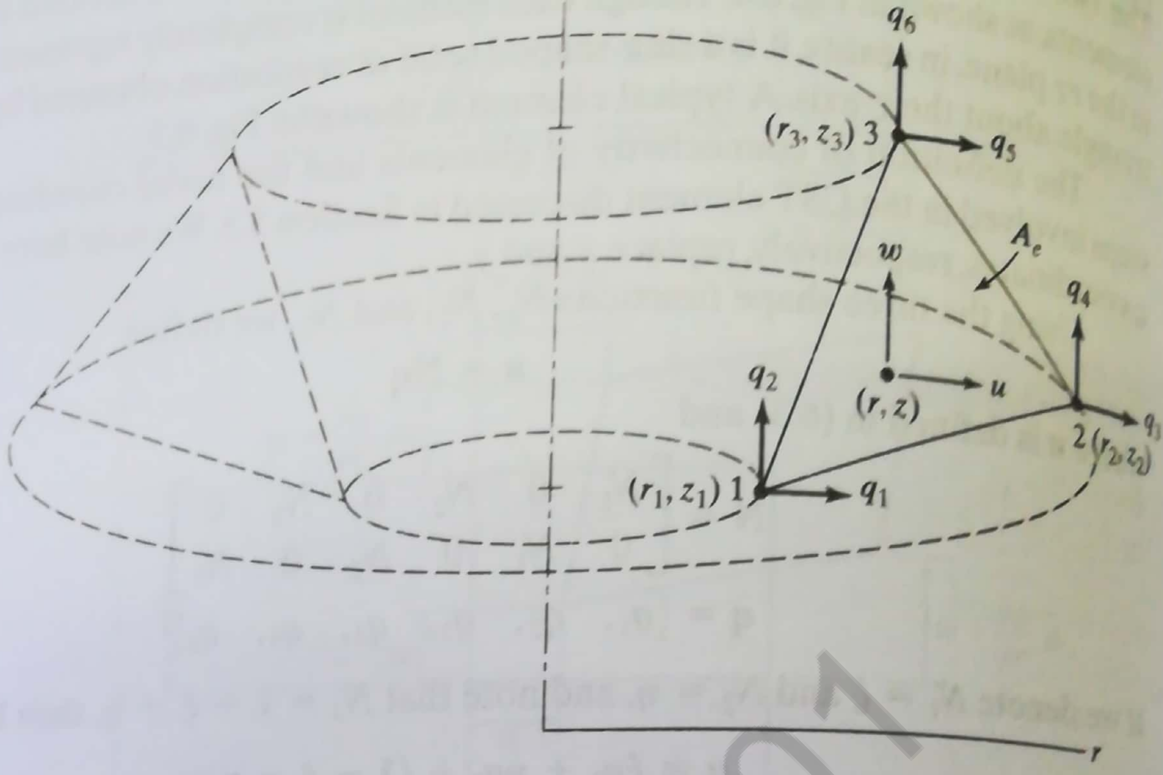


FIGURE 6.5 Axisymmetric triangular element.

By using the isoparametric representation, we find

$$r = \xi r_1 + \eta r_2 + (1 - \xi - \eta) r_3$$
$$z = \xi z_1 + \eta z_2 + (1 - \xi - \eta) z_3 \quad (6.17)$$

The chain rule of differentiation gives

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} \quad (6.18)$$

and

$$\begin{Bmatrix} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \end{Bmatrix} \quad (6.19)$$

where the Jacobian is given by

$$\mathbf{J} = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix} \quad (6.20)$$

In the definition of $\mathbf{J}$ earlier, we have used the notation $r_{ij} = r_i - r_j$ and $z_{ij} = z_i - z_j$. The determinant of $\mathbf{J}$ is

$$\det \mathbf{J} = r_{13} z_{23} - r_{23} z_{13} \quad (6.21)$$

Recall that $|\det \mathbf{J}| = 2A_e$. That is, the absolute value of the determinant of $\mathbf{J}$ equals twice the area of the element. The inverse relations for Eqs. 6.18 and 6.19 are given by

$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{Bmatrix} \tag{6.22}$$

where

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & z_{13} \end{bmatrix} \tag{6.23}$$

Introducing these transformation relationships into the strain–displacement relations in Eq. 6.6 and using Eqs. 6.16, we get

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{z_{23}(q_1 - q_5) - z_{13}(q_3 - q_5)}{\det \mathbf{J}} \\ \dfrac{-r_{23}(q_2 - q_6) + r_{13}(q_4 - q_6)}{\det \mathbf{J}} \\ \dfrac{-r_{23}(q_1 - q_5) + r_{13}(q_3 - q_5) + z_{23}(q_2 - q_6) - z_{13}(q_4 - q_6)}{\det \mathbf{J}} \\ \dfrac{N_1 q_1 + N_2 q_3 + N_3 q_5}{r} \end{Bmatrix}$$

