CHAPTER 5

Two-Dimensional Problems Using Constant Strain Triangles

5.1 INTRODUCTION

The two-dimensional finite element formulation in this chapter follows the steps used in the one-dimensional problem. The displacements, traction components, and distributed body force values are functions of the position indicated by (x, y). The displacement vector $\mathbf{u}$ is given as

$$\mathbf{u} = [u, v]^{\mathrm{T}} \tag{5.1}$$

where u and v are the x and y components of $\mathbf{u}$, respectively. The stresses and strains are given by

$$\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \tau_{xy}]^{\mathrm{T}} \tag{5.2}$$

$$\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \gamma_{xy}]^{\mathrm{T}} \tag{5.3}$$

From Fig. 5.1, representing the two-dimensional problem in a general setting, the body force, traction vector, and elemental volume are given by

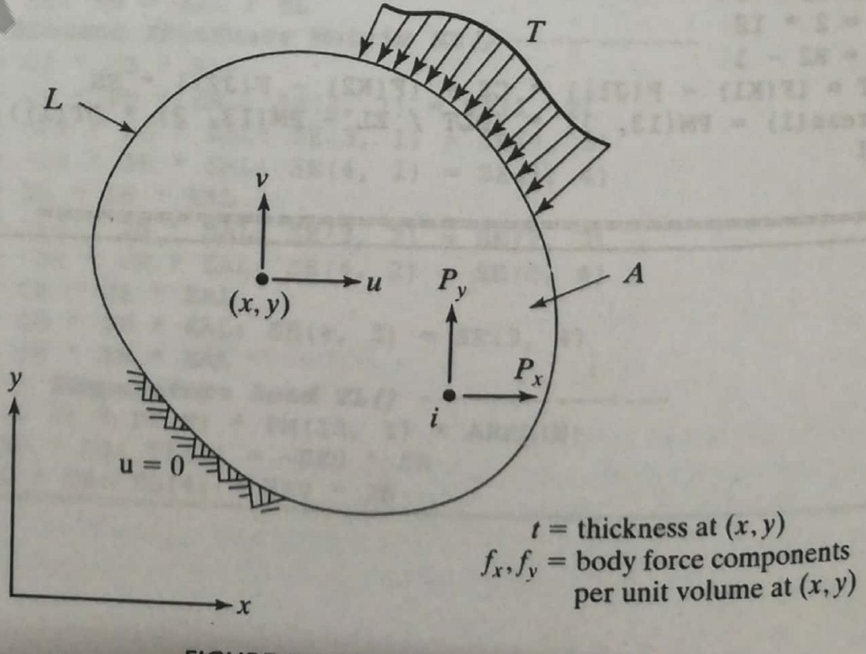

FIGURE 5.1 Two-dimensional problem.

$$\mathbf{f} = [f_x, f_y]^{\mathrm{T}} \qquad \mathbf{T} = [T_x, T_y]^{\mathrm{T}} \quad \text{and} \quad dV = t\,dA \tag{5.4}$$

where t is the thickness along the z direction. The body force $\mathbf{f}$ has the units force/unit volume, while the traction force $\mathbf{T}$ has the units force/unit area. The strain–displacement relations are given by

$$\boldsymbol{\epsilon} = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]^{\mathrm{T}} \tag{5.5}$$

Stresses and strains are related by (see Eqs. 1.18 and 1.19)

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon} \tag{5.6}$$

The region is discretized with the idea of expressing the displacements in terms of values at discrete points. Triangular elements are introduced first. Stiffness and load concepts are then developed using energy and Galerkin approaches.

5.2 FINITE ELEMENT MODELING

The two-dimensional region is divided into straight-sided triangles. Figure 5.2 shows a typical triangulation. The points where the corners of the triangles meet are called *nodes*, and each triangle formed by three nodes and three sides is called an *element*. The elements fill the entire region except a small region at the boundary. This unfilled region exists for curved boundaries, and it can be reduced by choosing smaller elements or elements with curved boundaries. The idea of the finite element method is to solve the continuous problem approximately, and this unfilled region contributes to some part of this

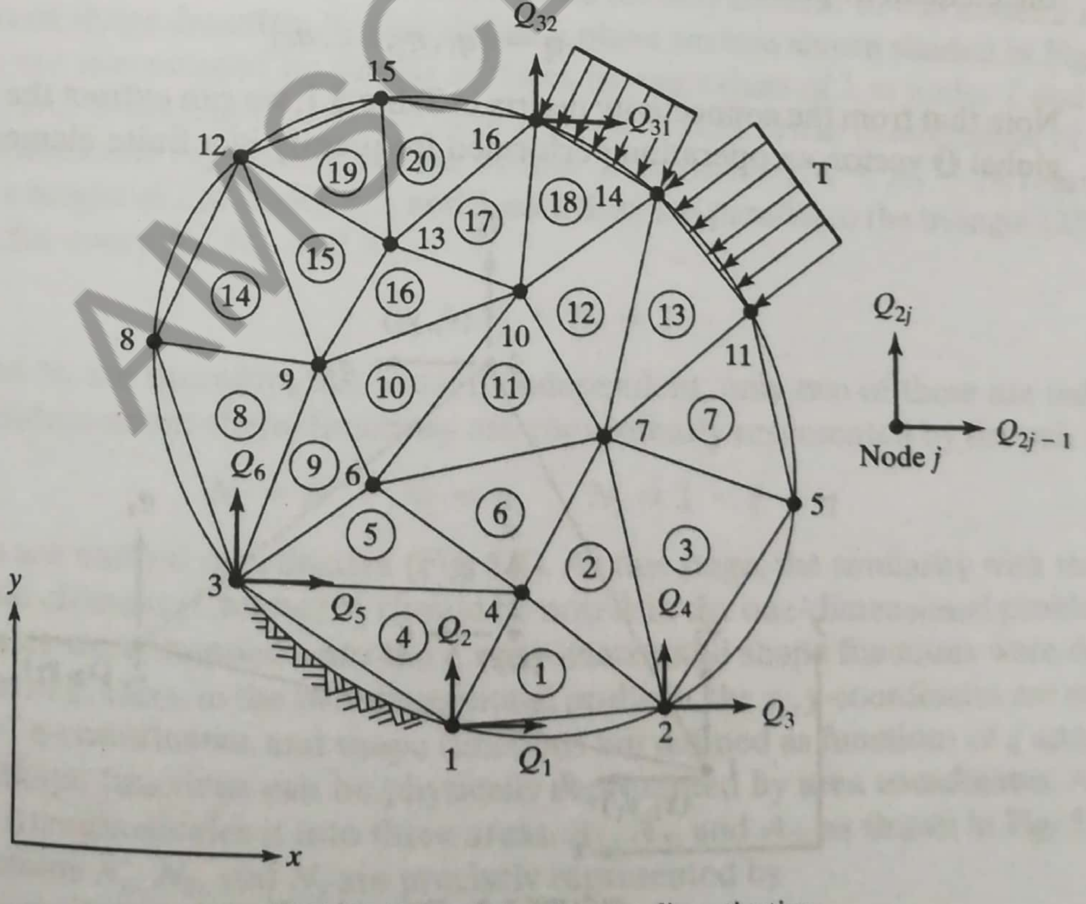

FIGURE 5.2 Finite element discretization.

approximation. For the triangulation shown in Fig. 5.2, the node numbers are indicated at the corners and element numbers are circled.

In the two-dimensional problem discussed here, each node is permitted to displace in the two directions x and y. Thus, each node has two degrees of freedom (dofs). As seen from the numbering scheme used in trusses, the displacement components of node j are taken as Q_{2j-1} in the x direction and Q_{2j} in the y direction. We denote the global displacement vector as

$$\mathbf{Q} = [Q_1, Q_2, \ldots, Q_N]^{\mathrm{T}} \tag{5.7}$$

where N is the number of degrees of freedom.

Computationally, the information on the triangulation is to be represented in the form of *nodal coordinates* and *connectivity*. The nodal coordinates are stored in a two-dimensional array represented by the total number of nodes and the two coordinates per node. The connectivity may be clearly seen by isolating a typical element, as shown in Fig. 5.3. For the three nodes designated locally as 1, 2, and 3, the corresponding global node numbers are defined in Fig. 5.2. This element connectivity information becomes an array of the size and number of elements and three nodes per element. A typical connectivity representation is shown in Table 5.1. Most standard finite element codes use the convention of going around the element in a counterclockwise direction to avoid calculating a negative area. However, in the program that accompanies this chapter, ordering is not necessary.

Table 5.1 establishes the correspondence of local and global node numbers and the corresponding degrees of freedom. The displacement components of a local node j in Fig. 5.3 are represented as q_{2j-1} and q_{2j} in the x and y directions, respectively. We denote the element displacement vector as

$$\mathbf{q} = [q_1, q_2, \ldots, q_6]^{\mathrm{T}} \tag{5.8}$$

Note that from the connectivity matrix in Table 5.1, we can extract the **q** vector from the global **Q** vector, an operation performed frequently in a finite element program. Also,

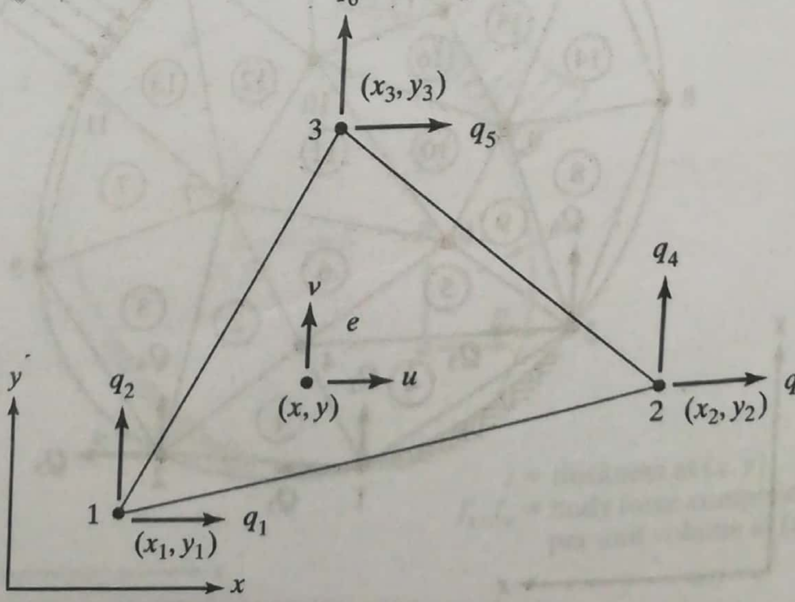

FIGURE 5.3 Triangular element.

TABLE 5.1 Element Connectivity

Element number e	Three nodes		
	1	2	3
1	1	2	4
2	4	2	7
⋮			
11	6	7	10
⋮			
20	13	16	15

the nodal coordinates designated by (x_1, y_1) (x_2, y_2) and (x_3, y_3) have the global correspondence established through Table 5.1. The local representation of nodal coordinates and degrees of freedom provides a setting for a simple and clear representation of element characteristics.

5.3 CONSTANT-STRAIN TRIANGLE (CST)

The displacements at points inside an element need to be represented in terms of the nodal displacements of the element. As discussed earlier, the finite element method uses the concept of shape functions in systematically developing these interpolations. For the constant strain triangle, the shape functions are linear over the element. The three shape functions N_1, N_2, and N_3 corresponding to nodes 1, 2, and 3, respectively, are shown in Fig. 5.4. Shape function N_1 is 1 at node 1 and linearly reduces to 0 at nodes 2 and 3. The values of shape function N_1 thus define a plane surface shown shaded in Fig. 5.4a. N_2 and N_3 are represented by similar surfaces having values of 1 at nodes 2 and 3, respectively, and dropping to 0 at the opposite edges. Any linear combination of these shape functions also represents a plane surface. In particular, $N_1 + N_2 + N_3$ represents a plane at a height of 1 at nodes 1, 2, and 3, and, thus, it is parallel to the triangle 123. Consequently, for every N_1, N_2, and N_3,

$$N_1 + N_2 + N_3 = 1 \tag{5.9}$$

N_1, N_2, and N_3 are therefore not linearly independent; only two of these are independent. The independent shape functions are conveniently represented by the pair ξ, η as

$$N_1 = \xi \qquad N_2 = \eta \qquad N_3 = 1 - \xi - \eta \tag{5.10}$$

where ξ, η are natural coordinates (Fig. 5.4). At this stage, the similarity with the one-dimensional element (Chapter 3) should be noted: in the one-dimensional problem the x-coordinates were mapped onto the ξ coordinates, and shape functions were defined as functions of ξ. Here, in the two-dimensional problem, the x-, y-coordinates are mapped onto the ξ-, η-coordinates, and shape functions are defined as functions of ξ and η.

The shape functions can be physically represented by **area coordinates**. A point (x, y) in a triangle divides it into three areas, A_1, A_2, and A_3, as shown in Fig. 5.5. The shape functions N_1, N_2, and N_3 are precisely represented by

2.1.2 Variational Formulations

The classical use of the phrase "variational formulations" refers to the construction of a functional (whose meaning will be made clear shortly) or a variational principle that is equivalent to the governing equations of the problem. The modern use of the phrase refers to the formulation in which the governing equations are translated into equivalent weighted-integral statements that are not necessarily equivalent to a variational principle. Even those problems that do not admit variational principles in the classical sense (e.g., the Navier–Stokes equations governing the flow of viscous or inviscid fluids) can now be formulated using weighted-integral statements.

The importance of variational formulations of physical laws, in the modern or general sense of the phrase, goes far beyond its use as simply an alternate to other formulations [Oden and Reddy (1983)]. In fact, variational forms of the laws of continuum physics may be the only natural and rigorously correct way to think of them. While all sufficiently smooth fields lead to meaningful variational forms, the converse is not true: There exist physical phenomena which can be adequately modeled mathematically only in a variational setting; they are nonsensical when viewed locally.

The starting point for the discussion of the finite element method is differential equations governing the physical phenomena under study. As such, we shall first discuss why integral statements of the differential equations are needed.

