CHAPTER 1

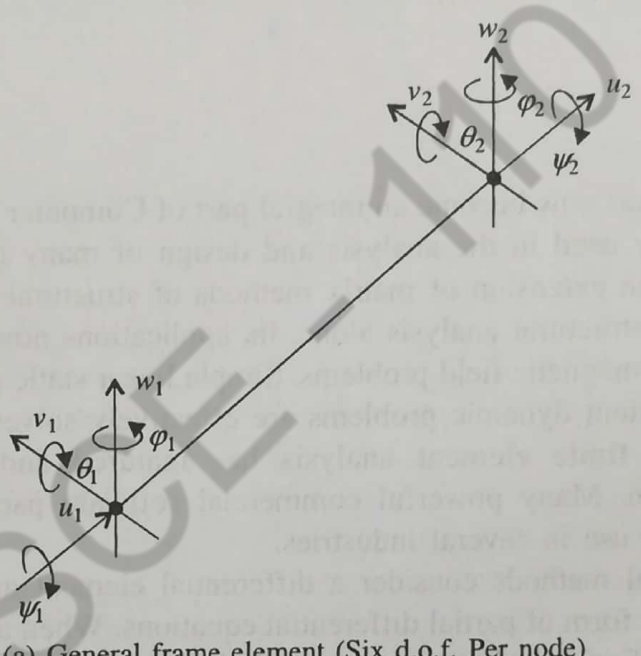
Introduction

Finite element analysis has now become an integral part of Computer Aided Engineering (CAE) and is being extensively used in the analysis and design of many complex real-life systems. While it started off as an extension of matrix methods of structural analysis and was initially perceived as a tool for structural analysis alone, its applications now range from structures to bio-mechanics to electromagnetic field problems. Simple linear static problems as well as highly complex nonlinear transient dynamic problems are effectively solved using the finite element method. The field of finite element analysis has matured and now rests on rigorous mathematical foundation. Many powerful commercial software packages are now available, enabling its widespread use in several industries.

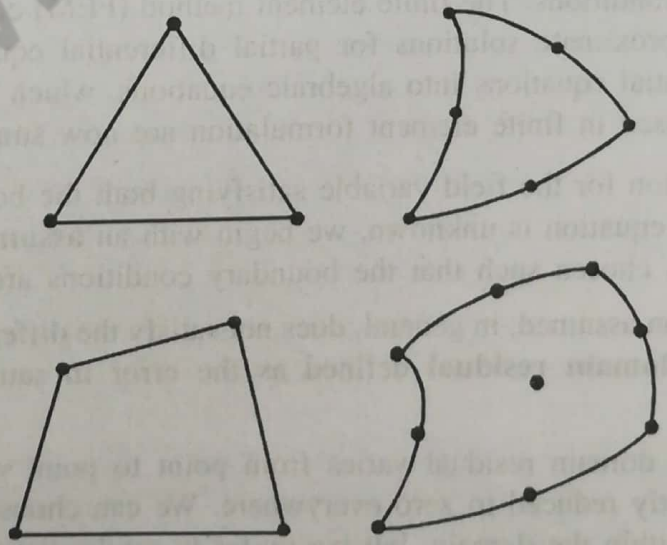
Classical analytical methods consider a differential element and develop the governing equations, usually in the form of partial differential equations. When applied to real-life problem situations, it is often difficult to obtain an exact solution to these equations in view of complex geometry and boundary conditions. The finite element method (FEM) can be viewed simply as a method of finding approximate solutions for partial differential equations or as a tool to transform partial differential equations into algebraic equations, which are then easily solved. Some of the key ideas used in finite element formulation are now summarised:

- Since the solution for the field variable satisfying both the boundary conditions and the differential equation is unknown, we begin with an **assumed trial solution**. The trial solution is chosen such that the boundary conditions are satisfied.
- The trial solution assumed, in general, does not satisfy the differential equation exactly and leaves a **domain residual** defined as the error in satisfying the differential equation.
- In general, the domain residual varies from point to point within the domain and cannot be exactly reduced to zero everywhere. We can choose to make it vanish at select points within the domain, but we prefer to render the residual very small, in some measure, over the entire domain. Thus, **the weighted sum of the domain residual** computed over the entire domain is rendered zero.

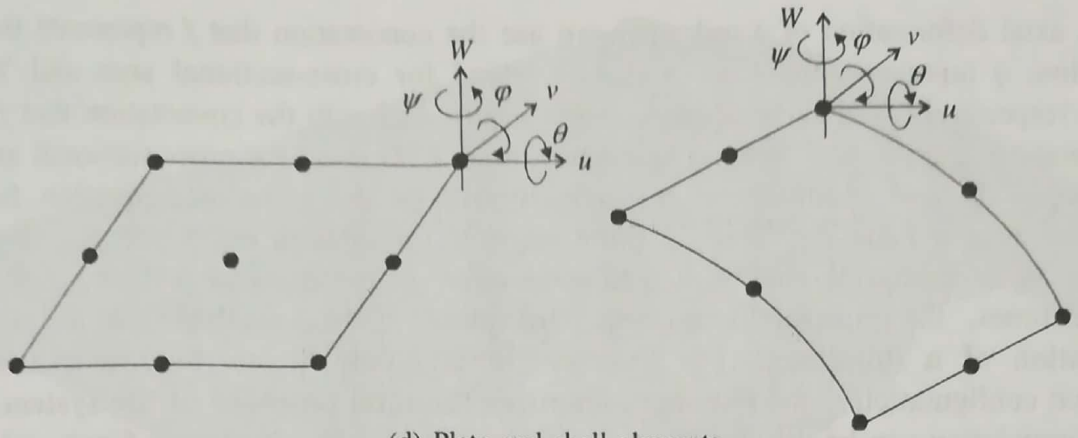
- The accuracy of the assumed trial solution can be improved by taking additional, higher order terms, but the computations become tedious and do not readily render themselves for automation. Also, for complex real-life problems, choosing a single continuous trial function valid over the entire domain satisfying the boundary conditions is not a trivial task. We therefore prefer to **discretise the domain into several segments** (called **finite elements**) and use several **piece-wise continuous trial functions**, each valid within a segment (finite element).
- Trial functions used in each segment (finite element) are known as **element level shape functions**. These are defined in the form of interpolation functions used to interpolate the value of the field variable at an interior point within the element from its value at certain key points (called the **nodes**) in the element. Typical elements commonly used in finite element analysis are shown in Figure 1.1.