This can be written in the matrix form as

$$\boldsymbol{\epsilon} = \mathbf{Bq} \tag{6.24}$$

where the element strain–displacement matrix, of dimension (4×6), is given by

$$\mathbf{B} = \begin{bmatrix} \dfrac{z_{23}}{\det \mathbf{J}} & 0 & \dfrac{z_{31}}{\det \mathbf{J}} & 0 & \dfrac{z_{12}}{\det \mathbf{J}} & 0 \\ 0 & \dfrac{r_{32}}{\det \mathbf{J}} & 0 & \dfrac{r_{13}}{\det \mathbf{J}} & 0 & \dfrac{r_{21}}{\det \mathbf{J}} \\ \dfrac{r_{32}}{\det \mathbf{J}} & \dfrac{z_{23}}{\det \mathbf{J}} & \dfrac{r_{13}}{\det \mathbf{J}} & \dfrac{z_{31}}{\det \mathbf{J}} & \dfrac{r_{21}}{\det \mathbf{J}} & \dfrac{z_{12}}{\det \mathbf{J}} \\ \dfrac{N_1}{r} & 0 & \dfrac{N_2}{r} & 0 & \dfrac{N_3}{r} & 0 \end{bmatrix} \tag{6.25}$$

Potential-Energy Approach

The potential energy Π on the discretized region is given by

$$\Pi = \sum_e \left[\frac{1}{2} \left(2\pi \int_e \boldsymbol{\epsilon}^{\mathrm{T}} \mathbf{D} \boldsymbol{\epsilon} r \, dA \right) - 2\pi \int_e \mathbf{u}^{\mathrm{T}} \mathbf{f} r \, dA - 2\pi \int_e \mathbf{u}^{\mathrm{T}} \mathbf{T} r \, d\ell \right] \tag{6.26}$$

The element strain energy U_e given by the first term can be written as

$$U_e = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\left(2\pi \int_e \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}r\, dA\right)\mathbf{q}$$

The quantity inside the parentheses is the element stiffness matrix,

$$\mathbf{k}^e = 2\pi \int_e \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}r\, dA \qquad (6.2$$

The fourth row in $\mathbf{B}$ has terms of the type N_i/r. Further, this integral also has an additional r in it. As a simple approximation, $\mathbf{B}$ and r can be evaluated at the centroid of the triangle and used as representative values for the triangle. At the centroid of the triangle,

$$N_1 = N_2 = N_3 = \tfrac{1}{3} \qquad (6.2$$

and

$$\bar{r} = \frac{r_1 + r_2 + r_3}{3} \qquad (6.2$$

where $\bar{r}$ is the radius of the centroid. Denoting $\bar{\mathbf{B}}$ as the element strain–displacement matrix $\mathbf{B}$ evaluated at the centroid, we get

$$\mathbf{k}^e = 2\pi\bar{r}\bar{\mathbf{B}}^{\mathrm{T}}\mathbf{D}\bar{\mathbf{B}}\int_e dA$$

or

$$\mathbf{k}^e = 2\pi\bar{r}A_e\bar{\mathbf{B}}^{\mathrm{T}}\mathbf{D}\bar{\mathbf{B}} \qquad (6.30)$$

We note here that $2\pi\bar{r}A_e$ is the volume of the ring-shaped element shown in Fig. 6.5. Also, A_e is given by

$$A_e = \tfrac{1}{2}|\det \mathbf{J}| \qquad (6.31)$$

We also use this centroid or midpoint rule* for body forces and surface tractions as discussed in the following section. Caution must be exerted for elements close to the axis of symmetry. For better results, smaller elements need to be chosen close to the axis of symmetry. Another approach is to introduce $r = N_1r_1 + N_2r_2 + N_3r_3$ in the following equations and perform elaborate integration. More elaborate methods of numerical integration are discussed in Chapter 7.

Body Force Term

We first consider the body force term $2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}r\, dA$. We have

$$2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}r\, dA = 2\pi \int_e (uf_r + wf_z)r\, dA$$

$$= 2\pi \int_e [(N_1q_1 + N_2q_3 + N_3q_5)f_r + (N_1q_2 + N_2q_4 + N_3q_6)f_z]r\, dA$$

*Suggested by O. C. Zienkiewicz, *The Finite Element Method*, 3d ed. New York: McGraw-Hill, 1983.

Once again, approximating the variable quantities by their values at the centroid of the triangle, we get

$$2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}r\, dA = \mathbf{q}^{\mathrm{T}}\mathbf{f}^e \tag{6.32}$$

where the element body force vector $\mathbf{f}^e$ is given by

$$\mathbf{f}^e = \frac{2\pi \bar{r} A_e}{3}[\bar{f}_r, \bar{f}_z, \bar{f}_r, \bar{f}_z, \bar{f}_r, \bar{f}_z]^{\mathrm{T}} \tag{6.33}$$

The bar on the **f** terms indicates that they are evaluated at the centroid. Where body force is the primary load, greater accuracy may be obtained by substituting $r = N_1r_1 + N_2r_2 + N_3r_3$ into Eq. 6.32 and integrating to get nodal loads.