2.1.3 Need for Weighted-Integral Statements

In almost all approximate methods used to determine the solution of differential and/or integral equations, we seek a solution in the form

$$u(\mathbf{x}) \approx U_N(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}) \qquad (2.1.1)$$

where u represents the solution of a particular differential equation and associated boundary conditions, and U_N is its approximation that is represented as a linear combination of unknown parameters c_j and known functions ϕ_j of position $\mathbf{x}$ in the domain Ω on which the problem is posed. We shall shortly discuss the conditions on ϕ_j. The approximate solution U_N is completely known only when c_j are known. Thus, we must find a means to determine c_j such that U_N satisfies the equations governing u. If somehow we can find U_N that satisfies the differential equation at every point $\mathbf{x}$ of the domain Ω

6.1.3 Finite Element Formulation

Comparison of Eqs. (6.1.6) and (6.1.16) with the model equation (3.2.1) reveals that the equations governing eigenvalue problems are special cases of the model equations studied in Chapters 3 and 5. Here we summarize the steps in the finite element formulation of eigenvalue problems for the sake of completeness and ready reference. We will consider eigenvalue problems described by (*a*) a single equation in a single unknown (e.g., heat transfer, bar, and Euler–Bernoulli beam problems), and (*b*) a pair of equations in two variables (e.g., Timoshenko beam theory).

Heat Transfer and Bar-Like Problems

Consider the problem of solving the equation

$$-\frac{d}{dx}\left[a(x)\frac{dU}{dx}\right]+c(x)U(x)=\lambda c_0(x)U(x) \tag{6.1.21}$$

for λ and $U(x)$. Here a, c, and c_0 are known quantities that depend on the physical problem (i.e., data), λ is the eigenvalue, and U is the eigenfunction. Special cases of Eq. (6.1.21) are given below.

$$\text{Heat transfer:} \quad a = kA, \quad c = P\beta, \quad c_0 = \rho c A \tag{6.1.22}$$

$$\text{Bars:} \quad a = EA, \quad c = 0, \quad c_0 = \rho A \tag{6.1.23}$$

Over a typical element Ω_e, we seek a finite element approximation of U in the form

$$U_h^e(x) = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \tag{6.1.24}$$

The weak form of (6.1.21) is

$$0 = \int_{x_a}^{x_b} \left(a\frac{dw}{dx}\frac{dU}{dx} + cwU(x) - \lambda c_0 w U \right) dx - Q_1^e w(x_a) - Q_n^e w(x_b) \tag{6.1.25}$$

where w is the weight function, and Q_1^e and Q_n^e are the secondary variables at node 1 and node n, respectively (assume that $Q_i^e = 0$ when $i \neq 1$ and $i \neq n$)

$$Q_1^e = -\left[a\frac{dU}{dx}\right]_{x_a}, \quad Q_n^e = \left[a\frac{dU}{dx}\right]_{x_b} \tag{6.1.26}$$

Substitution of the finite element approximation into the weak form gives the finite element model of the eigenvalue equation (6.1.21):

$$[K^e]\{u^e\} - \lambda[M^e]\{u^e\} = \{Q^e\} \tag{6.1.27a}$$

where

$$K_{ij}^e = \int_{x_a}^{x_b} \left[a(x)\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + c(x)\psi_i^e\psi_j^e \right] dx, \quad M_{ij}^e = \int_{x_a}^{x_b} c_0(x)\psi_i^e\psi_j^e \, dx \tag{6.1.27b}$$

Equation (6.1.27a) contains the finite element models of the eigenvalue equations (6.1.6) and (6.1.16) as special cases.

The assembly of element equations and imposition of boundary conditions on the assembled equations remain the same as in static problems of Chapter 3. However, the solution of the condensed equations for the unknown primary nodal variables is reduced to an algebraic eigenvalue problem in which the determinant of the coefficient matrix is set to zero to determine the values of λ and subsequently the nodal values of the eigenfunction $U(x)$. These ideas are illustrated through examples.

9.2 ELEMENT LIBRARY

9.2.1 Triangular Elements

The linear (three-node) triangular element was developed in Section 8.2.5. Higher-order triangular elements (i.e., triangular elements with interpolation functions of higher degree) can be systematically developed with the help of the so-called *Pascal's triangle*, which

Pascal's triangle	Degree of the complete polynomial	Number of terms in the polynomial	Element with nodes
1	0	1	
$x \quad y$	1	3	
$x^2 \quad xy \quad y^2$	2	6	
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3	10	
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$	4	15	
$x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5$	5	21	(Figure not shown)

Figure 9.2.1 Topmost six rows of Pascal's triangle for the generation of the Lagrange family of triangular elements.

contains the terms of polynomials of various degrees in the two coordinates x and y, as shown in Fig. 9.2.1. Here x and y denote some local coordinates; they do not, in general, represent the global coordinates of the problem. We can view the position of the terms as the nodes of the triangle, with the constant term and the first and last terms of a given row being the vertices of the triangle. Of course, the shape of the triangle is arbitrary—not necessarily an equilateral triangle, as might appear from the position of the terms in Pascal's triangle. For example, a triangular element of order 2 (i.e., the degree of the polynomial is 2) contains six nodes, as can be seen from the third row of Pascal's triangle. The position of the six nodes in the triangle is at the three vertices and at the midpoints of the three sides. The polynomial involves six constants, which can be expressed in terms of the nodal values of the variable being interpolated:

$$u = \sum_{i=1}^{6} u_i \psi_i(x, y) \qquad (9.2.1)$$

where ψ_i are the quadratic interpolation functions obtained following the same procedure as that used for the linear element in Section 8.2. In general, a pth-order triangular element has a number of n nodes

$$n = \frac{1}{2}(p+1)(p+2) \qquad (9.2.2)$$

and a complete polynomial of pth degree is given by

$$u(x, y) = \sum_{i=1}^{n} a_i x^r y^s = \sum_{j=1}^{n} u_j \psi_j, \qquad r + s \leq p \qquad (9.2.3)$$

The location of the entries in Pascal's triangle gives a symmetric location of nodal points in elements that will produce exactly the right number of nodes to define a Lagrange interpolation of any degree. It should be noted that the Lagrange family of triangular elements (of order greater than zero) should be used for second-order problems that require only the dependent variables (not their derivatives) of the problem to be continuous at interelement boundaries. It can be easily seen that the pth-degree polynomial associated with the pth-order Lagrange element, when evaluated on the boundary of the element, yields a pth-degree polynomial in the boundary coordinate. For example, the quadratic polynomial associated with the quadratic (six-node) triangular element shown in Fig. 9.2.2(a) is given by

$$u^e(x, y) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2 \tag{9.2.4}$$

The derivatives of u^e are

$$\frac{\partial u^e}{\partial x} = a_2 + a_4 y + 2a_5 x, \qquad \frac{\partial u^e}{\partial y} = a_3 + a_4 x + 2a_6 y \tag{9.2.5}$$

The element shown in Fig. 9.2.2(a) is an arbitrary quadratic triangular element. By rotating and translating the (x, y) coordinate system, we obtain the (s, t) coordinate system [see Fig. 9.2.2(b)]. Since the transformation from the (x, y) system to the (s, t) system involves only rotation (which is linear) and translation, a kth-degree polynomial in the (x, y)

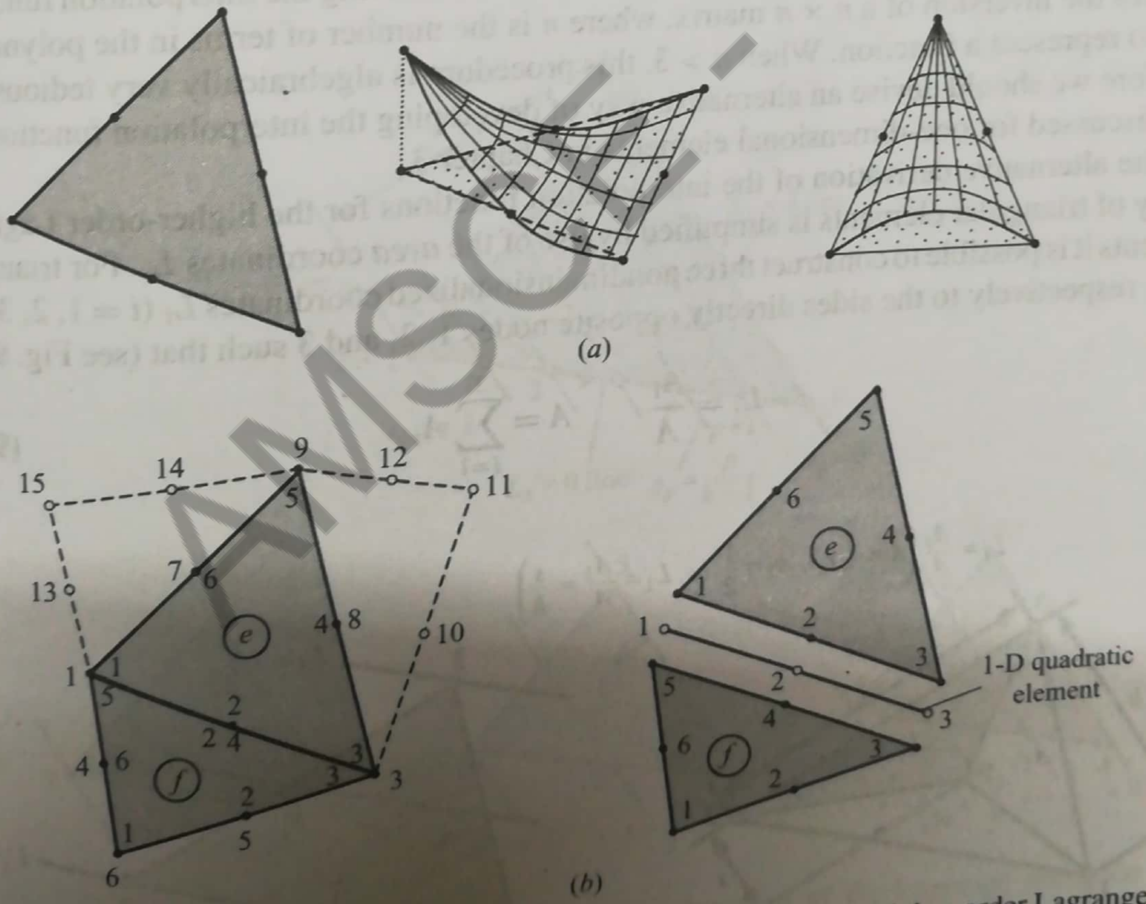

Figure 9.2.2 Variation of a function along the interelement boundaries of higher-order Lagrange elements: (a) a quadratic triangular element and (b) interelement continuity of a quadratic interpolation function.

coordinate system is still a kth-degree polynomial in the (s, t) system:

$$u^e(s,t) = \hat{a}_1 + \hat{a}_2 s + \hat{a}_3 t + \hat{a}_4 st + \hat{a}_5 s^2 + \hat{a}_6 t^2 \tag{9.2.6}$$

where $\hat{a}_i$ $(i = 1, 2, \ldots, 6)$ are constants that depend on a_i and the angle of rotation α. Now by setting $t = 0$, we get the restriction of u to side 1–2–3 of element Ω_e:

$$u^e(s,0) = \hat{a}_1 + \hat{a}_2 s + \hat{a}_5 s^2 \tag{9.2.7}$$

which is a quadratic polynomial in s. If a neighboring element Ω_f has its side 5–4–3 in common with side 1–2–3 of element Ω_e, then the function u on side 5–4–3 of element Ω_f is also a quadratic polynomial

$$u^f(s,0) = \hat{b}_1 + \hat{b}_2 s + \hat{b}_5 s^2 \tag{9.2.8}$$

Since the polynomials are uniquely defined by the same nodal values $U_1 = u_1^e = u_5^f$, $U_2 = u_2^e = u_4^f$, and $U_3 = u_3^e = u_3^f$, we have $u^e(s, 0) = u^f(s, 0)$ and hence the function u is uniquely defined on the interelement boundary of elements e and f.

The ideas discussed above can be easily extended to three dimensions, in which case Pascal's triangle takes the form of a Christmas tree and the elements are of a pyramid shape, called tetrahedral elements. We shall not elaborate on this any further because the scope of the present study is limited to two-dimensional elements only. A brief introduction to three-dimensional elements is presented in Chapter 14.

Recall from (8.2.21)–(8.2.25) that the procedure for deriving the interpolation functions involves the inversion of a $n \times n$ matrix, where n is the number of terms in the polynomial used to represent a function. When $n > 3$, this procedure is algebraically very tedious, and therefore we should devise an alternative way of developing the interpolation functions, as was discussed for one-dimensional elements in Chapter 3.

The alternative derivation of the interpolation functions for the higher-order Lagrange family of triangular elements is simplified by use of the *area* coordinates L_i. For triangular elements it is possible to construct three nondimensionalized coordinates L_i $(i = 1, 2, 3)$ that relate respectively to the sides directly opposite nodes 1, 2, and 3 such that (see Fig. 9.2.3)

$$L_i = \frac{A_i}{A} \qquad A = \sum_{i=1}^{3} A_i \tag{9.2.9}$$

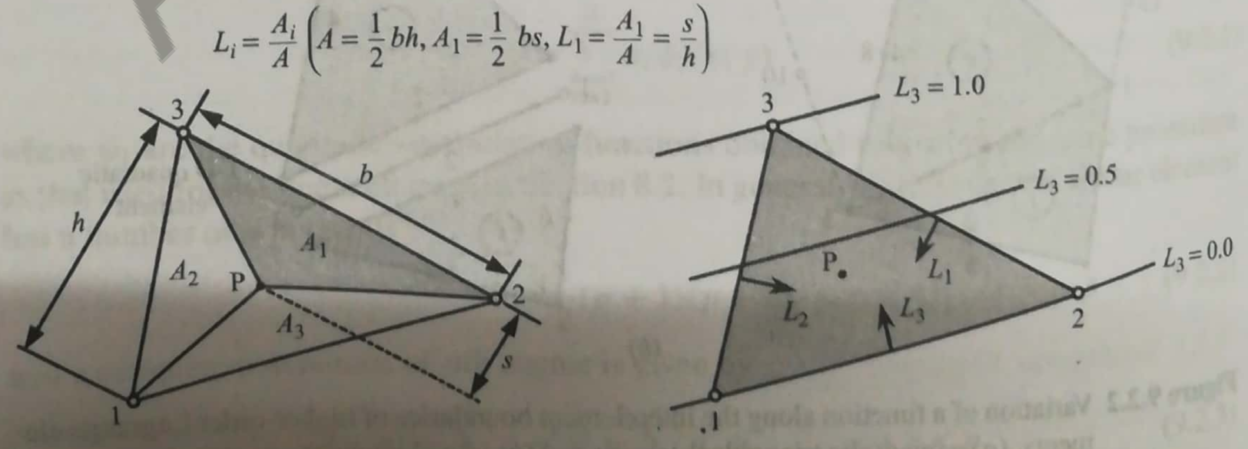

Figure 9.2.3 Definition of the natural coordinates of a triangular element.

where A_i is the area of the triangle formed by nodes j and k and an arbitrary point P in the element, and A is the total area of the element. For example, A_1 is the area of the shaded triangle, which is formed by nodes 2 and 3 and point P. The point P is at a perpendicular distance of s from the side connecting nodes 2 and 3. We have $A_1 = \frac{1}{2}bs$ and $A = \frac{1}{2}bh$. Hence,

$$L_1 = \frac{A_1}{A} = \frac{s}{h}$$

Clearly, L_1 is zero on side 2–3 (hence, zero at nodes 2 and 3) and has a value of unity at node 1. Thus, L_1 is the interpolation function associated with node 1. Similarly, L_2 and L_3 are the interpolation functions associated with nodes 2 and 3, respectively. In summary, we have

$$\psi_i = L_i \tag{9.2.10}$$

for a linear triangular element. We shall use L_i to construct interpolation functions for higher-order triangular elements.