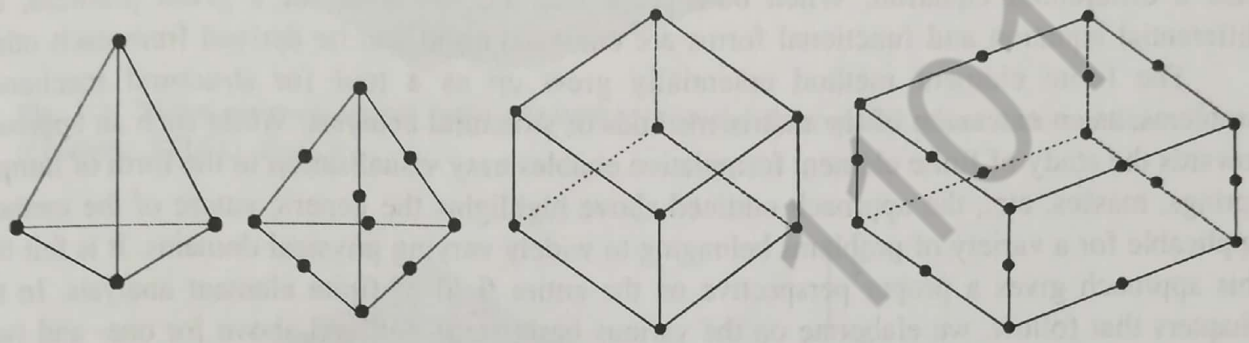
(a) General frame element (Six d.o.f. Per node)



(b) Common 2-d elements



(d) Plate and shell elements



(e) Common 3-d elements

Fig. 1.1 Typical finite elements

- With these shape functions, the weighted sum of the domain residual is **computed for each element** and then **summed up over all the elements** to obtain the weighted sum for the entire domain.
- For all elements using the same shape functions, the computations will be identical and, thus, for each type of element we have **element level characteristic matrices**. These characteristic matrices for several types of elements are derived a *priori* and programmed into a finite element software such as ANSYS, NASTRAN, IDEAS, etc. The user can choose to discretise (model) his domain with a variety of different finite elements. The computer program sets up the characteristic matrices for each element and then sums them all up for the entire finite element mesh to set up and solve the system level equations.

The basic steps of finite element analysis, as outlined above, are quite generic and can be applied to any problem—be it from the field of structural mechanics or heat transfer, or fluid flow or electromagnetic fields, given the appropriate differential equation and boundary conditions. In view of the similarity in the form of governing differential equations, the finite element formulation for a particular *type* of differential equation can be used to solve a *class* of problems. For example, a differential equation of the type

$$AC \frac{d^2 f}{dx^2} + q = 0$$

describes axial deformation of a rod when we use the connotation that f represents the axial deformation, q represents the load, and A, C stand for cross-sectional area and Young's modulus, respectively. The same equation, when interpreted with the connotation that f stands for temperature, q represents internal heat source and A, C stand for cross-sectional area and coefficient of thermal conductivity, respectively will be the governing equation for one-dimensional heat conduction. Thus, a finite element formulation developed for the above differential equation can be readily used to solve either of the physical problems.

Sometimes, the governing equations are more readily available in the form of **minimization of a functional**. For example, in problems of structural mechanics, the equilibrium configuration is the one that **minimizes the total potential of the system**. Finite element formulation can be developed readily for a problem described by a functional, rather than a differential equation. When both the forms are available for a given problem, the differential equation and functional forms are equivalent and can be derived from each other.

The finite element method essentially grew up as a tool for structural mechanics problems, as an extension of the matrix methods of structural analysis. While such an approach towards the study of finite element formulation enables easy visualisation in the form of lumped springs, masses, etc., the approach outlined above highlights the generic nature of the method, applicable for a variety of problems belonging to widely varying physical domains. It is felt that this approach gives a proper perspective on the entire field of finite element analysis. In the chapters that follow, we elaborate on the various basic steps outlined above for one- and two-dimensional, static and dynamic problems.

We now present several examples of application of finite element analysis to real-life problems, to give an overview of the capabilities of the method. Our application examples are drawn from the fields of structural mechanics, aerospace, manufacturing processes, electromagnetics, etc.

1.1 Typical Application Examples

1.1.1 Automotive Applications

In a vehicle having monocoque construction, the body itself is connected to the suspension. Therefore, the body panels are subjected to road loads. Hence, stresses and strains in these body panels are of interest. Figure 1.2 shows a FE mesh of a floor panel from the rear end of the vehicle. Provision for spare wheel as well as the various depressions used as stiffeners can be seen in the figure. A total of about 13,000 quadrilateral and triangular shell elements have been used to perform modal analysis, torsional stiffness analysis, and service load analysis. The same finite element mesh is also used for crash analysis using LS-DYNA software.

An automotive engine cylinder block experiences severe pressures and temperature gradients and other transient loads. It is essential to predict accurately the stresses and the vibration levels for further correlation with noise predictions. Figure 1.3 shows a typical finite element (shell element) model of a four cylinder in-line diesel engine cylinder block. Such a model is used to predict the system natural frequencies and mode shapes, response to combustion gas pressure, etc.

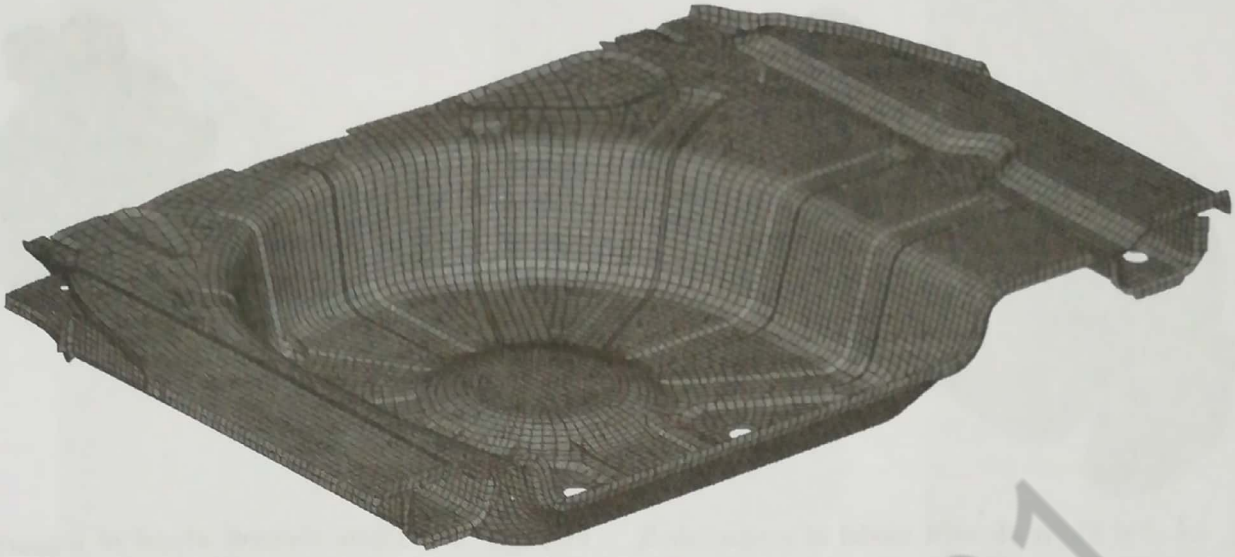


Fig. 1.2 Finite element model (MSC/NASTRAN) of the floor panel of an automobile.
(*Courtesy:* TELCO, *Pune.*)