Rotating Flywheel

As an example, let us consider a rotating flywheel with its axis in the z direction. We consider the flywheel to be stationary and apply the equivalent radial centrifugal (inertial) force per unit volume of $\rho r\omega^2$, where ρ is the density (mass per unit volume), and ω the angular velocity in rad/s. In addition, if gravity acts along the negative z-axis, then

$$\mathbf{f} = [f_r, f_z]^{\mathrm{T}} = [\rho r\omega^2, -\rho g]^{\mathrm{T}} \tag{6.34}$$

and

$$\bar{f}_r = \rho\bar{r}\omega^2, \bar{f}_z = -\rho g \tag{6.35}$$

For more precise results with coarse meshes, we need to use $r = N_1r_1 + N_2r_2 + N_3r_3$ and integrate.

Surface Traction

For a uniformly distributed load with components T_r and T_z, shown in Fig. 6.6, on the edge connecting nodes 1 and 2, we get

$$2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{T}r\, d\ell = \mathbf{q}^{\mathrm{T}}\mathbf{T}^e \tag{6.36}$$

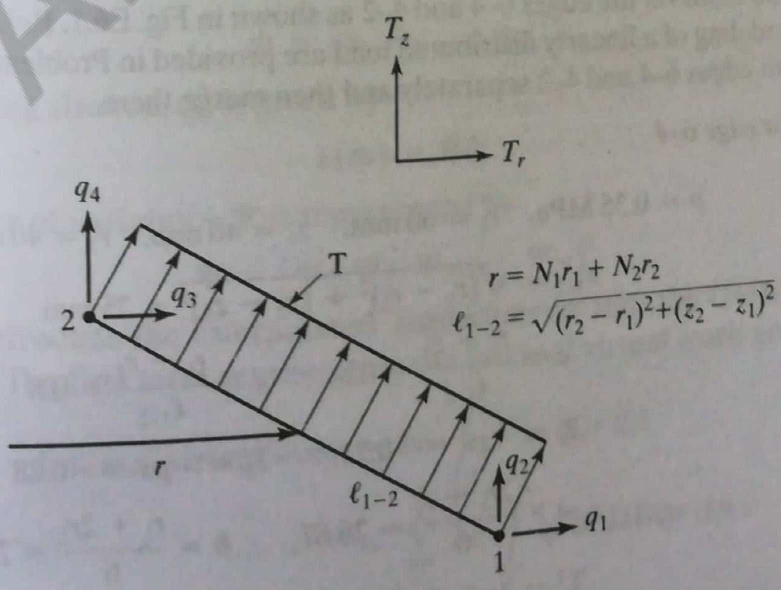


FIGURE 6.6 Surface traction.

Solution

Consider the following table:

Element	Connectivity 1	2	3
1	1	2	4
2	2	3	4

Node	Coordinates r	z
1	40	10
2	40	0
3	60	0
4	60	10

We will use the units of millimeters for length, newtons for force, and megapascals for stress and E. These units are consistent. On substituting $E = 200\,000$ MPa and $\nu = 0.3$, we have

$$\mathbf{D} = \begin{bmatrix} 2.69 \times 10^5 & 1.15 \times 10^5 & 0 & 1.15 \times 10^5 \\ 1.15 \times 10^5 & 2.69 \times 10^5 & 0 & 1.15 \times 10^5 \\ 0 & 0 & 0.77 \times 10^5 & 0 \\ 1.15 \times 10^5 & 1.15 \times 10^5 & 0 & 2.69 \times 10^5 \end{bmatrix}$$

for both elements, det $\mathbf{J} = 200\ \text{mm}^2$ and $A_e = 100\ \text{mm}^2$. From Eq. 6.31, forces F_1 and F_3 are given by

$$F_1 = F_3 = \frac{2\pi r_1 \ell_e p_i}{2} = \frac{2\pi(40)(10)(2)}{2} = 2514\ \text{N}$$

The **B** matrices relating element strains to nodal displacements are obtained first. For element 1, $\bar{r} = \frac{1}{3}(40 + 40 + 60) = 46.67$ mm and

$$\overline{\mathbf{B}}^1 = \begin{bmatrix} -0.05 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0.1 & 0 & -0.1 & 0 & 0 \\ 0.1 & -0.05 & -0.1 & 0 & 0 & 0.05 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

For element 2, $\bar{r} = \frac{1}{3}(40 + 60 + 60) = 53.33$ mm and

$$\overline{\mathbf{B}}^2 = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0 & 0.1 \\ 0 & -0.05 & -0.1 & 0.05 & 0.1 & 0 \\ 0.00625 & 0 & 0.00625 & 0 & 0.00625 & 0 \end{bmatrix}$$