Consider a higher-order element with k nodes (equally spaced) per side [see Fig. 9.2.4(a)]. Then the total number of nodes in the element is given by

$$n = \sum_{i=0}^{k-1}(k-i) = k + (k-1) + \cdots + 1 = \frac{1}{2}k(k+1) \tag{9.2.11}$$

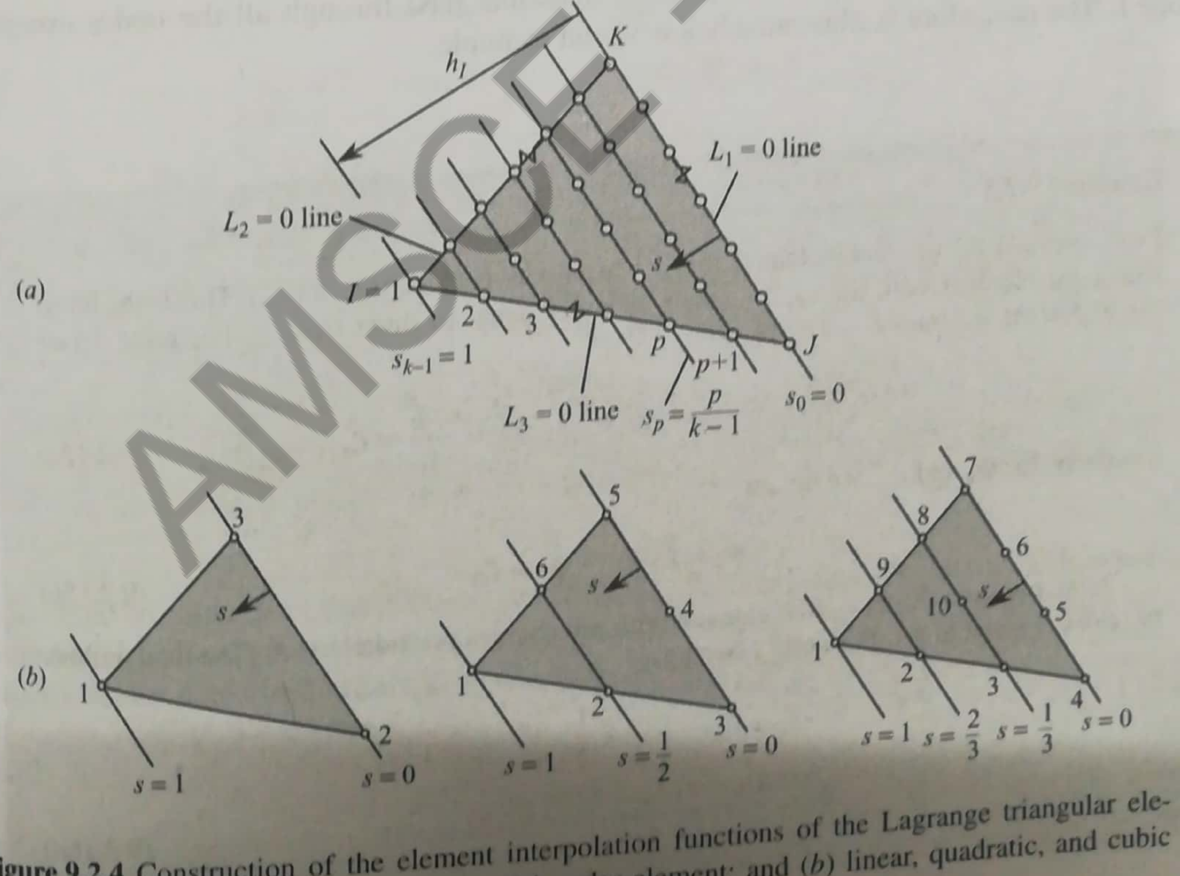

Figure 9.2.4 Construction of the element interpolation functions of the Lagrange triangular elements: (a) an arbitrary ($k - 1$)th-order element; and (b) linear, quadratic, and cubic elements.

DEVELOPMENT OF THE LINEAR-STRAIN TRIANGLE EQUATIONS ▲

CHAPTER OBJECTIVES

- To develop the linear-strain triangular (LST) element stiffness matrix.
- To describe how the LST stiffness matrix can be determined.
- To compare the difference in results using the CST and LST elements.

Introduction

In this chapter, we consider the development of the stiffness matrix and equations for a higher-order triangular element, called the *linear-strain triangle* (LST). This element is available in many commercial computer programs and has some advantages over the constant-strain triangle described in Chapter 6.

The LST element has six nodes and twelve unknown displacement degrees of freedom. The displacement functions for the element are quadratic instead of linear (as in the CST).

The procedures for development of the equations for the LST element follow the same steps as those used in Chapter 6 for the CST element. However, the number of equations now becomes twelve instead of six, making a longhand solution extremely cumbersome. Hence, we will use a computer to perform many of the mathematical operations.

After deriving the element equations, we will compare results from problems solved using the LST element with those solved using the CST element. The introduction of the higher-order LST element will illustrate the possible advantages of higher-order elements and should enhance your general understanding of the concepts involved with finite element procedures.

8.1 Derivation of the Linear-Strain Triangular Element Stiffness Matrix and Equations

We will now derive the LST stiffness matrix and element equations. The steps used here are identical to those used for the CST element, and much of the notation is the same.

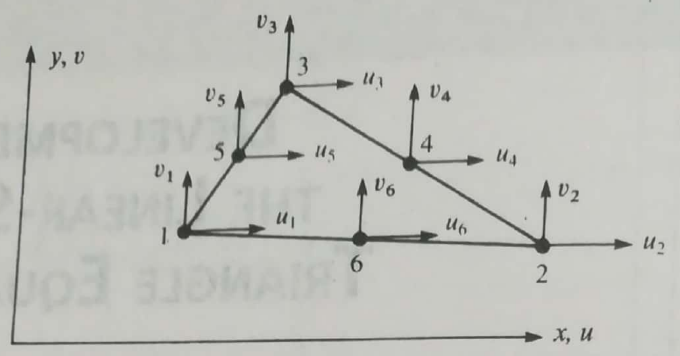

Figure 8–1 Basic six-node triangular element showing degrees of freedom

Step 1 Select Element Type

Consider the triangular element shown in Figure 8–1 with the usual end nodes and three additional nodes conveniently located at the midpoints of the sides. Thus, a computer program can automatically compute the midpoint coordinates once the coordinates of the corner nodes are given as input.

The unknown nodal displacements are now given by

$$\{d\} = \begin{Bmatrix} \{d_1\} \\ \{d_2\} \\ \{d_3\} \\ \{d_4\} \\ \{d_5\} \\ \{d_6\} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix} \tag{8.1.1}$$

Step 2 Select a Displacement Function

We now select a quadratic displacement function in each element as

$$\begin{aligned} u(x, y) &= a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 \\ v(x, y) &= a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2 \end{aligned} \tag{8.1.2}$$

Again, the number of coefficients $a_i (12)$ equals the total number of degrees of freedom for the element. The displacement compatibility among adjoining elements is satisfied because three nodes are located along each side and a parabola is defined by three points on its path. Since adjacent elements are connected at common nodes, their displacement compatibility across the boundaries will be maintained.

In general, when considering triangular elements, we can use a complete polynomial in Cartesian coordinates to describe the displacement field within an element.

Terms in Pascal Triangle	*Polynomial Degree*	*Number of Terms*	*Triangle*
1	0 (constant)	1	
$x \quad y$	1 (linear)	3	CST (Chap. 6) 
$x^2 \quad xy \quad y^2$	2 (quadratic)	6	LST (Chap. 8) 
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3 (cubic)	10	QST 

Figure 8–2 Relation between type of plane triangular element and polynomial coefficients based on a Pascal triangle

Using internal nodes as necessary for the higher-order cubic and quartic elements, we use all terms of a truncated Pascal triangle in the displacement field or, equivalently, the shape functions, as shown by Figure 8–2; that is, a complete linear function is used for the CST element considered previously in Chapter 6. The complete quadratic function is used for the LST of this chapter. The complete cubic function is used for the quadratic-strain triangle (QST), with an internal node necessary as the tenth node.

The general displacement functions, Eqs. (8.1.2), expressed in matrix form are now

$$\{\psi\}=\begin{Bmatrix} u \\ v \end{Bmatrix}=\begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{Bmatrix} \tag{8.1.3}$$

Alternatively, we can express Eq. (8.1.3) as

$$\{\psi\}=[M^*]\{a\} \tag{8.1.4}$$

where $[M^*]$ is defined to be the first matrix on the right side of Eq. (8.1.3). The coefficients a_1 through a_{12} can be obtained by substituting the coordinates into u and v as follows:

$$\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ \vdots \\ v_5 \\ v_6 \end{Bmatrix}=\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \\ a_7 \\ \vdots \\ a_{11} \\ a_{12} \end{Bmatrix} \tag{8.1.5}$$

Solving for the a_i's, we have

$$\begin{Bmatrix} a_1 \\ \vdots \\ a_6 \\ a_7 \\ \vdots \\ a_{12} \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ \vdots \\ u_6 \\ v_1 \\ \vdots \\ v_6 \end{Bmatrix} \tag{8.1.6}$$

or, alternatively, we can express Eq. (8.1.6) as

$$\{a\} = [X]^{-1}\{d\} \tag{8.1.7}$$

where $[X]$ is the 12×12 matrix on the right side of Eq. (8.1.6). It is best to invert the $[X]$ matrix by using a digital computer. Then the a_i's, in terms of nodal displacements, are substituted into Eq. (8.1.4). Note that only the 6×6 part of $[X]$ in Eq. (8.1.6) really must be inverted. Finally, using Eq. (8.1.7) in Eq. (8.1.4), we can obtain the general displacement expressions in terms of the shape functions and the nodal degrees of freedom as

$$\{\psi\} = [N]\{d\} \tag{8.1.8}$$

where

$$[N] = [M^*][X]^{-1} \tag{8.1.9}$$

Step 3 Define the Strain–Displacement and Stress–Strain Relationships

The element strains are again given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \end{Bmatrix} \tag{8.1.10}$$

or, using Eq. (8.1.3) for u and v in Eq. (8.1.10), we obtain the strain-generalized displacement equations as

$$\{\varepsilon\} = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{Bmatrix} \tag{8.1.11}$$

We observe that Eq. (8.1.11) yields a linear strain variation in the element. Therefore, the element is called a *linear-strain triangle* (LST). Rewriting Eq. (8.1.11), we have

$$\{\varepsilon\} = [M']\{a\} \tag{8.1.12}$$

where $[M']$ is the first matrix on the right side of Eq. (8.1.11). Substituting Eq. (8.1.6) for the a_i's into Eq. (8.1.12), we have $\{\varepsilon\}$ in terms of the nodal displacements as

$$\{\varepsilon\} = [B]\{d\} \tag{8.1.13}$$

where $[B]$ is a function of the variables x and y and the coordinates (x_1, y_1) through (x_6, y_6) given by

$$[B] = [M'][X]^{-1} \tag{8.1.14}$$

where Eq. (8.1.7) has been used in expressing Eq. (8.1.14). Note that $[B]$ is now a matrix of order 3 × 12.

The stresses are again given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [D][B]\{d\} \tag{8.1.15}$$

where $[D]$ is given by Eq. (6.1.8) for plane stress or by Eq. (6.1.10) for plane strain. These stresses are now linear functions of x and y coordinates.

Step 4 Derive the Element Stiffness Matrix and Equations

We determine the stiffness matrix in a manner similar to that used in Section 6.2 by using Eq. (6.2.50) repeated here as

$$[k] = \iiint_V [B]^T [D][B]\, dV \tag{8.1.16}$$

However, the $[B]$ matrix is now a function of x and y as given by Eq. (8.1.14). Therefore, we must perform the integration in Eq. (8.1.16). Finally, the $[B]$ matrix is of the form

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{bmatrix} \tag{8.1.17}$$

where the β's and γ's are now functions of x and y as well as of the nodal coordinates, as is illustrated for a specific linear-strain triangle in Section 8.2 by Eq. (8.2.8). The stiffness matrix is then seen to be a 12 × 12 matrix on multiplying the matrices in Eq. (8.1.16). The stiffness matrix, Eq. (8.1.16), is very cumbersome to obtain in explicit form, so it will not be given here. However, if the origin of the coordinates is considered to be at the centroid of the element, the integrations become amenable [9]. Alternatively, area coordinates [3, 8, 9] can be used to obtain an explicit form of the stiffness matrix. However, even the use of area coordinates usually involves tedious calculations. Therefore, the integration is best carried out numerically. (Numerical integration is described in Section 10.3.)