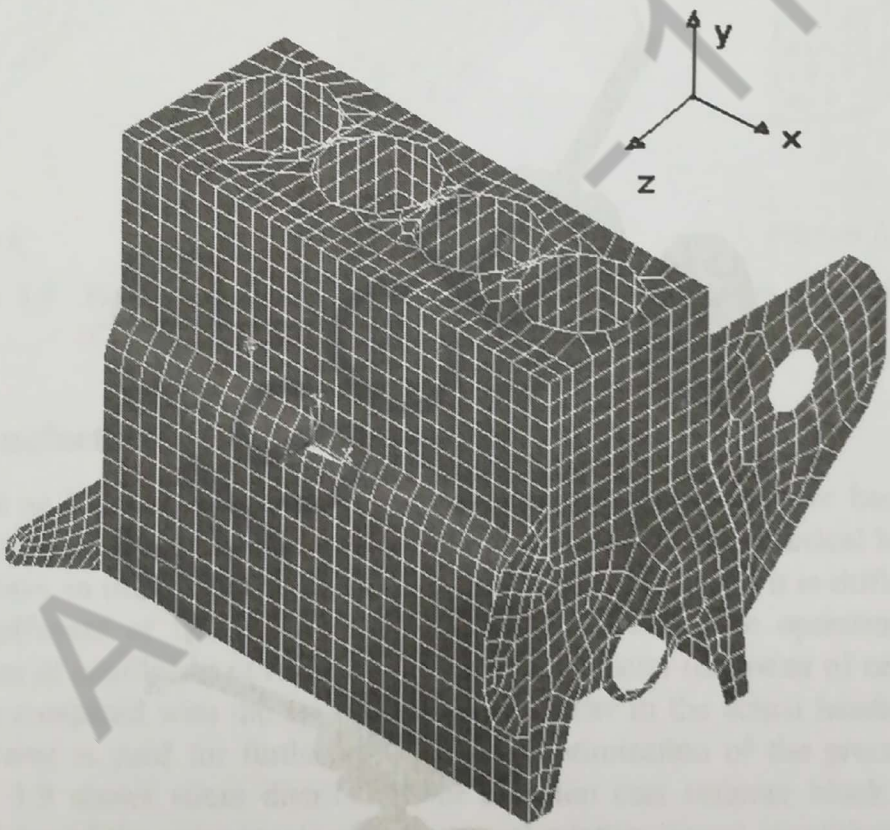


Fig. 1.3 Finite element model of an automotive engine cylinder block.
(*Courtesy: Mahindra & Mahindra Ltd., Nasik.*)

Figures 1.4–1.7 show representative finite element models of various components of a driveline where the gears have been modelled as friction wheels. Such a model can be used for studying the dynamic response of the entire driveline.

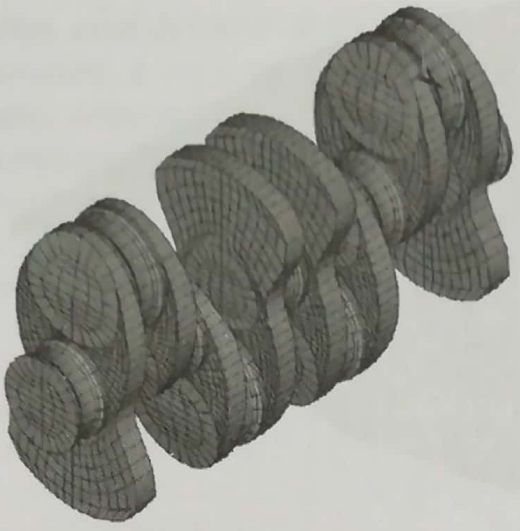


Fig. 1.4 3-d Finite element model of a crankshaft.
(*Courtesy: Mahindra & Mahindra Ltd., Mumbai.*)

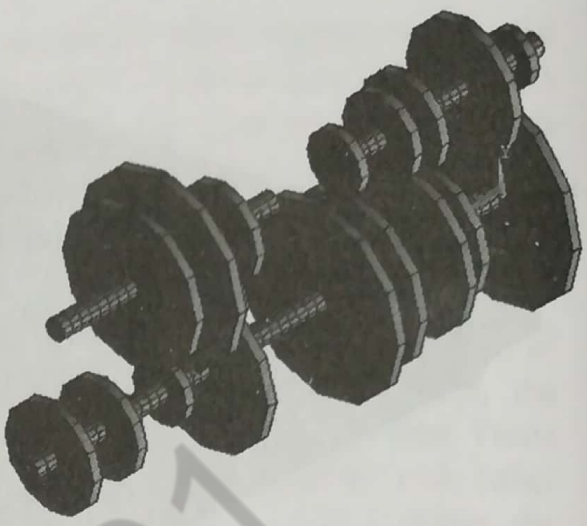


Fig. 1.5 3-d Finite element model of a gearbox.
(*Courtesy: Mahindra & Mahindra Ltd., Mumbai.*)

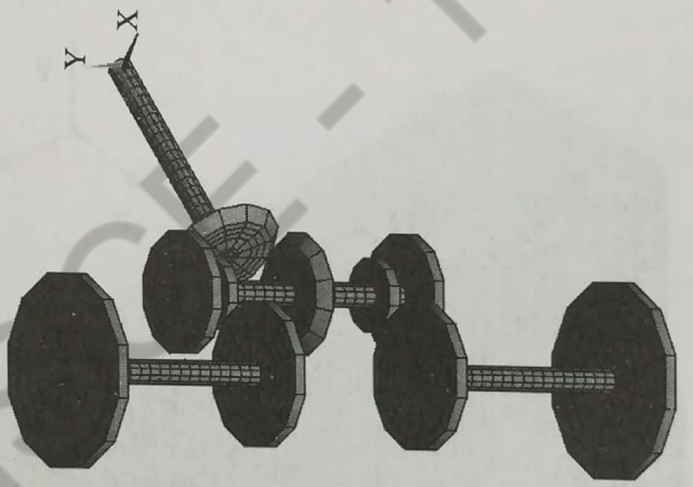


Fig. 1.6 3-d Finite element model of a differential.
(*Courtesy: Mahindra & Mahindra Ltd., Mumbai.*)