The element stress–displacement matrices are obtained by multiplying **DB**:

$$\mathbf{D}\overline{\mathbf{B}}^1 = 10^4 \begin{bmatrix} -1.26 & 1.15 & 0.082 & -1.15 & 1.43 & 0 \\ -0.49 & 2.69 & 0.082 & -2.69 & 0.657 & 0.1 \\ 0.77 & -0.385 & -0.77 & 0 & 0 & 0.385 \\ -0.384 & 1.15 & 0.191 & -1.15 & 0.766 & 0 \end{bmatrix}$$

$$\mathbf{D}\overline{\mathbf{B}}^2 = 10^4 \begin{bmatrix} -1.27 & 0 & 1.42 & -1.15 & 0.072 & 1.15 \\ -0.503 & 0 & 0.647 & -2.69 & 0.072 & 2.69 \\ 0 & -0.385 & -0.77 & 0.385 & 0.77 & 0 \\ -0.407 & 0 & 0.743 & -1.15 & 0.168 & 1.15 \end{bmatrix}$$

The stiffness matrices are obtained by finding $2\pi\bar{r}A_e\bar{\mathbf{B}}^T\mathbf{D}\bar{\mathbf{B}}$ for each element:

$$
\begin{array}{l}
\text{Global dof} \rightarrow \quad 1 \qquad\quad 2 \qquad\quad 3 \qquad\quad 4 \qquad\quad 7 \qquad\quad 8
\end{array}
$$

$$
\mathbf{k}^1 = 10^7\begin{bmatrix}
4.03 & -2.58 & -2.34 & 1.45 & -1.932 & 1.13 \\
 & 8.45 & 1.37 & -7.89 & 1.93 & -0.565 \\
 & & 2.30 & -0.24 & 0.16 & -1.13 \\
 & & & 7.89 & -1.93 & 0 \\
 & \text{Symmetric} & & & 2.25 & 0 \\
 & & & & & 0.565
\end{bmatrix}
$$

$$
\begin{array}{l}
\text{Global dof} \rightarrow \quad 3 \qquad\quad 4 \qquad\quad 5 \qquad\quad 6 \qquad\quad 7 \qquad\quad 8
\end{array}
$$

$$
\mathbf{k}^2 = 10^7\begin{bmatrix}
2.05 & 0 & -2.22 & 1.69 & -0.085 & -1.69 \\
 & 0.645 & 1.29 & -0.645 & -1.29 & 0 \\
 & & 5.11 & -3.46 & -2.42 & 2.17 \\
 & & & 9.66 & 1.05 & -9.01 \\
 & \text{Symmetric} & & & 2.62 & 0.241 \\
 & & & & & 9.01
\end{bmatrix}
$$

Using the elimination approach, on assembling the matrices with reference to the degrees of freedom 1 and 3, we get

$$
10^7 = \begin{bmatrix} 4.03 & -2.34 \\ -2.34 & 4.35 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 2514 \\ 2514 \end{Bmatrix}
$$

so that

$$
Q_1 = 0.014 \times 10^{-2}\,\text{mm}
$$

$$
Q_3 = 0.0133 \times 10^{-2}\,\text{mm}
$$

Stress Calculations

From the set of nodal displacements obtained above, the element nodal displacements $\mathbf{q}$ can be found using the connectivity. Then, using stress–strain relation in Eq. 6.8 and strain–displacement relation in Eq. 6.24, we have

$$
\boldsymbol{\sigma} = \mathbf{D}\bar{\mathbf{B}}\mathbf{q} \tag{6.50}
$$

where $\bar{\mathbf{B}}$ is $\mathbf{B}$, given in Eq. 6.25, evaluated at the centroid of the element. We also note that σ_θ is a principal stress. The two principal stresses σ_1 and σ_2 corresponding to σ_r, σ_z, and τ_{rz} can be calculated using Mohr's circle.

Example 6.3

Calculate the element stresses in the problem discussed in Example 6.2.

Solution We need to find $\boldsymbol{\sigma}^{e^T} = [\sigma_r, \sigma_z, \tau_{rz}, \sigma_\theta]^e$ for each element. From the connectivity established in Example 6.2,

$$
\mathbf{q}^1 = [0.0140,\ \ 0,\ \ 0.0133,\ \ 0,\ \ 0,\ \ 0]^T \times 10^{-2}
$$

$$
\mathbf{q}^2 = [0.0133,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0]^T \times 10^{-2}
$$

Using the product matrices $\mathbf{DB}^e$ and $\mathbf{q}$ in the formula

$$\boldsymbol{\sigma}^e = \mathbf{D}\overline{\mathbf{B}}^e\mathbf{q}$$

we get

$$\boldsymbol{\sigma}^1 = [-166, \quad -58.2, \quad 5.4, \quad -28.4]^{\mathrm{T}} \times 10^{-2}\,\mathrm{MPa}$$
$$\boldsymbol{\sigma}^2 = [-169.3, \quad -66.9, \quad 0, \quad -54.1]^{\mathrm{T}} \times 10^{-2}\,\mathrm{MPa}$$

■

Temperature Effects

Uniform increase in temperature of ΔT introduces initial normal strains $\boldsymbol{\epsilon}_0$ given as

$$\boldsymbol{\epsilon}_0 = [\alpha\Delta T, \quad \alpha\Delta T, \quad 0, \quad \alpha\Delta T]^{\mathrm{T}} \tag{6.51}$$

The stresses are given by

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) \tag{6.52}$$

where $\boldsymbol{\epsilon}$ is the total strain.