The element body forces and surface forces should not be automatically lumped at the nodes, but for a consistent formulation (one that is formulated from the same shape functions used to formulate the stiffness matrix), Eqs. (6.3.1) and (6.3.7), respectively, should be used. (Problems 8.3 and 8.4 illustrate this concept.) These forces can be added to any concentrated nodal forces to obtain the element force matrix. Here the element force matrix is of order 12 × 1 because, in general, there could be an x and a y component of force at each of the six nodes associated with the element. The element equations are then given by

$$\underset{(12\times 1)}{\begin{Bmatrix} f_{1x} \\ f_{1y} \\ \vdots \\ f_{6y} \end{Bmatrix}} = \underset{(12\times 12)}{\begin{bmatrix} k_{11} & \dots & k_{1,12} \\ k_{21} & & k_{2,12} \\ \vdots & & \vdots \\ k_{12,1} & \dots & k_{12,12} \end{bmatrix}} \underset{(12\times 1)}{\begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ v_6 \end{Bmatrix}} \tag{8.1.18}$$

Steps 5 through 7

Steps 5 through 7, which involve assembling the global stiffness matrix and equations, determining the unknown global nodal displacements, and calculating the stresses, are identical to those in Section 6.2 for the CST. However, instead of constant stresses in each element, we now have a linear variation of the stresses in each element. Common practice was to use the centroidal element stresses. Current practice is to use the average of the nodal element stresses.

8.2 Example LST Stiffness Determination

To illustrate some of the procedures outlined in Section 8.1 for deriving an LST stiffness matrix, consider the following example. Figure 8–3 shows a specific LST and its coordinates. The triangle is of base dimension b and height h, with midside nodes.

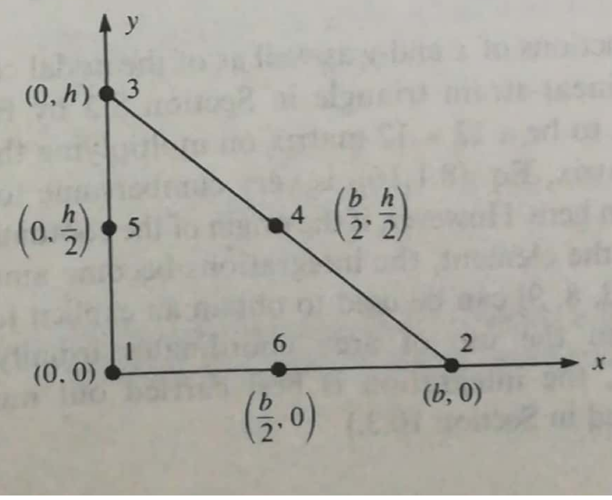

Figure 8–3 LST triangle for evaluation of a stiffness matrix

6.2 Derivation of the Constant-Strain Triangular Element Stiffness Matrix and Equations

To illustrate the steps and introduce the basic equations necessary for the plane triangular element, consider the thin plate subjected to tensile surface traction loads T_S in Figure 6–6(a).

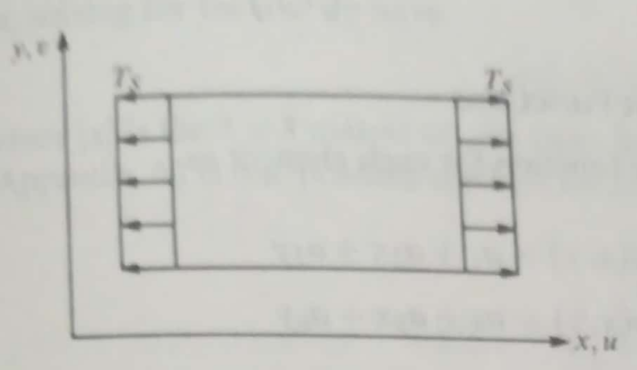

Figure 6–6(a) Thin plate in tension

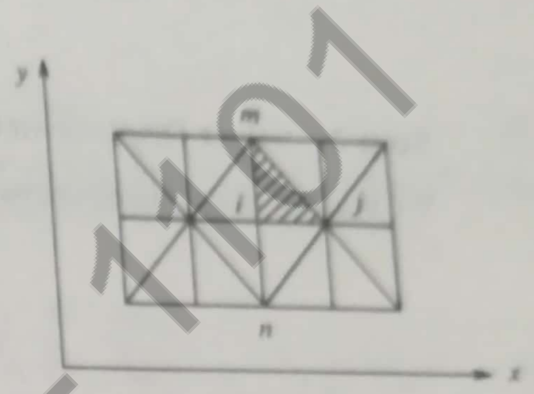

Figure 6–6(b) Discretized plate of Figure 6–6(a) using triangular elements

Step 1 Select Element Type

To analyze the plate, we consider the basic triangular element in Figure 6–7 taken from the discretized plate, as shown in Figure 6–6(b). The discretized plate has been

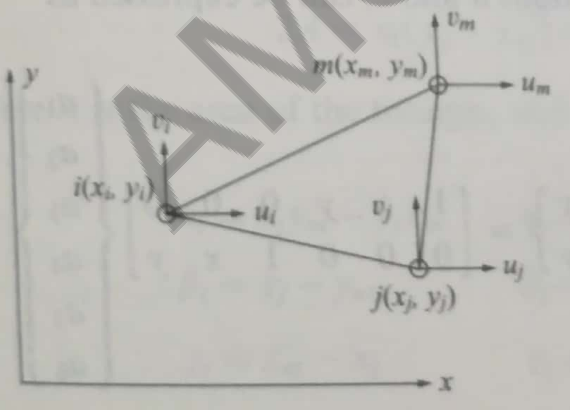

Figure 6–7 Basic triangular element showing degrees of freedom

divided into triangular elements, each with nodes such as i, j, and m. We use triangular elements because boundaries of irregularly shaped bodies can be closely approximated in this way, and because the expressions related to the triangular element are comparatively simple. This discretization is called a *coarse-mesh generation* if a few large elements are used. Each node has two degrees of freedom—an x and a y displacement. We will let u_i and v_i represent the node i displacement components in the x and y directions, respectively.

Here all formulations are based on this counterclockwise system of labeling of nodes, although a formulation based on a clockwise system of labeling could be used. Remember that a consistent labeling procedure for the whole body is necessary to avoid problems in the calculations such as negative element areas. Here (x_i, y_i), (x_j, y_j), and (x_m, y_m) are the known nodal coordinates of nodes i, j, and m, respectively.

The nodal displacement matrix is given by

$$\{d\} = \begin{Bmatrix} \{d_i\} \\ \{d_j\} \\ \{d_m\} \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \tag{6.2.1}$$

Step 2 Select Displacement Functions

We select a linear displacement function for each element as

$$\begin{aligned} u(x,y) &= a_1 + a_2x + a_3y \\ v(x,y) &= a_4 + a_5x + a_6y \end{aligned} \tag{6.2.2}$$

where $u(x, y)$ and $v(x, y)$ describe displacements at any interior point (x_i, y_i) of the element.

The linear function ensures that compatibility will be satisfied. A linear function with specified endpoints has only one path through which to pass—that is, through the two points. Hence, the linear function ensures that the displacements along the edge and at the nodes shared by adjacent elements, such as edge i-j of the two elements shown in Figure 6–6(b), are equal. Using Eqs. (6.2.2), the general displacement function $\{\psi\}$, which stores the functions u and v, can be expressed as

$$\{\psi\} = \begin{Bmatrix} a_1 + a_2x + a_3y \\ a_4 + a_5x + a_6y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \tag{6.2.3}$$

To obtain the a's in Eqs. (6.2.2), we begin by substituting the coordinates of the nodal points into Eqs. (6.2.2) to yield

$$\begin{aligned}
u_i &= u(x_i, y_i) = a_1 + a_2 x_i + a_3 y_i \\
u_j &= u(x_j, y_j) = a_1 + a_2 x_j + a_3 y_j \\
u_m &= u(x_m, y_m) = a_1 + a_2 x_m + a_3 y_m \\
v_i &= v(x_i, y_i) = a_4 + a_5 x_i + a_6 y_i \\
v_j &= v(x_j, y_j) = a_4 + a_5 x_j + a_6 y_j \\
v_m &= v(x_m, y_m) = a_4 + a_5 x_m + a_6 y_m
\end{aligned} \tag{6.2.4}$$

We can solve for the a's beginning with the first three of Eqs. (6.2.4) expressed in matrix form as

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{6.2.5}$$

or, solving for the a's, we have

$$\{a\} = [x]^{-1}\{u\} \tag{6.2.6}$$

where $[x]$ is the 3 × 3 matrix on the right side of Eq. (6.2.5). The method of cofactors (Appendix A) is one possible method for finding the inverse of $[x]$. Thus,

$$[x]^{-1} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \tag{6.2.7}$$

where

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} \tag{6.2.8}$$

is the determinant of $[x]$, which on evaluation is

$$2A = x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j) \tag{6.2.9}$$

Here A is the area of the triangle, and

$$\begin{aligned}
\alpha_i &= x_j y_m - y_j x_m & \alpha_j &= y_i x_m - x_i y_m & \alpha_m &= x_i y_j - y_i x_j \\
\beta_i &= y_j - y_m & \beta_j &= y_m - y_i & \beta_m &= y_i - y_j \\
\gamma_i &= x_m - x_j & \gamma_j &= x_i - x_m & \gamma_m &= x_j - x_i
\end{aligned} \tag{6.2.10}$$

Having determined $[x]^{-1}$, we can now express Eq. (6.2.6) in expanded matrix form as

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \tag{6.2.11}$$

Similarly, using the last three of Eqs. (6.2.4), we can obtain

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix} \tag{6.2.12}$$

We will derive the general x displacement function $u(x, y)$ of $\{\psi\}$ (v will follow analogously) in terms of the coordinate variables x and y, known coordinate variables $\alpha_i, \alpha_j, \ldots, \gamma_m$, and unknown nodal displacements u_i, u_j, and u_m. Beginning with Eqs. (6.2.2) expressed in matrix form, we have

$$\{u\} = [1 \quad x \quad y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{6.2.13}$$

Substituting Eq. (6.2.11) into Eq. (6.2.13), we obtain

$$\{u\} = \frac{1}{2A}[1 \quad x \quad y] \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \tag{6.2.14}$$

Expanding Eq. (6.2.14), we have

$$\{u\} = \frac{1}{2A}[1 \quad x \quad y] \begin{Bmatrix} \alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{Bmatrix} \tag{6.2.15}$$

Multiplying the two matrices in Eq. (6.2.15) and rearranging, we obtain

$$u(x, y) = \frac{1}{2A}\{(\alpha_i + \beta_i x + \gamma_i y)u_i + (\alpha_j + \beta_j x + \gamma_j y)u_j + (\alpha_m + \beta_m x + \gamma_m y)u_m\} \tag{6.2.16}$$

Similarly, replacing u_i by v_i, u_j by v_j, and u_m by v_m in Eq. (6.2.16), we have the y displacement given by

$$v(x, y) = \frac{1}{2A}\{(\alpha_i + \beta_i x + \gamma_i y)v_i + (\alpha_j + \beta_j x + \gamma_j y)v_j + (\alpha_m + \beta_m x + \gamma_m y)v_m\} \tag{6.2.17}$$

To express Eqs. (6.2.16) and (6.2.17) for u and v in simpler form, we define

$$
\begin{aligned}
N_i &= \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y) \\
N_j &= \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y) \\
N_m &= \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)
\end{aligned}
\qquad (6.2.18)
$$

Thus, using Eqs. (6.2.18), we can rewrite Eqs. (6.2.16) and (6.2.17) as

$$
\begin{aligned}
u(x,y) &= N_i u_i + N_j u_j + N_m u_m \\
v(x,y) &= N_i v_i + N_j v_j + N_m v_m
\end{aligned}
\qquad (6.2.19)
$$

Expressing Eqs. (6.2.19) in matrix form, we obtain

$$
\{\psi\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}
$$

or

$$
\{\psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \qquad (6.2.20)
$$

Finally, expressing Eq. (6.2.20) in abbreviated matrix form, we have

$$
\{\psi\} = [N]\{d\} \qquad (6.2.21)
$$

where $[N]$ is given by

$$
[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \qquad (6.2.22)
$$

We have now expressed the general displacements as functions of $\{d\}$, in terms of the shape functions N_i, N_j, and N_m. The shape functions represent the shape of $\{\psi\}$ when plotted over the surface of a typical element. For instance, N_i represents the shape of the variable u when plotted over the surface of the element for $u_i = 1$ and all other degrees of freedom equal to zero; that is, $u_j = u_m = v_i = v_j = v_m = 0$. In addition, $u(x_i, y_i)$ must be equal to u_i. Therefore, we must have $N_i = 1$, $N_j = 0$, and $N_m = 0$ at (x_i, y_i). Similarly, $u(x_j, y_j) = u_j$. Therefore, $N_i = 0$, $N_j = 1$, and $N_m = 0$ at (x_j, y_j). Figure 6–8 shows the shape variation of N_i plotted over the surface of a typical element. Note that N_i does not equal zero except along a line connecting and including nodes j and m.

Example 6.1

Evaluate the stiffness matrix for the element shown in Figure 6–11. The coordinates are shown in units of mm. Assume plane stress conditions. Let $E = 210$ GPa, $\nu = 0.25$, and thickness $t = 20$ mm. Assume the element nodal displacements have been determined to be $u_1 = 0.0$, $v_1 = 0.05$ mm, $u_2 = 0.025$ mm, $v_2 = 0.0$, $u_3 = 0.0$, and $v_3 = 0.05$ mm. Determine the element stresses.

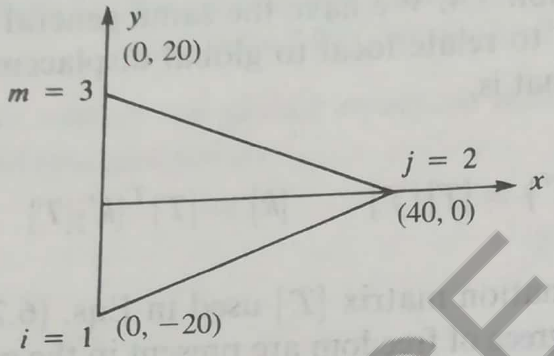

Figure 6–11 Plane stress element for stiffness matrix evaluation

SOLUTION:

We use Eq. (6.2.52) to obtain the element stiffness matrix. To evaluate $[k]$, we first use Eqs. (6.2.10) to obtain the β's and γ's as follows:

$$\begin{aligned}
\beta_i &= y_j - y_m = 0 - 20 = -20 & \gamma_i &= x_m - x_j = 0 - 40 = -40 \\
\beta_j &= y_m - y_i = 20 - (-20) = 40 & \gamma_j &= x_i - x_m = 0 - 0 = 0 \\
\beta_m &= y_i - y_j = -20 - 0 = -20 & \gamma_m &= x_j - x_i = 40 - 0 = 40
\end{aligned} \tag{6.2.61}$$

Using Eqs. (6.2.32) and (6.2.34), we obtain matrix $[B]$ as

$$[B] = \frac{10^2}{2(8)}\begin{bmatrix} -2 & 0 & 4 & 0 & -2 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & -2 & 0 & 4 & 4 & -1 \end{bmatrix}\frac{1}{\text{m}} \tag{6.2.62}$$

where we have used $A = 8 \times 10^{-4}\ \text{m}^2$ in Eq. (6.2.62).