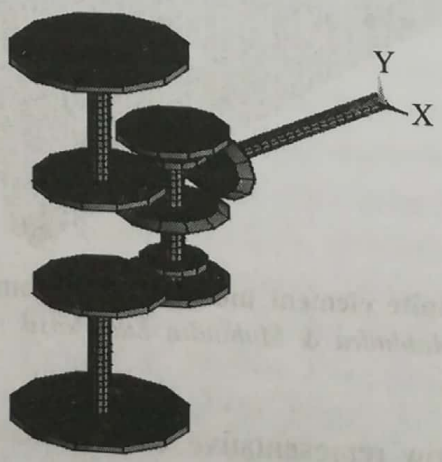


Fig. 1.7 3-d Finite element model of a rear axle.
(*Courtesy: Mahindra & Mahindra Ltd., Mumbai.*)

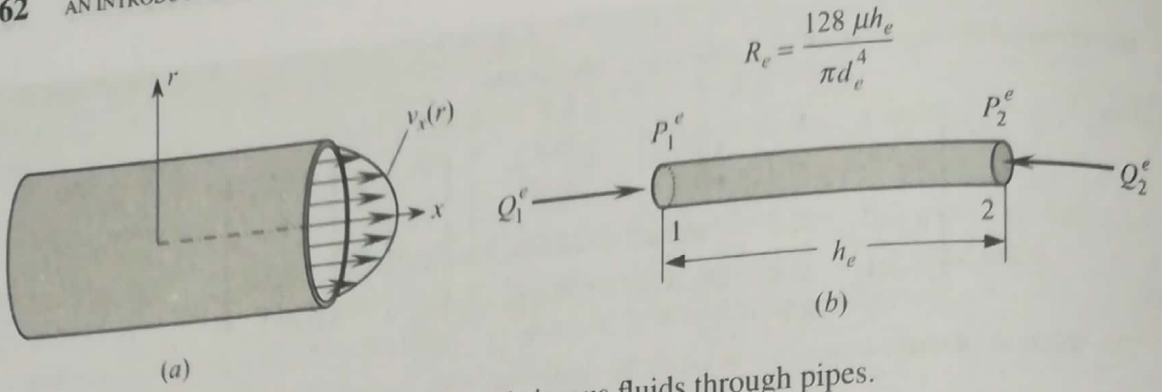


Figure 4.2.5 Flow of viscous fluids through pipes.

flow of viscous fluids through circular pipes is given by

$$v_x = -\frac{1}{4\mu}\frac{dP}{dx}\left[1 - \left(\frac{2r}{d}\right)^2\right] \tag{4.2.13a}$$

where dP/dx is the pressure gradient, d is the diameter of the pipe, and μ is the viscosity of the fluid [see Fig. 4.2.5(a)]. The volume rate of flow, Q, is obtained by integrating v_x over the pipe cross section. Thus, the relationship between Q and the pressure gradient dP/dx is given by the equation

$$Q = -\frac{\pi d^4}{128\mu}\frac{dP}{dx} \tag{4.2.13b}$$

The negative sign indicates that the flow is in the direction of negative pressure gradient.

Equation (4.2.13b) can be used to develop a relationship between the nodal values of the volume rate of flow, (Q_1^e, Q_2^e) and the pressure, (P_1^e, P_2^e), of a pipe element of length h_e and diameter d_e. The volume rate of flow entering node 1 is given by [see Fig. 4.2.5(b)]

$$Q_1^e = -\frac{\pi d_e^4}{128\mu h_e}\left(P_2^e - P_1^e\right)$$

Similarly, the volume rate of flow entering node 2 is

$$Q_2^e = -\frac{\pi d^4}{128\mu h_e}\left(P_1^e - P_2^e\right)$$

Thus, we have

$$\frac{\pi d^4}{128\mu h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} P_1^e \\ P_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{4.2.14}$$

The constant, $R_e = 128\mu h_e/\pi d_e^4$ is called the *pipe resistance*, in analogy with the electrical resistance [see Eq. (4.2.11)].

4.3 HEAT TRANSFER

4.3.1 Governing Equations

The equations governing conduction heat transfer were discussed in Example 1.2.2. Here we briefly-review the pertinent equations for our use.

The Fourier heat conduction law for one-dimensional systems states that the heat flow $q(x)$ is relate the temperature gradient $\partial T/\partial x$ by the relation (with heat flow in the positive direction of x)

$$q = -kA\frac{\partial T}{\partial x} \tag{4.3.1}$$

where k is the thermal conductivity of the material, A the cross-sectional area, and T the temperature. The negative sign in (4.3.1) indicates that heat flows downhill on the temperature scale. The balance of energy requires that

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + Ag = \rho cA\frac{\partial T}{\partial t} \tag{4.3.2}$$

where g is the heat energy generated per unit volume, ρ is the density, c is the specific heat of the material, and t is time. Equation (4.3.2) governs the transient heat conduction in a slab or fin (i.e., a one-dimensional system) when the heat flow in the normal to the x-direction is zero. For a plane wall, we take $A = 1$.

In the case of radially symmetric problems with cylindrical geometries, (4.3.2) takes a different form. Consider a long cylinder of inner radius R_i, outer radius R_o, and length L. When L is very large compared with the diameter, it is assumed that heat flows in the radial direction r. The transient radially symmetric heat flow in a cylinder is governed by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + g = \rho c\frac{\partial T}{\partial t} \tag{4.3.3}$$

A cylindrical fuel element of a nuclear reactor, a current-carrying electrical wire, and a thick-walled circular tube provide examples of one-dimensional radial systems.

The boundary conditions for heat conduction involve specifying either the temperature T or the heat flow Q at a point:

$$T = T_0 \quad \text{or} \quad Q \equiv -kA\frac{\partial T}{\partial x} = Q_0 \tag{4.3.4}$$

It is known that when a heated surface is exposed to a cooling medium, such as air or liquid, the surface will cool faster. We say that the heat is convected away. The *convection heat transfer* between the surface and the medium in contact is given by *Newton's law of cooling*:

$$Q = \beta A(T_s - T_\infty) \tag{4.3.5}$$

where T_s is the surface temperature, T_∞ is the temperature of the surrounding medium, called the *ambient temperature*; and β is the *convection heat transfer coefficient* or *film conductance* (or film coefficient). The heat flow due to conduction and convection at a boundary point must be in balance with the applied flow Q_0:

$$\pm kA\frac{\partial T}{\partial x} + \beta A(T - T_\infty) + Q_0 = 0 \tag{4.3.6}$$

The sign of the first term in (4.3.6) is negative when the heat flow is from the fluid at T_∞ to the surface at the left end of the element, and it is positive when the heat flow is from the fluid at T_∞ to the surface at the right end.