On substitution into the strain energy, this yields an additional term of $-\boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}_0$ in the potential energy Π. Using the element strain–displacement relations in Eq. 6.24, we find that

$$2\pi \int_A \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}_0 r\, dA = \sum_e \mathbf{q}^{\mathrm{T}}(2\pi\bar{r}A_e\overline{\mathbf{B}}^{\mathrm{T}}\mathbf{D}\bar{\boldsymbol{\epsilon}}_0) \tag{6.53}$$

The consideration of the temperature effect in the Galerkin approach is rather simple. The term $\boldsymbol{\epsilon}^{\mathrm{T}}$ in Eq. (6.53) is replaced by $\boldsymbol{\epsilon}^{\mathrm{T}}(\boldsymbol{\phi})$.

The expression in parentheses gives element nodal load contributions. The vector $\bar{\boldsymbol{\epsilon}}_0$ is the initial strain evaluated at the centroid, representing the average temperature rise of the element. We have

$$\boldsymbol{\Theta}^e = 2\pi\bar{r}A_e\overline{\mathbf{B}}^{\mathrm{T}}\mathbf{D}\bar{\boldsymbol{\epsilon}}_0 \tag{6.54}$$

where

$$\boldsymbol{\Theta}^e = [\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6]^{\mathrm{T}} \tag{6.55}$$

6.4 PROBLEM MODELING AND BOUNDARY CONDITIONS

We have seen that the axisymmetric problem simply reduces to consideration of the revolving area. The boundary conditions need to be enforced on this area. θ independence arrests the rotation. Axisymmetry also implies that points lying on the z-axis remain radially fixed. Let us now consider some typical problems with a view to modeling them.

Cylinder Subjected to Internal Pressure

Figure 6.7 shows a hollow cylinder of length L subjected to an internal pressure. One end of the cylindrical pipe is attached to a rigid wall. In this, we need to model only the rectangular region of the length L bound between r_i and r_0. Nodes on the fixed end are constrained in the z and r directions. Stiffness and force modifications will be made for these nodes.

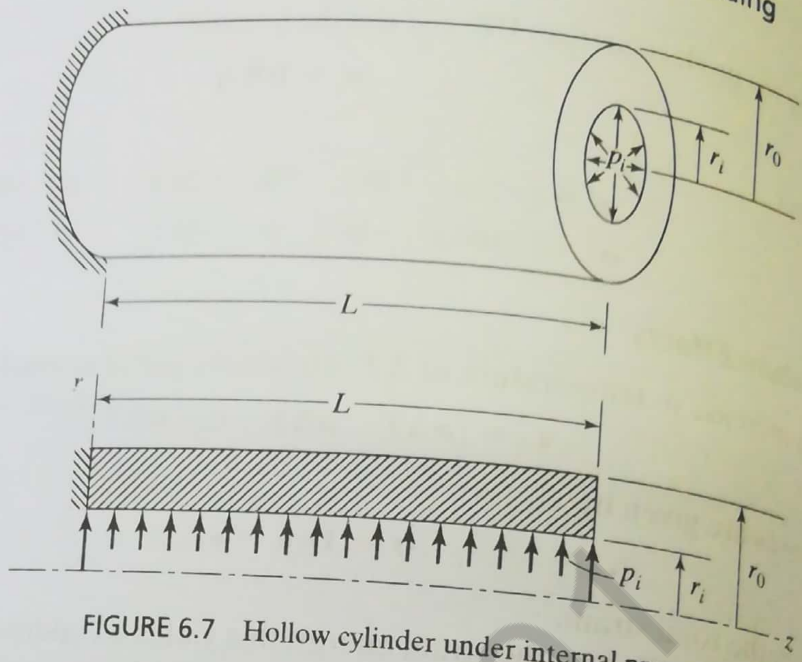


FIGURE 6.7 Hollow cylinder under internal pressure.

Infinite Cylinder

In Fig. 6.8, modeling of a cylinder of infinite length subjected to external pressure is shown. The length dimensions are assumed to remain constant. This plane strain condition is modeled by considering a unit length and restraining the end surfaces in the z direction.

Press Fit on a Rigid Shaft

Press fit of a ring of length L and internal radius r_i onto a rigid shaft of radius $r_i + \delta$ is considered in Fig. 6.9. When symmetry is assumed about the midplane, this plane is restrained in the z direction. When we impose the condition that nodes at the internal radius have to displace radially by δ, a large stiffness C is added to the diagonal locations for the radially constrained dofs and a force $C\delta$ is added to the corresponding force components. Solution of the equations gives displacements at nodes; stresses can then be evaluated.