Using Eq. (6.1.8) for plane stress conditions,

$$[D] = \frac{210 \times 10^9}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \dfrac{1-0.25}{2} \end{bmatrix}\frac{\text{N}}{\text{m}^2} \tag{6.2.63}$$

Substituting Eqs. (6.2.62) and (6.2.63) into Eq. (6.2.52), we obtain

$$[k] = \frac{(20 \times 10^{-3})(8 \times 10^{-4})(210 \times 10^9)(10^2)}{16(0.9375)}\begin{bmatrix} -2 & 0 & -4 \\ 0 & -4 & -2 \\ 4 & 0 & 0 \\ 0 & 0 & 4 \\ -2 & 0 & 4 \\ 0 & 4 & -2 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}\frac{10^2}{2(8)}\begin{bmatrix} -2 & 0 & 4 & 0 & -2 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & -2 & 0 & 4 & 4 & -2 \end{bmatrix}$$

Performing the matrix triple product, we have

$$[k] = (56 \times 10^7)\begin{bmatrix} 2.5 & 1.25 & -2 & -1.5 & -0.5 & 0.25 \\ 1.25 & 4.375 & -1 & -0.75 & -0.25 & -3.625 \\ -2 & -1 & 4 & 0 & -2 & 1 \\ -1.5 & -0.75 & 0 & 1.5 & 1.5 & -0.75 \\ -0.5 & -0.25 & -2 & 1.5 & 2.5 & -1.25 \\ 0.25 & -3.625 & 1 & -0.75 & -1.25 & 4.375 \end{bmatrix}\frac{\text{N}}{\text{m}} \tag{6.2.64}$$

To evaluate the stresses, we use Eq. (6.2.36). Substituting Eqs. (6.2.62) and (6.2.63), along with the given nodal displacements, into Eq. (6.2.36), we obtain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{210 \times 10^9}{1-(0.25)^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$\times \frac{10^2}{2(8)} \begin{bmatrix} -2 & 0 & 4 & 0 & -2 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & -2 & 0 & 4 & 4 & -2 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.05 \\ 0.025 \\ 0.0 \\ 0.0 \\ 0.05 \end{Bmatrix} \times 10^{-3} \tag{6.2.65}$$

Performing the matrix triple product in Eq. (6.2.65), we have

$$\sigma_x = 140 \text{ MPa} \qquad \sigma_y = 35 \text{ MPa} \qquad \tau_{xy} = -105 \text{ MPa} \tag{6.2.66}$$

Finally, the principal stresses and principal angle are obtained by substituting the results from Eqs. (6.2.66) into Eqs. (6.1.2) and (6.1.3) as follows:

$$\begin{aligned} \sigma_1 &= \frac{140+35}{2} + \left[\left(\frac{140-35}{2}\right)^2 + (-105)^2\right]^{1/2} \\ &= 204.89 \text{ MPa} \\ \sigma_2 &= \frac{140+35}{2} - \left[\left(\frac{140-35}{2}\right)^2 + (-105)^2\right]^{1/2} \\ &= -29.89 \text{ MPa} \\ \theta_p &= \frac{1}{2}\tan^{-1}\left[\frac{2(-105)}{140-35}\right] = -31.7^\circ \end{aligned} \tag{6.2.67}$$

■

dimensional heat transfer.

13.1 Derivation of the Basic Differential Equation

One-Dimensional Heat Conduction (without Convection)

We now consider the derivation of the basic differential equation for the one-dimensional problem of heat conduction without convection. The purpose of this derivation is to present a physical insight into the heat-transfer phenomena, which must be understood so that the finite element formulation of the problem can be fully understood. (For additional information on heat transfer, consult texts such as References [1] and [2].) We begin with the control volume shown in Figure 13–2. By conservation of energy, we have

$$E_{\text{in}} + E_{\text{generated}} = \Delta U + E_{\text{out}} \tag{13.1.1}$$

or

$$q_x A\,dt + QA\,dx\,dt = \Delta U + q_{x+dx} A\,dt \tag{13.1.2}$$

where

E_{in} is the energy entering the control volume, in units of joules (J) or kW · h.

ΔU is the change in stored energy, in units of kW · h (kWh).

q_x is the heat conducted (heat flux) into the control volume at surface edge x, in units of kW/m^2.

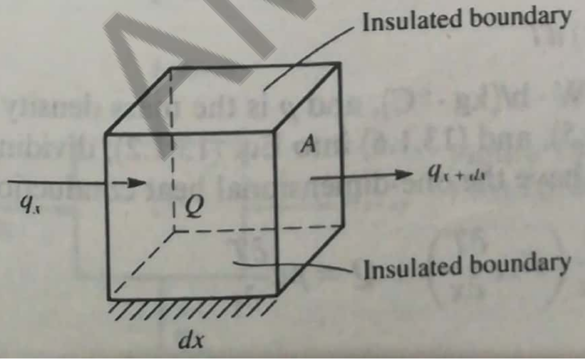

Figure 13–2 Control volume for one-dimensional heat conduction

q_{x+dx} is the heat conducted
$x + dx$.

t is time, in h or s.

Q is the internal heat source (heat generated per unit time per unit volume is positive), in kW/m^3 (a heat sink, heat drawn out of the volume, is negative).

A is the cross-sectional area perpendicular to heat flow q, in m^2.

By Fourier's law of heat conduction,

$$q_x = -K_{xx}\frac{dT}{dx} \tag{13.1.3}$$

where

K_{xx} is the thermal conductivity in the x direction, in kW/(m · °C).

T is the temperature, in °C.

dT/dx is the temperature gradient, in °C/m.

Equation (13.1.3) states that the heat flux in the x direction is proportional to the gradient of temperature in the x direction. The minus sign in Eq. (13.1.3) implies that, by convention, heat flow is positive in the direction opposite the direction of temperature increase. Equation (13.1.3) is analogous to the one-dimensional stress–strain law for the stress analysis problem—that is, to $\sigma_x = E(du/dx)$. Similarly,

$$q_{x+dx} = -K_{xx}\frac{dT}{dx}\bigg|_{x+dx} \tag{13.1.4}$$

where the gradient in Eq. (13.1.4) is evaluated at $x + dx$. By Taylor series expansion for any general function $f(x)$, we have

$$f_{x+dx} = f_x + \frac{df}{dx}dx + \frac{d^2f}{dx^2}\frac{dx^2}{2} + \cdots$$

Therefore, using a two-term Taylor series, Eq. (13.1.4) becomes

$$q_{x+dx} = -\left[K_{xx}\frac{dT}{dx} + \frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right)dx\right] \tag{13.1.5}$$

The change in stored energy can be expressed by

$$\begin{aligned}\Delta U &= \text{specific heat} \times \text{mass} \times \text{change in temperature}\\ &= c(\rho A\,dx)\,dT\end{aligned} \tag{13.1.6}$$

where c is the specific heat in kW · h/(kg · °C), and ρ is the mass density in kg/m^3. On substituting Eqs. (13.1.3), (13.1.5), and (13.1.6) into Eq. (13.1.2), dividing Eq. (13.1.2) by $A\,dx\,dt$, and simplifying, we have the one-dimensional heat conduction equation as

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial T}{\partial x}\right) + Q = \rho c\frac{\partial T}{\partial t} \tag{13.1.7}$$

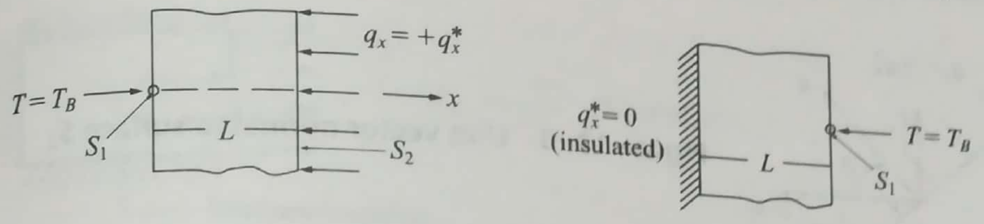

Figure 13–3 Examples of boundary conditions in one-dimensional heat conduction

For steady state, any differentiation with respect to time is equal to zero, so Eq. (13.1.7) becomes

$$\frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right) + Q = 0 \tag{13.1.8}$$

For constant thermal conductivity and steady state, Eq. (13.1.7) becomes

$$K_{xx}\frac{d^2T}{dx^2} + Q = 0 \tag{13.1.9}$$

The boundary conditions are of the form

$$T = T_B \qquad \text{on } S_1 \tag{13.1.10}$$

where T_B represents a known boundary temperature and S_1 is a surface where the temperature is known, and

$$q_x^* = -K_{xx}\frac{dT}{dx} = \text{constant} \qquad \text{on } S_2 \tag{13.1.11}$$

where S_2 is a surface where the prescribed heat flux q_x^* or temperature gradient is known. On an insulated boundary, $q_x^* = 0$. These different boundary conditions are shown in Figure 13–3, where by sign convention, positive q_x^* occurs when heat is flowing into the body, and negative q_x^* when heat is flowing out of the body.

Two-Dimensional Heat Conduction (Without Convection)

Consider the two-dimensional heat conduction problem in Figure 13–4. In a manner similar to the one-dimensional case, for steady-state conditions, we can show that for material properties coinciding with the global x and y directions,

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{yy}\frac{\partial T}{\partial y}\right) + Q = 0 \tag{13.1.12}$$

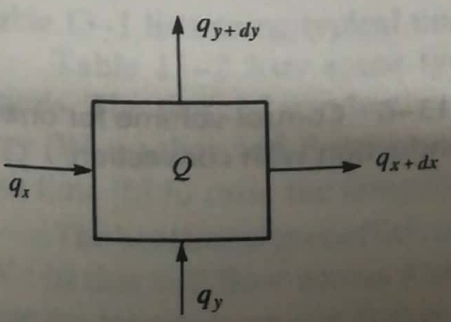

Figure 13–4 Control volume for two-dimensional heat conduction

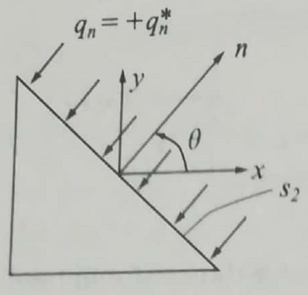

Figure 13–5 Unit vector normal to surface S_2

with boundary conditions

$$T = T_B \qquad \text{on } S_1 \tag{13.1.13}$$

$$q_n = q_n^* = K_{xx}\frac{\partial T}{\partial x}C_x + K_{yy}\frac{\partial T}{\partial y}C_y = \text{constant} \qquad \text{on } S_2 \tag{13.1.14}$$

where C_x and C_y are the direction cosines of the unit vector $\boldsymbol{n}$ normal to the surface S_2 shown in Figure 13–5. Again, q_n^* is by sign convention, positive if heat is flowing into the edge of the body.

13.2 Heat Transfer with Convection

For a conducting solid in contact with a fluid, there will be a heat transfer taking place between the fluid and solid surface when a temperature difference occurs.

The fluid will be in motion either through external pumping action (**forced convection**) or through the buoyancy forces created within the fluid by the temperature differences within it (**natural** or **free convection**).

We will now consider the derivation of the basic differential equation for one-dimensional heat conduction with convection. Again we assume the temperature change is much greater in the x direction than in the y and z directions. Figure 13–6 shows the control volume used in the derivation. Again, by Eq. (13.1.1) for conservation of energy, we have

$$q_x A\,dt + QA\,dx\,dt = c(\rho A\,dx)\,dT + q_{x+dx}A\,dt + q_h P\,dx\,dt \tag{13.2.1}$$

In Eq. (13.2.1), all terms have the same meaning as in Section 13.1, except the heat flow by convective heat transfer is given by Newton's law of cooling

$$q_h = h(T - T_\infty) \tag{13.2.2}$$

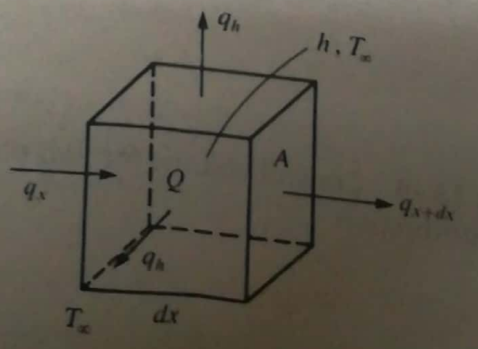

Figure 13–6 Control volume for one-dimensional heat conduction with convection

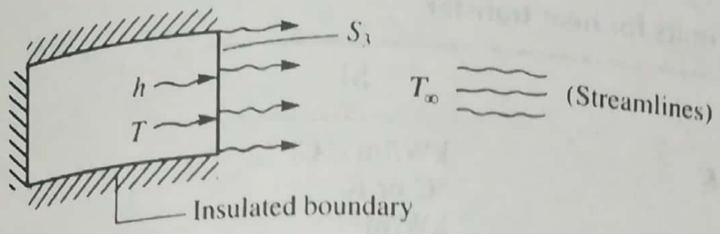

Figure 13–7 Model illustrating convective heat transfer (arrows on surface S_3 indicate heat transfer by convection)

where

h is the heat-transfer or convection coefficient, in kW/(m² . °C).

T is the temperature of the solid surface at the solid/fluid interface.

T_∞ is the temperature of the fluid (here the free-stream fluid temperature).

P in Eq. (13.2.1) denotes the perimeter around the constant cross-sectional area A.