Convection of heat from a surface to the surrounding fluid can be increased by attaching thin strips of conducting metal to the surface. The metal strips are called *fins*. For a fin with heat flow along its length, heat can convect across the lateral surface of the fin [see Fig. 4.3.1(*a*)]. To account for the convection of heat through the surface, we must add the rate of heat loss by convection to the right-hand side of (4.3.2):

$$\frac{\partial}{\partial x}\left(Ak\frac{\partial T}{\partial x}\right) + Aq = \rho cA\frac{\partial T}{\partial t} + P\beta(T - T_\infty) \tag{4.3.7a}$$

where P is the perimeter and β is the film coefficient. Equation (4.3.7*a*) can be expressed in the alternative form

$$\rho cA\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + P\beta T = Ag + P\beta T_\infty \tag{4.3.7b}$$

The units of various quantities (in metric system) are as follows:

T	°C (celsius)	k	$W/(m \cdot °C)$
g	W/m^3	ρ	kg/m^3
c	$J/(kg \cdot °C)$	β	$W/(m^2 \cdot °C)$

For a steady state, we set the time derivatives in (4.3.2), (4.3.3), (4.3.7*a*), and (4.3.7*b*) equal to zero. The steady-state equations for various one-dimensional systems are summarized below [see Fig. 4.3.1(*b*) and (*c*); see Eqs. (1.2.14) and (1.2.17)].

Plane Wall $[Q = k(dT/dx)]$

$$-\frac{d}{dx}\left(k\frac{dT}{dx}\right) = Ag \tag{4.3.8}$$

Fin $[Q = kA(dT/dx)]$

$$-\frac{d}{dx}\left(kA\frac{dT}{dx}\right) + cT = Ag + cT_\infty, \quad c = P\beta \tag{4.3.9}$$

Cylindrical System $[Q = k(dT/dr)]$

$$-\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = g(r) \tag{4.3.10}$$

The essential and natural boundary conditions associated with these equations are of the form

$$T = T_0, \quad Q + \beta A(T - T_\infty) + Q_0 = 0$$

Equations (4.3.8)–(4.3.10) are a special case of the model equation (3.2.1) discussed in Section 3.2, with $a = kA$, $c = P\beta$, and $f \to Ag + P\beta T_\infty$. We immediately have the finite element model of Eqs. (4.3.8) and (4.3.9) from (3.2.31*a*) and (3.2.31*b*):

$$[K^e]\{T^e\} = \{f^e\} + \{Q^e\} \tag{4.3.11a}$$

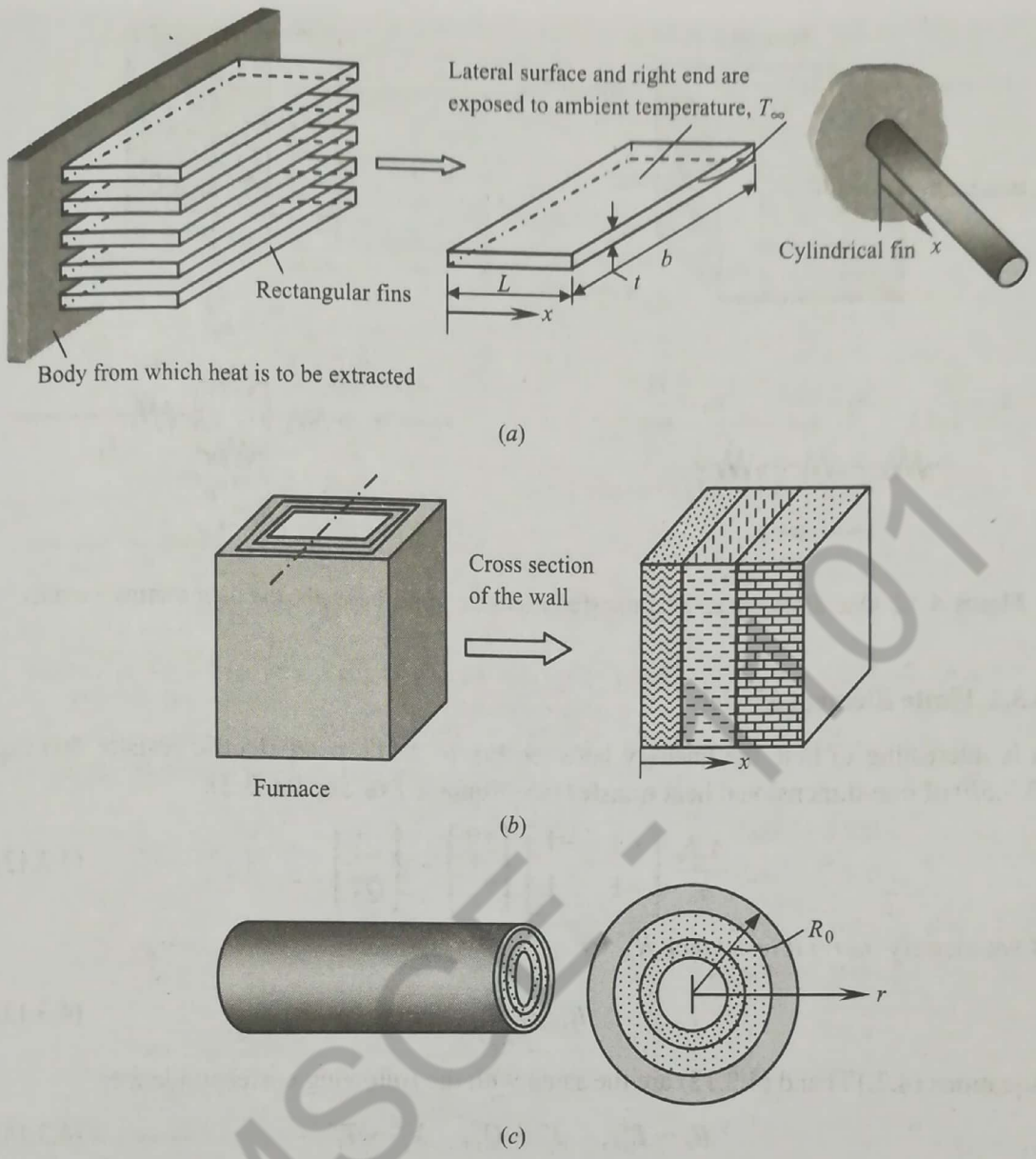


Figure 4.3.1 Heat transfer in (*a*) fins, (*b*) plane wall, and (*c*) radially symmetric system.

Here

$$K_{ij}^e = \int_{x_a}^{x_b} \left(kA \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + P\beta \psi_i^e \psi_j^e \right) dx, \quad f_i^e = \int_{x_a}^{x_b} \psi_i^e (Ag + P\beta T_\infty)\, dx$$

$$Q_1^e = \left(-kA \frac{dT}{dx} \right) \Big|_{x_a}, \quad Q_2^e = \left(kA \frac{dT}{dx} \right) \Big|_{x_b} \tag{4.3.11b}$$

where Q_1^e and Q_2^e denote heat flow *into* the element at the nodes.

Equation (4.3.10) is also a special case of the model boundary value problem. However, in developing the weak forms of (4.3.10), integration must be carried over a typical volume element of each system, as discussed in Section 3.4 [see Eq. (3.4.2)].