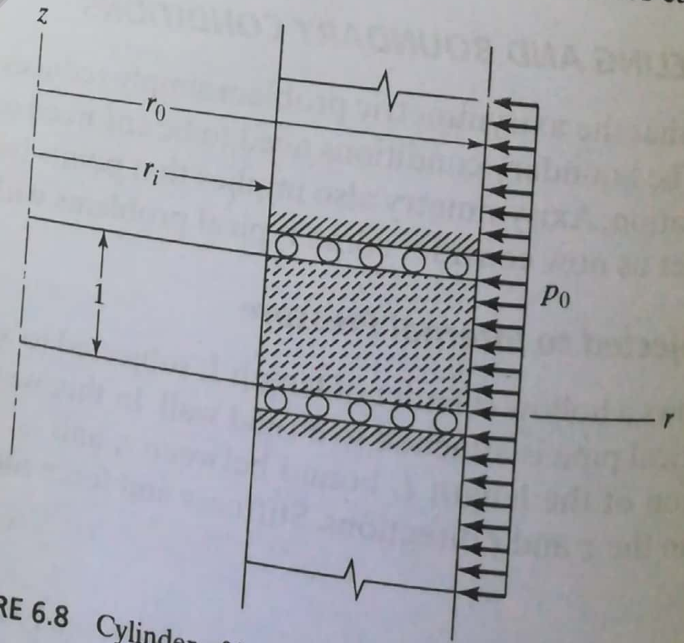


FIGURE 6.8 Cylinder of infinite length under external pressure.

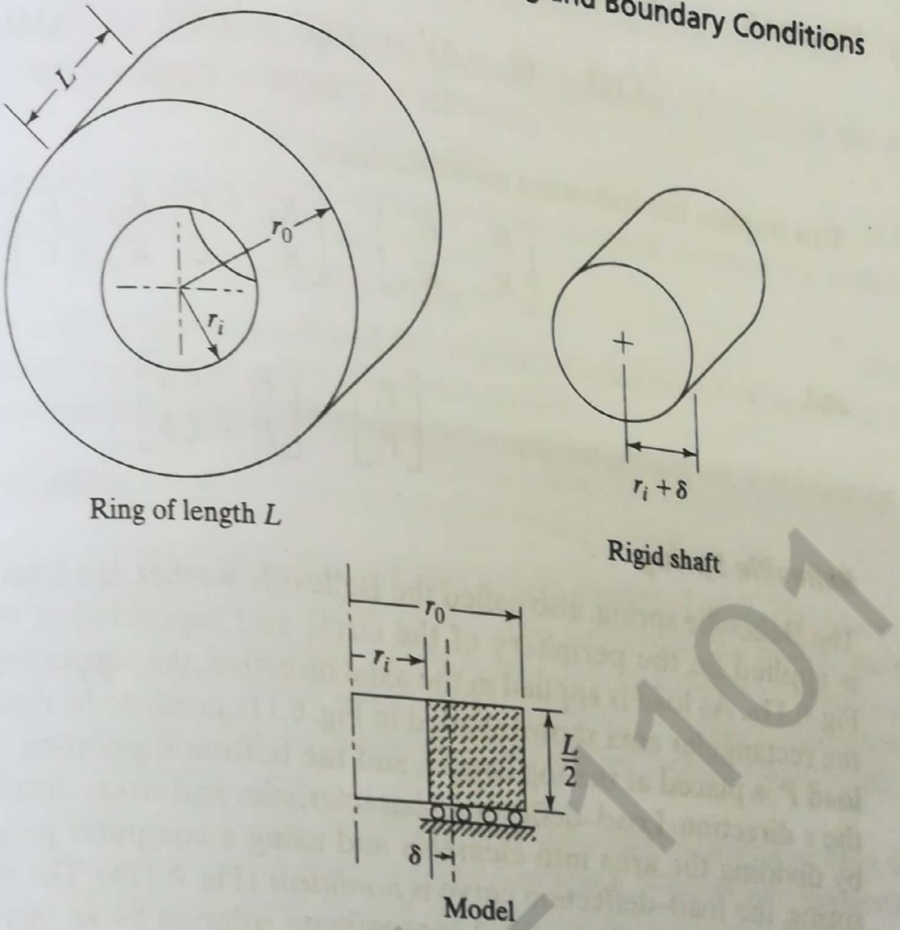


FIGURE 6.9 Press fit on a rigid shaft.