Again, using Eqs. (13.1.3) through (13.1.6) and (13.2.2) in Eq. (13.2.1), dividing by $A\,dx\,dt$, and simplifying, we obtain the differential equation for one-dimensional heat conduction with convection as

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial T}{\partial x}\right)+Q=\rho c\frac{\partial T}{\partial t}+\frac{hP}{A}(T-T_\infty) \tag{13.2.3}$$

with possible boundary conditions on (1) temperature, given by Eq. (13.1.10), and/or (2) temperature gradient, given by Eq. (13.1.11), and/or (3) loss of heat by convection from the ends of the one-dimensional body, as shown in Figure 13–7. Equating the heat flow in the solid wall to the heat flow in the fluid at the solid/fluid interface, we have

$$-K_{xx}\frac{dT}{dx}=h(T-T_\infty) \qquad \text{on } S_3 \tag{13.2.4}$$

as a boundary condition for the problem of heat conduction with convection.

13.3 Typical Units; Thermal Conductivities, *K*; and Heat-Transfer Coefficients, *h*

Table 13–1 lists some typical units used for the heat-transfer problem.

Table 13–2 lists some typical thermal conductivities of various solids and liquids. The thermal conductivity K, in W/(m · °C), measures the amount of heat energy (W · h) that will flow through a unit length (ft or m) of a given substance in a unit time (h) to raise the temperature one degree (°C).

The heat transfer coefficient h, in W/(m² · °C), measures the amount of heat energy (W · h) that will flow across a unit area (m²) of a given substance in a unit time (h) to raise the temperature one degree (C).

Table 13–1 Typical units for heat transfer

Variable	SI
Thermal conductivity, K	kW/(m · °C)
Temperature, T	°C or K
Internal heat source, Q	kW/m^3
Heat flux, q	kW/m^2
Heat flow, $\bar{q}$	kW
Convection coefficient, h	kW/(m^2 · °C)
Energy, E	kW · h
Specific heat, c	(kW · h)/(kg · °C)
Mass density, ρ	kg/m^3

Table 13–2 Typical thermal conductivities of some solids and fluids

Material	K [W/(m · °C)]
Solids	
Aluminum, 0 °C (32 °F)	202
Steel (1% carbon), 0 °C	35
Fiberglass, 20 °C (68 °F)	0.035
Concrete, 0 °C	0.81–1.40
Earth, coarse gravelly, 20 °C	0.520
Wood, oak, radial direction, 20 °C	0.17
Fluids	
Engine oil, 20 °C	0.145
Dry air, atmospheric pressure, 20 °C	0.0243

Natural or free convection occurs when, for instance, a heated plate is exposed to ambient room air without an external source of motion. This movement of the air, experienced as a result of the density gradients near the plate, is called *natural* or *free convection*. **Forced convection** is experienced, for instance, in the case of a fan blowing air over a plate.

13.4 One-Dimensional Finite Element Formulation Using a Variational Method

The temperature distribution influences the amount of heat moving into or out of a body and also influences the stresses in a body. Thermal stresses occur in all bodies that experience a temperature gradient from some equilibrium state but are not free to expand in all directions. To evaluate thermal stresses, we need to know the temperature distribution in the body. The finite element method is a realistic method for predicting quantities such as temperature distribution and thermal stresses in a body. In this section, we formulate the one-dimensional heat-transfer equations using a variational method. Examples are included to illustrate the solution of this type of problem.

10.3 TORSION

Consider a prismatic rod of arbitrary cross-sectional shape, which is subjected to a twisting moment M as shown in Fig. 10.13. The problem is to determine shearing stresses τ_{xz} and τ_{yz} (Fig. 10.14) and the angle of twist per unit length, α. It can be shown that the solution of such problems, with simply connected cross sections, reduces to solving the two-dimensional equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + 2 = 0 \qquad \text{in } A \tag{10.73}$$

$$\theta = 0 \qquad \text{on } S \tag{10.74}$$

where A is interior and S is the boundary of the cross section. Again, we note that Eq. 10.73 is a special case of Helmholtz's equations given in Eq. 10.1. In Eq. 10.74, θ is called the **stress function**, since once θ is known, then shearing stresses are obtained as

$$\tau_{xz} = G\alpha\frac{\partial \theta}{\partial y} \qquad \tau_{yz} = -G\alpha\frac{\partial \theta}{\partial x} \tag{10.75}$$

with α determined from

$$M = 2G\alpha \int_A \int \theta \, dA \tag{10.76}$$

where G is the shear modulus of the material. The finite element method for solving Eqs. 10.73 and 10.74 will now be given.

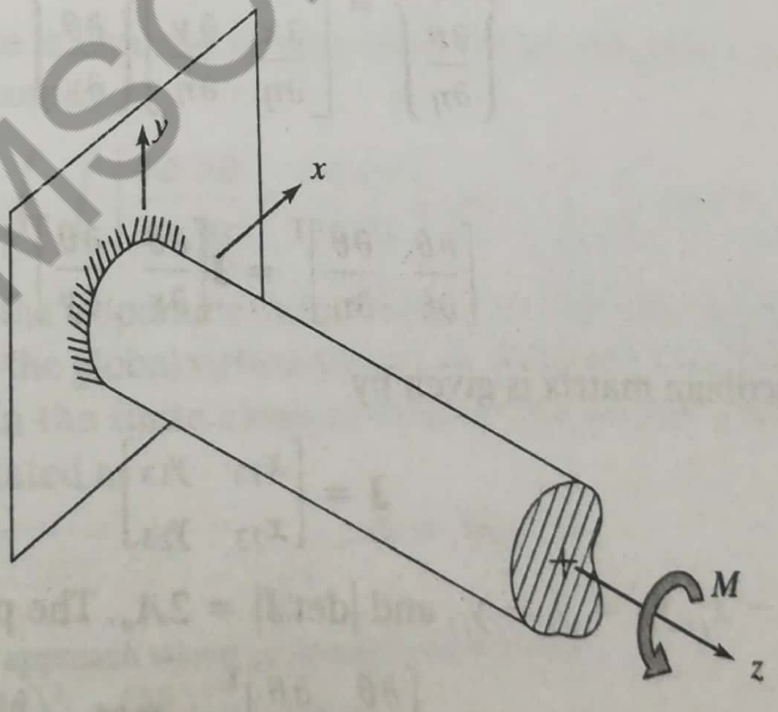

FIGURE 10.13 A rod of arbitrary cross section subjected to a torque.

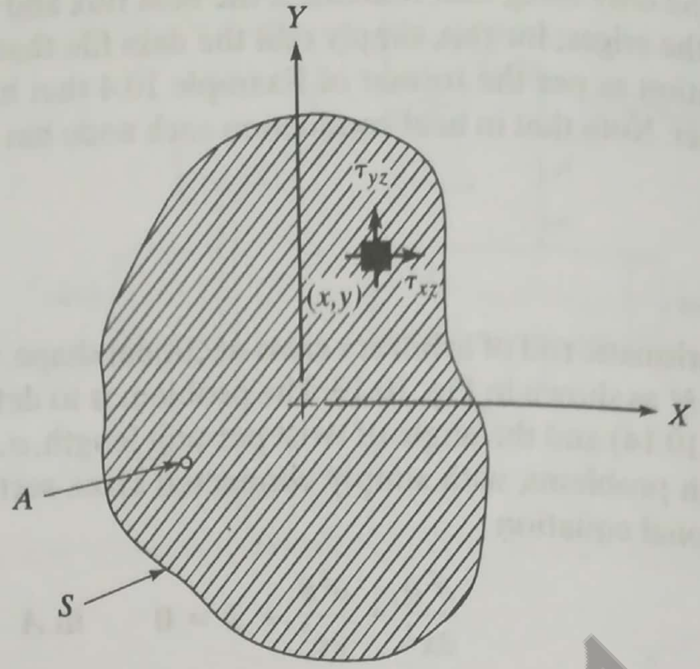

FIGURE 10.14 Shearing stresses in torsion.

Triangular Element

The stress function θ within a triangular element is interpolated as

$$\theta = \mathbf{N}\boldsymbol{\theta}^e \tag{10.77}$$

where $\mathbf{N} = [\xi, \quad \eta, \quad 1 - \xi - \eta]$ are the usual shape functions, and $\boldsymbol{\theta}^e = [\theta_1, \quad \theta_2, \quad \theta_3]^{\mathrm{T}}$ are the nodal values of θ. Furthermore, we have the isoparametric relations (Chapter 5)

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$\begin{Bmatrix} \dfrac{\partial \theta}{\partial \xi} \\ \dfrac{\partial \theta}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial \theta}{\partial y} \end{Bmatrix} \tag{10.78}$$

or

$$\left[\frac{\partial \theta}{\partial \xi} \quad \frac{\partial \theta}{\partial \eta}\right]^{\mathrm{T}} = \mathbf{J}\left[\frac{\partial \theta}{\partial x} \quad \frac{\partial \theta}{\partial y}\right]^{\mathrm{T}}$$

where the Jacobian matrix is given by

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \tag{10.79}$$

with $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$, and $|\det \mathbf{J}| = 2A_e$. The preceding equations yield

$$\left[\frac{\partial \theta}{\partial x} \quad \frac{\partial \theta}{\partial y}\right]^{\mathrm{T}} = \mathbf{B}\boldsymbol{\theta}^e \tag{10.80a}$$

or

$$[-\tau_{yz} \quad \tau_{xz}]^{\mathrm{T}} = G\alpha\, \mathbf{B}\boldsymbol{\theta}^e \tag{10.80b}$$

where

$$\mathbf{B} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \tag{10.81}$$

The fact that identical relations also apply to the heat-conduction problem in the previous section show the similarity of treating all field problems by the finite element method.

Galerkin Approach†

The problem in Eqs. 10.73–10.74 will now be solved using Galerkin's approach. The problem is to find the approximate solution θ such that

$$\int_A\int \phi\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + 2\right) dA = 0 \tag{10.82}$$

for every $\phi(x, y)$ constructed from the same basis as θ and satisfying $\phi = 0$ on S. Since

$$\phi\frac{\partial^2\theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\phi\frac{\partial\theta}{\partial x}\right) - \frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial x}$$

we have

$$\int_A\int \left[\frac{\partial}{\partial x}\left(\phi\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi\frac{\partial\theta}{\partial y}\right)\right] dA - \int_A\int\left(\frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\theta}{\partial y}\right) dA$$
$$+ \int_A\int 2\phi\, dA = 0 \tag{10.83}$$

Using the divergence theorem, the first term in the previous expression reduces to

$$\int_A\int \left[\frac{\partial}{\partial x}\left(\phi\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi\frac{\partial\theta}{\partial y}\right)\right] dA = \int_S \phi\left(\frac{\partial\theta}{\partial x}n_x + \frac{\partial\phi}{\partial y}n_y\right) dS = 0 \tag{10.84}$$

where the right side is equated to zero owing to the boundary condition $\phi = 0$ on S. Equation 10.83 becomes

$$\int_A\int \left[\frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\theta}{\partial y}\right] dA - \int_A\int 2\phi\, dA = 0 \tag{10.85}$$

Now, we introduce the isoparametric relations $\theta = \mathbf{N}\boldsymbol{\theta}^e$, etc., as given in Eqs. 10.77–10.81. Further, we denote the global virtual-stress function vector as $\boldsymbol{\Psi}$ whose dimension equals number of nodes in the finite element model. The virtual-stress function within each element is interpolated as

$$\phi = \mathbf{N}\boldsymbol{\psi} \tag{10.86}$$

---

† The functional approach would be based on minimizing

$$\pi = G\alpha^2 \int_A\int \left\{\frac{1}{2}\left[\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2\right] - 2\theta\right\} dA$$

Moreover, we have

$$\begin{bmatrix} \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} \end{bmatrix}^{\mathrm{T}} = \mathbf{B}\boldsymbol{\psi} \tag{10.87}$$

Substituting these into Eq. 10.85 and noting that

$$\left(\frac{\partial \phi}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \theta}{\partial y}\right) = \left(\frac{\partial \phi}{\partial x} \quad \frac{\partial \phi}{\partial y}\right)\begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial \theta}{\partial y} \end{Bmatrix}$$

we get

$$\sum_e \boldsymbol{\psi}^{\mathrm{T}}\mathbf{k}\boldsymbol{\theta}^e - \sum_e \boldsymbol{\psi}^{\mathrm{T}}\mathbf{f} = 0 \tag{10.88}$$

where

$$\mathbf{k} = A_e\mathbf{B}^{\mathrm{T}}\mathbf{B} \tag{10.89}$$

$$\mathbf{f} = \frac{2A_e}{3}[1, \quad 1, \quad 1]^{\mathrm{T}} \tag{10.90}$$

Equation 10.88 can be written as

$$\boldsymbol{\Psi}^{\mathrm{T}}(\mathbf{K}\boldsymbol{\Theta} - \mathbf{F}) = 0 \tag{10.91}$$

which should hold for all $\boldsymbol{\Psi}$ satisfying $\Psi_i = 0$ at nodes i on the boundary. We thus have

$$\mathbf{K}\boldsymbol{\Theta} = \mathbf{F} \tag{10.92}$$

where rows and columns of **K** and **F** that correspond to boundary nodes have been deleted.

Example 10.5

Consider the shaft with a rectangular cross section shown in Fig. E10.5a. Determine, in terms of M and G, the angle of twist per unit length.