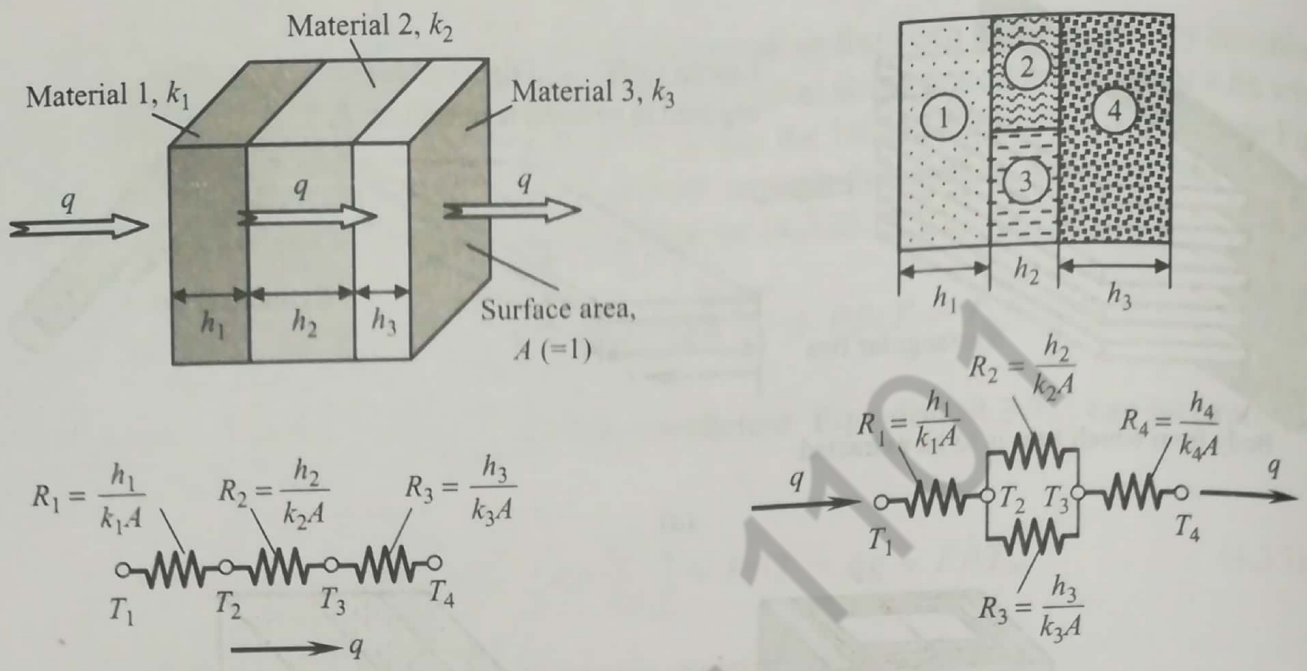


Figure 4.3.2 One-dimensional heat transfer through composite walls and their thermal circuits.

4.3.2 Finite Element Models

It is interesting to note the analogy between Eq. (4.2.11) of an electric resistor and Eq. (3.3.5*b*) of one-dimensional heat transfer (see Remark 7 of Section 3.3):

$$\frac{A_e k_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{4.3.12}$$

If we identify *thermal resistance* R_{th}^e by

$$R_{th}^e = \frac{h_e}{k_e A_e} \tag{4.3.13}$$

Equations (4.2.11) and (4.3.12) are the same with the following correspondence:

$$R_e \sim R_{th}^e, \quad I_i^e \sim Q_i^e, \quad V_i^e \sim T_i^e \tag{4.3.14}$$

This allows us to model complicated problems involving both series and parallel thermal resistances. Typical problems and their electrical analogies are shown in Figure 4.3.2.

4.4 FLUID MECHANICS

4.4.1 Governing Equations

All bulk matter in nature exists in one of two forms: solid or fluid. A solid body is characterized by the relative immobility of its molecules whereas a fluid state is characterized by relative mobility of its molecules. Fluids can exist either as gases or liquids. The field of fluid mechanics is concerned with the motion of fluids and the conditions affecting the motion (see Reddy and Gartling, 2001).

The basic equations of fluid mechanics are derived from the global laws of conservation of mass, momentum, and energy. Conservation of mass gives the continuity equation, while the conservation of momentum results in the equations of motion. The conservation of energy, considered in the last section, is the first law of thermodynamics, and it results in Eqs. (4.3.8)–(4.3.10) for one-dimensional systems when thermal-fluid coupling is omitted. For additional details, see Schlichting (1979), Bird et al. (1960), and Reddy and Gartling (2001). More details are provided in Chapter 10, which is dedicated to finite element models of two-dimensional flows of viscous incompressible fluids.

Here, we consider so-called parallel flow, where only one velocity component is different from zero resulting in all the fluid particles moving in one direction, i.e., $u = u(x, y, z)$, where u is the velocity component along the x coordinate. We assume that there are no body forces. The z-momentum equation requires that $u = u(x, y)$. The conservation of mass in this case reduces to

$$\frac{\partial u}{\partial x} = 0$$

which implies that $u = u(y)$. The y-momentum equation simplifies to

$$\frac{\partial P}{\partial y} = 0$$

which implies that $P = P(x)$, where P is the pressure. The x-momentum equation simplifies to

$$\mu \frac{d^2u}{dy^2} = \frac{dP}{dx} \tag{4.4.1}$$

The energy equation for this problem reduces to

$$\rho c u \frac{\partial T}{\partial x} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\left(\frac{du}{dy}\right)^2 \tag{4.4.2}$$

Here we are primarily interested in the finite element analysis of Eq. (4.4.1).

4.4.2 Finite Element Model

Equation (4.4.1) is a special case of the model equation (3.2.1) with the following correspondence:

$$f = -\frac{dP}{dx}, \quad a = \mu = \text{constant}, \quad c = 0, \quad x = y \tag{4.4.3}$$

5.1 INTRODUCTION

Here we consider the finite element formulation of the one-dimensional fourth-order differential equation that arises in the Euler–Bernoulli beam theory and the pair of one-dimensional second-order equations associated with the Timoshenko beam theory. The superposition of the beam and bar elements give rise to frame elements that can be used to analyze plane frame structures. The formulations of a fourth-order equation as well as coupled second-order equations (see Problems 3.1 and 3.3) involve the same steps as described in Section 3.2 for a second-order equation, but the mathematical details are somewhat different, especially in the finite element formulation of the equations.