Press Fit on an Elastic Shaft

The condition at the contacting boundary leads to an interesting problem when an elastic sleeve is press fitted onto an elastic shaft. Take the problem of Fig. 6.9 stated above with the shaft also treated as elastic. A method to handle this is considered by referring to Fig. 6.10. We may define pairs of nodes on the contacting boundary, each pair consisting of one node on the sleeve and one on the shaft. If Q_i and Q_j are displacements of a typical pair along the radial degrees of freedom, we need to satisfy the multipoint constraint

$$Q_j - Q_i = \delta \tag{6.56}$$

When the term $\frac{1}{2}C(Q_j - Q_i - \delta)^2$ is added to the potential energy, the constraint is approximately enforced. The penalty approach for handling multipoint constraints is discussed in Chapter 3. Note that C is a large number. We have

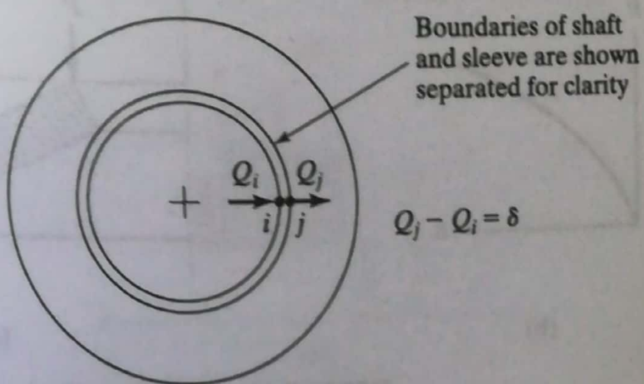


FIGURE 6.10 Elastic sleeve on an elastic shaft.

$$\tfrac{1}{2}C(Q_j - Q_i - \delta)^2 = \tfrac{1}{2}CQ_i^2 + \tfrac{1}{2}CQ_j^2 - \tfrac{1}{2}C(Q_iQ_j + Q_jQ_i) + CQ_i\delta - CQ_j\delta + \tfrac{1}{2}C\delta^2 \tag{6.57}$$

This implies the following modifications:

$$\begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix} \rightarrow \begin{bmatrix} K_{ii} + C & K_{ij} - C \\ K_{ji} - C & K_{jj} + C \end{bmatrix} \tag{6.58}$$

and

$$\begin{bmatrix} F_i \\ F_j \end{bmatrix} \rightarrow \begin{bmatrix} F_i - C\delta \\ F_j + C\delta \end{bmatrix} \tag{6.59}$$

Belleville Spring

The Belleville spring, also called the Belleville washer, is a conical disk spring. The load is applied on the periphery of the circle and supported at the bottom as shown in Fig. 6.11a. As load is applied in the axial direction, the supporting edge moves out. Only the rectangular area shown shaded in Fig. 6.11c needs to be modeled. An axisymmetric load P is placed at the top corner, and the bottom supporting corner is constrained in the z direction. Load–deflection characteristics and stress distribution can be obtained by dividing the area into elements and using a computer program. In the Belleville spring, the load–deflection curve is *nonlinear* (Fig. 6.11b). The stiffness depends on the geometry. We can find a good approximate solution by an incremental approach. We