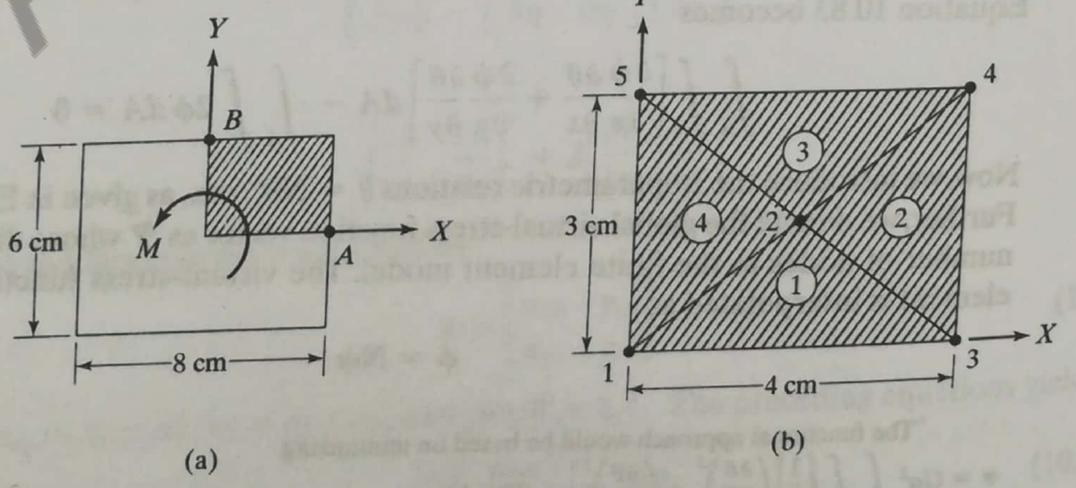

FIGURE E10.5

Solution A finite element model of a quadrant of this cross section is shown in Fig. E10.5b. We define the element connectivity as in the following table:

Element	1	2	3
1	1	3	2
2	3	4	2
3	4	5	2
4	5	1	2

Using the relations

$$\mathbf{B} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$$

and

$$\mathbf{k} = A_e \mathbf{B}^{\mathrm{T}} \mathbf{B}$$

we get

$$\mathbf{B}^{(1)} = \frac{1}{6}\begin{bmatrix} -1.5 & 1.5 & 0 \\ -2 & -2 & 4 \end{bmatrix} \qquad \mathbf{k}^{(1)} = \frac{1}{2}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.042 & 0.292 & -1.333 \\ & 1.042 & -1.333 \\ \text{Symmetric} & & 2.667 \end{bmatrix} \end{array}$$

Similarly,

$$\mathbf{k}^{(2)} = \frac{1}{2}\begin{array}{c} \begin{array}{ccc} 3 & 4 & 2 \end{array} \\ \begin{bmatrix} 1.042 & -0.292 & -0.75 \\ & 1.042 & -0.75 \\ \text{Symmetric} & & 1.5 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(3)} = \frac{1}{2}\begin{array}{c} \begin{array}{ccc} 4 & 5 & 2 \end{array} \\ \begin{bmatrix} 1.042 & 0.292 & -1.333 \\ & 1.042 & -1.333 \\ \text{Symmetric} & & 2.667 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(4)} = \frac{1}{2}\begin{array}{c} \begin{array}{ccc} 5 & 1 & 2 \end{array} \\ \begin{bmatrix} 1.042 & -0.292 & -0.75 \\ & 1.042 & -0.75 \\ \text{Symmetric} & & 1.5 \end{bmatrix} \end{array}$$

Similarly, the element load vector $\mathbf{f} = (2A_e/3)[1, \ 1, \ 1,]^{\mathrm{T}}$ for each element is

$$\mathbf{f}^{(i)} = \begin{Bmatrix} 2 \\ 2 \\ 2 \end{Bmatrix} \qquad i = 1, \ 2, \ 3, \ 4$$

We can now assemble **K** and **F**. Since the boundary conditions are

$$\Theta_3 = \Theta_4 = \Theta_5 = 0$$

we are interested only in degrees of freedom 1 and 2. Thus, the finite element equations are

$$\frac{1}{2}\begin{bmatrix} 2.084 & -2.083 \\ -2.083 & 8.334 \end{bmatrix}\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 8 \end{Bmatrix}$$

The solution is

$$[\Theta_1, \quad \Theta_2] = [7.676, \quad 3.838]$$

Consider the equation

$$M = 2G\alpha \int_A\int \theta\, dA$$

Using $\theta = \mathbf{N}\boldsymbol{\theta}^e$, and noting that $\int_e \mathbf{N}\, dA = (A_e/3)[1, \quad 1, \quad 1]$, we get

$$M = 2G\alpha\left[\sum_e \frac{A_e}{3}(\theta_1^e + \theta_2^e + \theta_3^e)\right] \times 4$$

This multiplication by 4 is because the finite element model represents only one-quarter of the rectangular cross section. Thus, we get the angle of twist per unit length to be

$$\alpha = 0.004\frac{M}{G}$$

For given values of M and G, we can thus determine the value of α. Further, the shearing stresses in each element can be calculated from Eq. 10.80b. ∎

10.5 Higher-Order Shape Functions

In general, higher-order element shape functions can be developed by adding additional nodes to the sides of the linear element. These elements result in higher-order strain variations within each element, and convergence to the exact solution thus occurs at a faster rate using fewer elements. (However, a trade-off exists because a more complicated element takes up so much computation time that even with few elements in the model, the computation time can become larger than for the simple linear element model.) Another advantage of the use of higher-order elements is that curved boundaries of irregularly shaped bodies can be approximated more closely than by the use of simple straight-sided linear elements.

Linear Strain Bar

To illustrate the concept of higher-order elements, we will begin with the three-noded linear strain quadratic displacement (and quadratic shape functions) shown in Figure 10–13. Figure 10–13 shows a quadratic isoparametric bar element (also called a linear strain bar) with three coordinates of nodes, x_1, x_2, and x_3, in the global coordinates.

Example 10.6

For the three-noded linear strain bar isoparametric element shown in Figure 10–13, determine (a) the shape functions, N_1, N_2, and N_3, and (b) the strain–displacement matrix [B]. Assume the general axial displacement function to be a quadratic taken as $u = a_1 + a_2 s + a_3 s^2$.

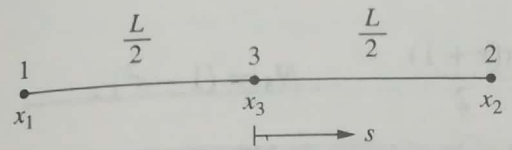

Figure 10–13 Three-noded linear strain bar element

SOLUTION:

(a) As we are formulating shape functions for an isoparametric element, we assume the following axial coordinate function for x as

$$x = a_1 + a_2 s + a_3 s^2 \tag{10.5.1}$$

Evaluating the a_i's in terms of the nodal coordinates, we obtain

$$\begin{aligned} x(-1) &= a_1 - a_2 + a_3 = x_1 \quad &\text{or} \quad x_1 &= a_1 - a_2 + a_3 \\ x(0) &= a_1 = x_3 \quad &\text{or} \quad x_3 &= a_1 \\ x(1) &= a_1 + a_2 + a_3 = x_2 \quad &\text{or} \quad x_2 &= a_1 - a_2 + a_3 \end{aligned} \tag{10.5.2}$$

Substituting $a_1 = x_3$ from the second of Eqs. (10.5.2), into the first and third of Eqs. (10.5.2), we obtain a_2 and a_3 as follows:

$$\begin{aligned} x_1 &= x_3 - a_2 + a_3 \\ x_2 &= x_3 + a_2 + a_3 \end{aligned} \tag{10.5.3}$$

Adding Eqs. (10.5.3) together and solving for a_3 gives the following:

$$a_3 = (x_1 + x_2 - 2x_3)/2 \tag{10.5.4}$$

$$\begin{aligned} x_1 &= x_3 - a_2 + ((x_1 + x_2 - 2x_3)/2) \\ a_2 &= x_3 - x_1 + ((x_1 + x_2 - 2x_3)/2) = (x_2 - x_1)/2 \end{aligned} \tag{10.5.5}$$

Substituting the values for a_1, a_2, and a_3 from Eqs. (10.5.2), (10.5.4), and (10.5.5) into the general equation for x given by Eq. (10.5.1), we obtain

$$x = a_1 + a_2 s + a_3 s^2 = x_3 + \frac{x_2 - x_1}{2} s + \frac{x_1 + x_2 - 2x_3}{2} s^2 \tag{10.5.6}$$

Combining like terms in x_1, x_2, and x_3, from Eq. (10.5.6), we obtain the final form of x as:

$$x = \left(\frac{s(s-1)}{2}\right) x_1 + \frac{s(s+1)}{2} x_2 + (1 - s^2) x_3 \tag{10.5.7}$$

Recall that the function x can be expressed in terms of the shape function matrix and the axial coordinates, we have from Eq. (10.5.7)

$$\{x\} = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \left[\left(\frac{s(s-1)}{2}\right) \quad \frac{s(s+1)}{2} \quad (1-s^2)\right]\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{10.5.8}$$

Therefore the shape functions are

$$N_1 = \frac{s(s-1)}{2} \qquad N_2 = \frac{s(s+1)}{2} \qquad N_3 = (1-s^2) \tag{10.5.9}$$

(b) We now determine the strain–displacement matrix $[B]$ as follows:
From our basic definition of axial strain we have

$$\{\varepsilon_x\} = \frac{du}{dx} = \frac{du}{ds}\frac{ds}{dx} = [B]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{10.5.10}$$

Using an isoparametric formulation means the displacement function is of the same form as the axial coordinate function. Therefore, using Eq. (10.5.6), we have

$$u = u_3 + \frac{u_2}{2}s - \frac{u_1}{2}s + \frac{u_1}{2}s^2 + \frac{u_2}{2}s^2 - \frac{2u_3}{2}s^2 \tag{10.5.11}$$

Differentiating u with respect to s, we obtain

$$\frac{du}{ds} = \frac{u_2}{2} - \frac{u_1}{2} + u_1 s + u_2 s - 2u_3 s = \left(s - \frac{1}{2}\right)u_1 + \left(s + \frac{1}{2}\right)u_2 + (-2s)u_3 \tag{10.5.12}$$

We have previously proven that $dx/ds = L/2 = |[J]|$ (see Eq. (10.1.9b). This relationship holds for the higher-order one-dimensional elements as well as for the two-noded constant strain bar element as long as node 3 is at the geometry center of the bar. Using this relationship and Eq. (10.5.12) in Eq. (10.5.10), we obtain

$$\begin{aligned} \frac{du}{dx} = \frac{du}{ds}\frac{ds}{dx} &= \left(\frac{2}{L}\right)\left(\left(s - \frac{1}{2}\right)u_1 + \left(s + \frac{1}{2}\right)u_2 + (-2s)u_3\right) \\ &= \left(\frac{2s-1}{L}\right)u_1 + \left(\frac{2s+1}{L}\right)u_2 + \left(\frac{-4s}{L}\right)u_3 \end{aligned} \tag{10.5.13}$$

In matrix form, Eq. (10.5.13) becomes

$$\frac{du}{dx} = \left[\frac{2s-1}{L} \quad \frac{2s+1}{L} \quad \frac{-4s}{L}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{10.5.14}$$

As Eq. (10.5.14) represents the axial strain, we have

$$\{\varepsilon_x\} = \frac{du}{dx} = \begin{bmatrix} \dfrac{2s-1}{L} & \dfrac{2s+1}{L} & \dfrac{-4s}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [B] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{10.5.15}$$

Therefore the gradient matrix $[B]$ is given by

$$[B] = \begin{bmatrix} \dfrac{2s-1}{L} & \dfrac{2s+1}{L} & \dfrac{-4s}{L} \end{bmatrix} \tag{10.5.16}$$

■

Example 10.7

For the three-noded bar element shown previously in Figure 10–13, evaluate the stiffness matrix analytically. Use the $[B]$ from Example 10.6.

SOLUTION:

From Example 10.6, Eq. (10.5.16), we have

$$[B] = \begin{bmatrix} \dfrac{2s-1}{L} & \dfrac{2s+1}{L} & \dfrac{-4s}{L} \end{bmatrix}, \quad [J] = \frac{L}{2} \quad \text{(see Eq. (10.1.9b))} \tag{10.5.17}$$

Substituting the expression for $[B]$ into Eq. (10.1.15) for the stiffness matrix, we obtain

$$[k] = \frac{L}{2}\int_{-1}^{1} [B]^T E[B]A\,ds = \frac{AEL}{2}\int_{-1}^{1} \begin{bmatrix} \dfrac{(2s-1)^2}{L^2} & \dfrac{(2s-1)(2s+1)}{L^2} & \dfrac{(2s-1)(-4s)}{L^2} \\ \dfrac{(2s+1)(2s-1)}{L^2} & \dfrac{(2s+1)^2}{L^2} & \dfrac{(2s+1)(-4s)}{L^2} \\ \dfrac{(-4s)(2s-1)}{L^2} & \dfrac{(-4s)(2s+1)}{L^2} & \dfrac{(-4s)^2}{L^2} \end{bmatrix} ds \tag{10.5.18}$$

Simplifying the terms in Eq. (10.5.18) for easier integration, we have

$$[k] = \frac{AE}{2L}\int_{-1}^{1} \begin{bmatrix} 4s^2-4s+1 & 4s^2-1 & -8s^2+4s \\ 4s^2-1 & 4s^2+4s+1 & -8s^2-4s \\ -8s^2+4s & -8s^2-4s & 16s^2 \end{bmatrix} ds \tag{10.5.19}$$

Upon explicit integration of Eq. (10.5.19), we obtain

$$[k] = \frac{AE}{2L} \left. \begin{bmatrix} \dfrac{4}{3}s^3-2s^2+s & \dfrac{4}{3}s^3-s & -\dfrac{8}{3}s^3+2s^2 \\ \dfrac{4}{3}s^3-s & \dfrac{4}{3}s^3+2s^2+s & -\dfrac{8}{3}s^3-2s^2 \\ -\dfrac{8}{3}s^3+2s^2 & -\dfrac{8}{3}s^3-2s^2 & \dfrac{16}{3}s^3 \end{bmatrix} \right|_{-1}^{1} \tag{10.5.20}$$

Evaluating Eq. (10.5.20) at the limits 1 and −1, we have

$$[k] = \frac{AE}{2L}\begin{bmatrix} \frac{4}{3}-2+1 & \frac{4}{3}-1 & -\frac{8}{3}+2 \\ \frac{4}{3}-1 & \frac{4}{3}+2+1 & -\frac{8}{3}-2 \\ -\frac{8}{3}+2 & -\frac{8}{3}-2 & \frac{16}{3} \end{bmatrix} - \begin{bmatrix} -\frac{4}{3}-2-1 & -\frac{4}{3}+1 & \frac{8}{3}+2 \\ -\frac{4}{3}+1 & -\frac{4}{3}+2-1 & \frac{8}{3}-2 \\ \frac{8}{3}+2 & \frac{8}{3}-2 & -\frac{16}{3} \end{bmatrix} \tag{10.5.21}$$

Simplifying Eq. (10.5.21), we obtain the final stiffness matrix as

$$[k] = \frac{AE}{2L}\begin{bmatrix} 4.67 & 0.667 & -5.33 \\ 0.667 & 4.67 & -5.33 \\ -5.33 & -5.33 & 10.67 \end{bmatrix} \tag{10.5.22}$$

■

Example 10.8

We now illustrate how to evaluate the stiffness matrix for the three-noded bar element shown in Figure 10–14 by using two-point Gaussian quadrature. We can then compare this result to that obtained by the explicit integration performed in Example 10.7.