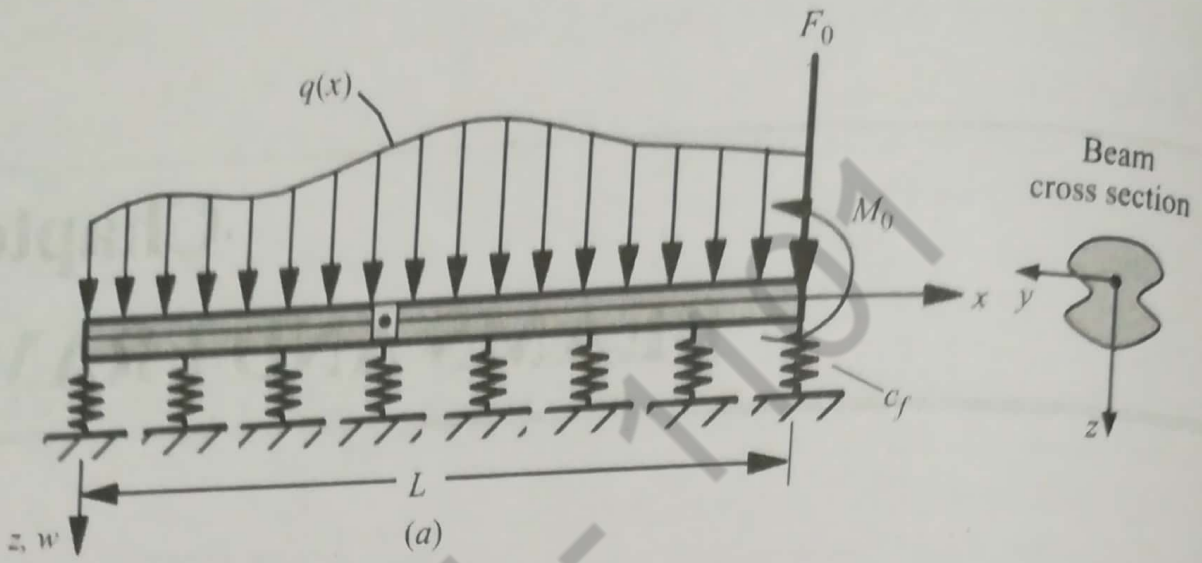
5.2 EULER–BERNOULLI BEAM ELEMENT

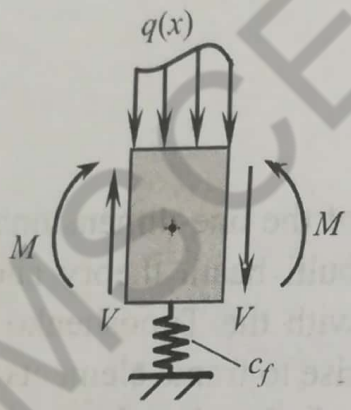
5.2.1 Governing Equation

In the Euler-Bernoulli beam theory, it is assumed that plane cross sections perpendicular to the axis of the beam remain plane and perpendicular to the axis after deformation. In this theory, the transverse deflection w of the beam is governed by the fourth-order differential equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w=q(x) \quad \text{for} \quad 0<x<L \tag{5.2.1}$$

where $EI=E(x)I(x)$, $c_f=c_f(x)$, $q=q(x)$ are given functions of x (i.e., data), and w is the dependent variable; E denotes the modulus of elasticity, I the second moment of area about the y axis of the beam, q is the distributed transverse load, c_f the elastic foundation modulus (if any), and w is the transverse deflection of the beam. The sign convention used in the derivation of (5.2.1) is shown in Fig. 5.2.1. In addition to satisfying the differential equation (5.2.1), w must also satisfy appropriate boundary conditions; since the equation is of fourth order, four boundary conditions are needed to solve it. The weak formulation of the equation will provide the form of these four boundary conditions. A step-by-step procedure for the finite element analysis of (5.2.1) is presented next (see Example 2.4.2).





Forces and their interrelationships at a point in the beam

$$-\frac{dV}{dx} + c_f w = q, \qquad -\frac{dM}{dx} + V = 0, \qquad M = -EI\frac{d^2w}{dx^2}$$

(b)

Figure 5.2.1 (*a*) Typical beam with distributed load $q(x)$ and point force F_0 and moment M_0. (*b*) The shear force-bending moment-deflection relations and the sign convention.

5.3 TIMOSHENKO BEAM ELEMENTS

5.3.1 Governing Equations

Recall that the Euler–Bernoulli beam theory is based on the assumption that plane cross sections remain plane and *normal* to the longitudinal axis after bending [see Fig. 5.3.1(*a*)]. This assumption results in zero transverse shear strain. When the normality assumption is not used, i.e., plane sections remain plane but not necessarily normal to the longitudinal axis after deformation, the transverse shear strain $\gamma_{xz} = 2\varepsilon_{xz}$ is not zero. Therefore, the rotation