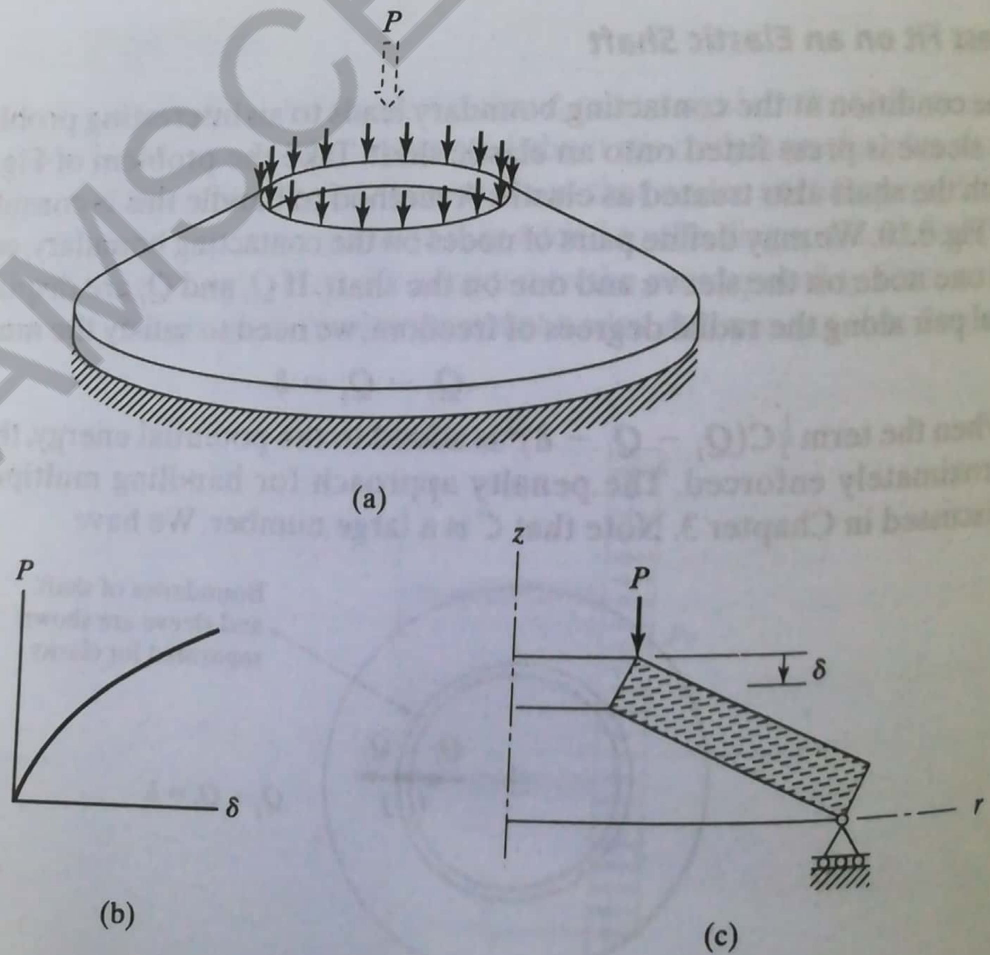


FIGURE 6.11 Belleville spring.

find the stiffness matrix $\mathbf{K}(\mathbf{x})$ for the given coordinate geometry. We obtain the displacements $\Delta\mathbf{Q}$ for an incremental loading of $\Delta\mathbf{F}$ from

$$\mathbf{K}(\mathbf{x})\,\Delta\mathbf{Q} = \Delta\mathbf{F} \tag{6.60}$$

The displacements $\Delta\mathbf{Q}$ are converted to the components Δu and Δw and are added to $\mathbf{x}$ to update the new geometry:

$$\mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{u} \tag{6.61}$$

$\mathbf{K}$ is recalculated for the new geometry, and the new set of equations 6.60 is solved. The process is continued until the full applied load is reached.

This example illustrates the incremental approach for geometric nonlinearity.

Thermal Stress Problem

Shown in Fig. 6.12a is a steel sleeve inserted into a rigid insulated wall. The sleeve fits snugly, and then the temperature is raised by ΔT. The stresses in the sleeve increase because of the constraint. The rectangular area of length $L/2$, bounded by r_i and r_0 is considered (Fig. 6.12b), with points on the outer radius constrained radially and points on r constrained axially. The load vector is modified using the load vector from Eq. 6.55, and the finite element equations are solved.

Modeling of simple to complex problems of engineering importance have been discussed. In real life, each problem poses its own challenge. With a clear understanding of the loading, boundary conditions, and the material behavior, the modeling of a problem can be broken down into simple and easy steps.

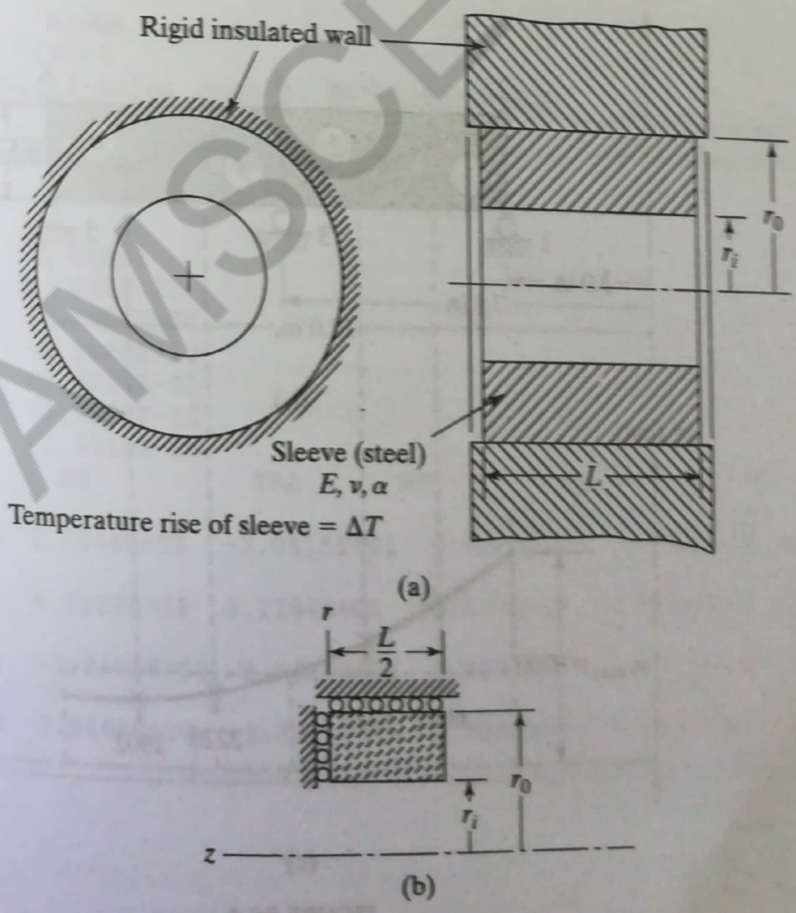


FIGURE 6.12 Thermal stress problem.