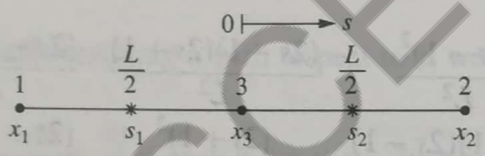

Figure 10–14 Three-noded bar with two Gauss points

SOLUTION:

Starting with Eq. (10.5.18), we have for the stiffness matrix

$$[k] = \frac{L}{2}\int_{-1}^{1}[B]^T E[B]A\,ds = \frac{AEL}{2}\int_{-1}^{1}\begin{bmatrix} \frac{(2s-1)^2}{L^2} & \frac{(2s-1)(2s+1)}{L^2} & \frac{(2s-1)(-4s)}{L^2} \\ \frac{(2s+1)(2s-1)}{L^2} & \frac{(2s+1)^2}{L^2} & \frac{(2s+1)(-4s)}{L^2} \\ \frac{(-4s)(2s-1)}{L^2} & \frac{(-4s)(2s+1)}{L^2} & \frac{(-4s)^2}{L^2} \end{bmatrix} ds \tag{10.5.23}$$

Using two-point Gaussian quadrature, we evaluate the stiffness matrix at the two points shown in Figure 10–14 (also based on Table 10–2):

$$s_1 = -0.57735, \quad s_2 = 0.57735 \tag{10.5.24}$$

with weights given by

$$W_1 = 1, \quad W_2 = 1 \tag{10.5.25}$$

We then evaluate each term in the integrand of Eq. (10.5.23) at each Gauss point and multiply each term by its weight (here each weight is 1). We then add those Gauss point evaluations together to obtain the final term for each element of the stiffness matrix. For two-point evaluation, there will be two terms added together to obtain each element of the stiffness matrix. We proceed to evaluate the stiffness matrix term by term as follows:

The one–one element:

$$\sum_{i=1}^{2} W_i(2s_i - 1)^2 = (1)[2(-0.57735) - 1]^2 + (1)[2(0.57735) - 1]^2 = 4.6667$$

The one–two element:

$$\sum_{i=1}^{2} W_i(2s_i - 1)(2s_i + 1) = (1)[(2)(-0.57735) - 1][(2)(-0.57735) + 1]$$
$$+ (1)[(2)(0.57735) - 1][(2)(0.57735) + 1] = 0.6667$$

The one–three element:

$$\sum_{i=1}^{2} W_i(-4s_i(2s_i - 1)) = (1)(-4)(-0.57735)[(2)(-0.57735) - 1]$$
$$+ (1)(-4)(0.57735)[(2)(0.57735) - 1] = -5.3333$$

The two–two element:

$$\sum_{i=1}^{2} W_i(2s_i + 1)^2 = (1)[(2)(-0.57735) + 1]^2 + (1)[(2)(0.57735) + 1]^2 = 4.6667$$

The two–three element:

$$\sum_{i=1}^{2} W_i[-4s_i(2s_i + 1)] = (1)(-4)(-0.57735)[(2)(-0.57735) + 1]$$
$$+ (1)(-4)(0.57735)[(2)(0.57735) + 1] = -5.3333$$

The three–three element:

$$\sum_{i=1}^{2} W_i(16{s_i}^2) = (1)(16)(-0.57735)^2 + (1)(16)(0.57735)^2 = 10.6667$$

By symmetry, the two–one element equals the one–two element, etc. Therefore, from the evaluations of the terms above, the final stiffness matrix is

$$[k] = \frac{AE}{2L}\begin{bmatrix} 4.67 & 0.667 & -5.33 \\ 0.667 & 4.67 & -5.33 \\ -5.33 & -5.33 & 10.67 \end{bmatrix} \tag{10.5.26}$$

Equation (10.5.26) is identical to Eq. (10.5.22) obtained analytically by direct explicit integration of each term in the stiffness matrix. ■

To further illustrate the concept of higher-order elements, we will consider the quadratic and cubic element shape functions as described in Reference [3].

Quadratic Rectangle (Q8 and Q9)

Figure 10–15 shows a quadratic isoparametric element with four corner nodes and four additional midside nodes. This eight-noded element is often called a "Q8" element.

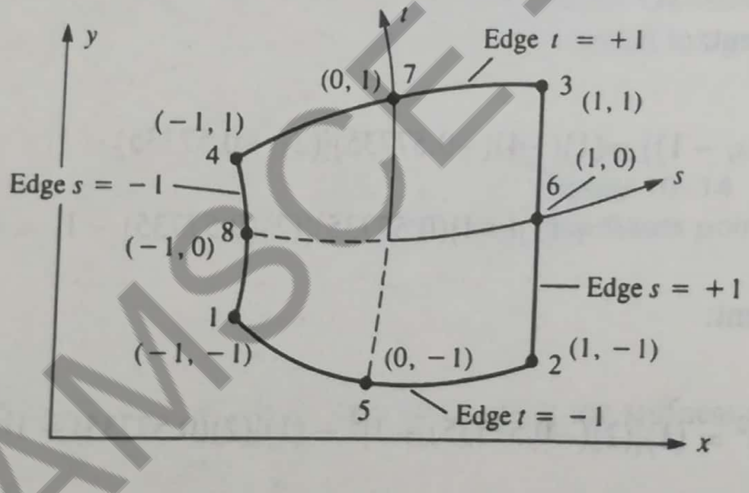

Figure 10–15 Quadratic (Q8) isoparametric element

The shape functions of the quadratic element are based on the incomplete cubic polynomial such that coordinates x and y are

$$x = a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2 + a_6 t^2 + a_7 s^2 t + a_8 st^2$$
$$y = a_9 + a_{10} s + a_{11} t + a_{12} st + a_{13} s^2 + a_{14} t^2 + a_{15} s^2 t + a_{16} st^2 \tag{10.5.27}$$

These functions have been chosen so that the number of generalized degrees of freedom (2 per node times 8 nodes equals 16) are identical to the total number of a_i's. The literature also refers to this eight-noded element as a "serendipity" element as it is based on an incomplete cubic, but it yields good results in such cases as beam bending. We are also reminded that because we are considering an isoparametric formulation, displacements u and v are of identical form as x and y, respectively, in Eq. (10.5.27).

To describe the shape functions, two forms are required—one for corner nodes and one for midside nodes, as given in Reference [3]. For the corner nodes $(i = 1, 2, 3, 4)$,

$$\begin{aligned} N_1 &= \frac{1}{4}(1-s)(1-t)(-s-t-1) \\ N_2 &= \frac{1}{4}(1+s)(1-t)(s-t-1) \\ N_3 &= \frac{1}{4}(1+s)(1+t)(s+t-1) \\ N_4 &= \frac{1}{4}(1-s)(1+t)(-s+t-1) \end{aligned} \tag{10.5.28}$$

or, in compact index notation, we express Eqs. (10.5.28) as

$$N_i = \frac{1}{4}(1+ss_i)(1+tt_i)(ss_i+tt_i-1) \tag{10.5.29}$$

where i is the number of the shape function and

$$\begin{aligned} s_i &= -1, 1, 1, -1 \qquad (i = 1, 2, 3, 4) \\ t_i &= -1, -1, 1, 1 \qquad (i = 1, 2, 3, 4) \end{aligned} \tag{10.5.30}$$

For the midside nodes $(i = 5, 6, 7, 8)$,

$$\begin{aligned} N_5 &= \frac{1}{2}(1-t)(1+s)(1-s) \\ N_6 &= \frac{1}{2}(1+s)(1+t)(1-t) \\ N_7 &= \frac{1}{2}(1+t)(1+s)(1-s) \\ N_8 &= \frac{1}{2}(1-s)(1+t)(1-t) \end{aligned} \tag{10.5.31}$$

or, in index notation,

$$\begin{aligned} N_i &= \frac{1}{2}(1-s^2)(1+tt_i) \qquad t_i = -1, 1 \qquad (i = 5, 7) \\ N_i &= \frac{1}{2}(1+ss_i)(1-t^2) \qquad s_i = 1, -1 \qquad (i = 6, 8) \end{aligned} \tag{10.5.32}$$

We can observe from Eqs. (10.5.28) and (10.5.31) that an edge (and displacement) can vary with s^2 (along t constant) or with t^2 (along s constant). Furthermore, $N_i = 1$ at node i and $N_i = 0$ at the other nodes, as it must be according to our usual definition of shape functions.

The displacement functions are given by

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_8 \end{Bmatrix} \tag{10.5.33}$$

and the strain matrix is now

$$\{\varepsilon\} = [D'][N]\{d\}$$

with

$$[B] = [D'][N]$$

We can develop the matrix $[B]$ using Eq. (10.2.17) with $[D']$ from Eq. (10.2.16) and with $[N]$ now the 2×16 matrix given in Eq. (10.5.33), where the N's are defined in explicit form by Eq. (10.5.28) and (10.5.31).

To evaluate the matrix $[B]$ and the matrix $[k]$ for the eight-noded quadratic isoparametric element, we now use the nine-point Gauss rule (often described as a 3×3 rule). Results using 2×2 and 3×3 rules have shown significant differences, and the 3×3 rule is recommended by Bathe and Wilson [7]. Table 10–2 indicates the locations of points and the associated weights. The 3×3 rule is shown in Figure 10–16.

By adding a ninth node at $s = 0$, $t = 0$ in Figure 10–15, we can create an element called a "Q9." This is an internal node that is not connected to any other nodes. We then add the $a_{17}s^2t^2$ and $a_{18}s^2t^2$ terms to x and y, respectively in Eq. (10.5.27) and to u and v. The element is then called a Lagrange element as the shape functions can be derived using Lagrange interpolation formulas. For more on this subject consult [8].

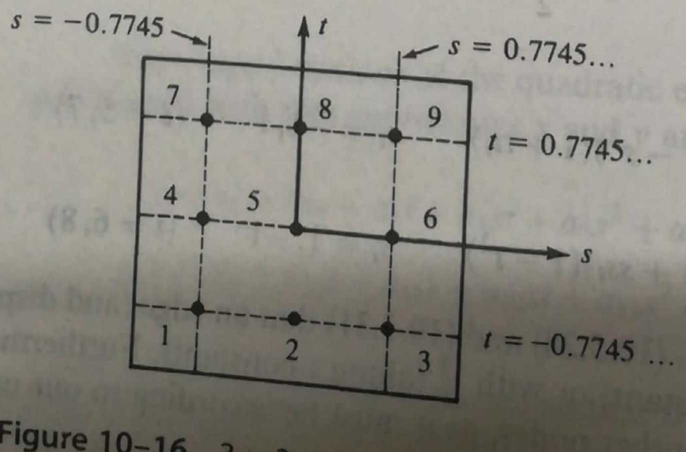

Figure 10–16 3 × 3 rule in two dimensions

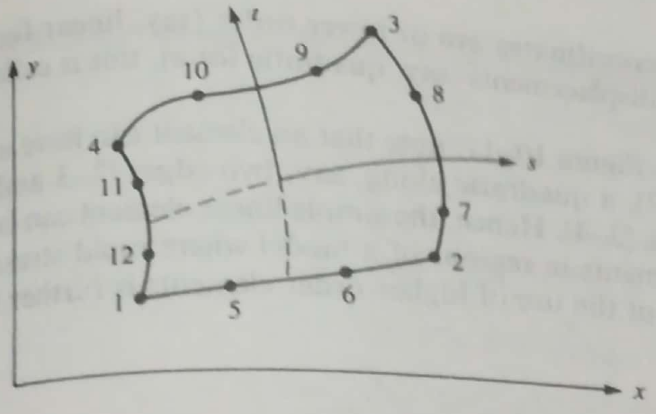

Figure 10–17 Cubic isoparametric element

Cubic Rectangle (Q12)

The cubic (Q12) element in Figure 10–17 has four corner nodes and additional nodes taken to be at one-third and two-thirds of the length along each side. The shape functions of the cubic element (as derived in Reference [3]) are based on the incomplete quartic polynomial such that

$$x = a_1 + a_2 s + a_3 t + a_4 s^2 + a_5 st + a_6 t^2 + a_7 s^2 t + a_8 st^2 + a_9 s^3 + a_{10} t^3 + a_{11} s^3 t + a_{12} st^3 \qquad (10.5.34)$$

with a similar polynomial for y. For the corner nodes ($i = 1, 2, 3, 4$),

$$N_i = \frac{1}{32}(1 + ss_i)(1 + tt_i)[9(s^2 + t^2) - 10] \qquad (10.5.35)$$

with s_i and t_i given by Eqs. (10.5.30). For the nodes on sides $s = \pm 1$ ($i = 7, 8, 11, 12$),

$$N_i = \frac{9}{32}(1 + ss_i)(1 + 9tt_i)(1 - t^2) \qquad (10.5.36)$$

with $s_i = \pm 1$ and $t_i = \pm\frac{1}{3}$. For the nodes on sides $t = \pm 1$ ($i = 5, 6, 9, 10$),

$$N_i = \frac{9}{32}(1 + tt_i)(1 + 9ss_i)(1 - s^2) \qquad (10.5.37)$$

with $t_i = \pm 1$ and $s_i = \pm\frac{1}{3}$.

Having the shape functions for the quadratic element given by Eqs. (10.5.28) and (10.5.31) or for the cubic element given by Eqs. (10.5.35) through (10.5.37), we can again use Eq. (10.2.17) to obtain $[B]$ and then Eq. (10.2.27) to set up $[k]$ for numerical integration for the plane element. The cubic element requires a 3 × 3 rule (nine points) to evaluate the matrix $[k]$ exactly. We then conclude that what is really desired is a library of shape functions that can be used in the general equations developed for stiffness matrices, distributed load, and body force and can be applied not only to stress analysis but to nonstructural problems as well.

Since in this discussion the element shape functions N_i relating x and y to nodal coordinates x_i and y_i are of the same form as the shape functions relating u and v to nodal displacements u_i and v_i, this is said to be an *isoparametric formulation*. For instance, for the linear element $x = \sum_{i=1}^{4} N_i x_i$ and the displacement function $u = \sum_{i=1}^{4} N_i u_i$, use the same shape functions N_i given by Eq. (10.2.5). If instead the