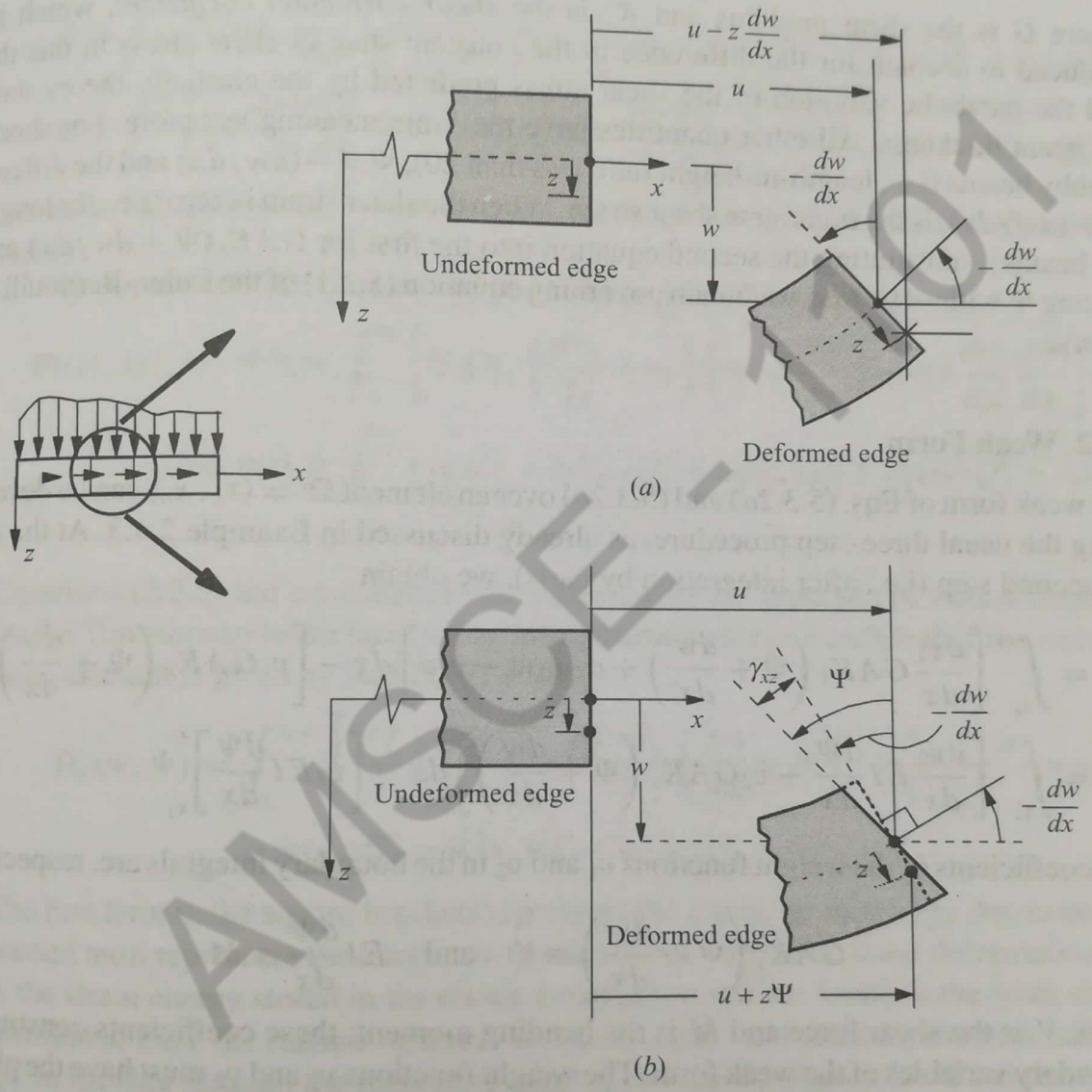


Figure 5.3.1 (*a*) Kinematics of the Euler–Bernoulli beam theory. (*b*) Kinematics of the Timoshenko beam theory where normals before deformation no longer remain normal after deformation.

of a transverse normal plane about the y-axis is not equal to $-dw/dx$ [see Fig. 5.3.1(*b*)]. Beam theory based on these relaxed assumptions is known as a *shear deformation beam theory*, most commonly known as the *Timoshenko beam theory*. We denote the rotation about the y-axis by an independent function $\Psi(x)$.

The equilibrium equations of the Timoshenko beam theory [see Example 2.4.3; also see Reddy (2002)] are the same as those of the Euler–Bernoulli beam theory (see Fig. 5.2.1), but the kinematic relations are different:

$$-\frac{dV}{dx}+c_f w=q, \quad -\frac{dM}{dx}+V=0, \quad M=EI\frac{d\Psi}{dx}, \quad V=GAK_s\left(\frac{dw}{dx}+\Psi\right) \tag{5.3.1}$$

Thus, the governing equations in terms of the deflection w and rotation Ψ become

$$-\frac{d}{dx}\left[GAK_s\left(\Psi+\frac{dw}{dx}\right)\right]+c_f w=q \tag{5.3.2a}$$

$$-\frac{d}{dx}\left(EI\frac{d\Psi}{dx}\right)+GAK_s\left(\Psi+\frac{dw}{dx}\right)=0 \tag{5.3.2b}$$

where G is the shear modulus and K_s is the *shear correction coefficient*, which is introduced to account for the difference in the constant state of shear stress in this theory and the parabolic variation of the shear stress predicted by the elasticity theory through the beam thickness. All other quantities have the same meaning as before. For short and chubby beams (i.e., length-to-height ratio less than 20), $\Psi \neq -(dw/dx)$ and the difference, $\Psi+(dw/dx)$, is the transverse shear strain. When the shear strain is zero (i.e., for long slender beams), substituting the second equation into the first for $GAK_s(\Psi+dw/dx)$ and replacing Ψ with $-dw/dx$, we obtain governing equation (5.2.1) of the Euler–Bernoulli beam theory.

and Rasmussen (1982) and Reddy (2002), among other references.

2.2.2 Boundary Value, Initial Value, and Eigenvalue Problems

The objective of most analyses is to determine unknown functions, called *dependent variables,* that are governed by a set of differential equations posed in a given domain Ω and some conditions on the boundary Γ of the domain. Often, a domain not including its boundary is called an open domain. A domain Ω with its boundary Γ is called a closed domain and is denoted by $\bar{\Omega} = \Omega \cup \Gamma$.

A function u of several variables $(x, y, \cdots)$ is said to be of class $C^m(\Omega)$ in a domain Ω if all its partial derivatives with respect to $(x, y, \cdots)$ of order up to and including m exist

and are *continuous* in Ω. Thus, if u is of class C^0 in a two-dimensional domain Ω, then u is continuous in Ω (i.e., $\partial u/\partial x$ and $\partial u/\partial y$ exist but may not be continuous). Similarly, if u is of class C^1, then u, $\partial u/\partial x$, and $\partial u/\partial y$ exist and are continuous (hence, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, and $\partial^2 u/\partial y \partial x$ exist but may not be continuous).

When the dependent variables are functions of one independent variable (say, x), the domain is a line segment (i.e., one-dimensional) and the end points of the domain are called boundary points. When the dependent variables are functions of two independent variables (say, x and y), the domain is two-dimensional and the boundary is the closed curve enclosing it. In a three-dimensional domain, dependent variables are functions of three coordinates (say x, y, and z) and the boundary is a two-dimensional surface.

As discussed in Section 1.2, a differential equation is said to describe a *boundary value problem* over the domain Ω if the dependent variable and possibly its derivatives are required to take specified values on the boundary Γ of Ω. An *initial value problem* is one in which the dependent variable and possibly its derivatives are specified initially (i.e., at time $t = 0$). Initial value problems are generally time-dependent problems. Examples of boundary and initial value problems were discussed in Section 1.2. A problem can be both a boundary value and initial value problem if the dependent variable is subject to both boundary and initial conditions. Another type of problem we encounter is one in which a differential equation governing the dependent unknown also contains an unknown parameter and we are required to find both the dependent variable and the parameter such that the differential equation and associated boundary conditions are satisfied. Such problems are called *eigenvalue problems*. Examples of various types of problems we encounter in science and engineering are given below (the mathematical classification of differential equations into elliptic, parabolic, and hyperbolic is of no interest at the moment).

Boundary Value Problems. *Steady State Heat Transfer in a Fin and Axial Deformation of a Bar* [Fig. 2.2.3(a)]: Find $u(x)$ that satisfies the second-order differential equation and *boundary conditions*:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu = f \quad \text{for} \quad 0 < x < L \tag{2.2.13a}$$

$$u(0) = u_0, \quad \left(a\frac{du}{dx}\right)_{x=L} = q_0 \tag{2.2.13b}$$

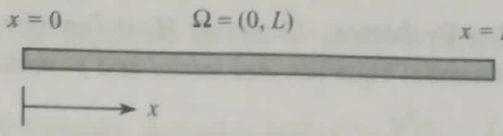
Bending of Elastic Beams under Transverse Load: Find $u(x)$ that satisfies the fourth-order differential equation and *boundary conditions*:

$$\frac{d^2}{dx^2}\left(b\frac{d^2u}{dx^2}\right) + cu = f \quad \text{for} \quad 0 < x < L \tag{2.2.14a}$$

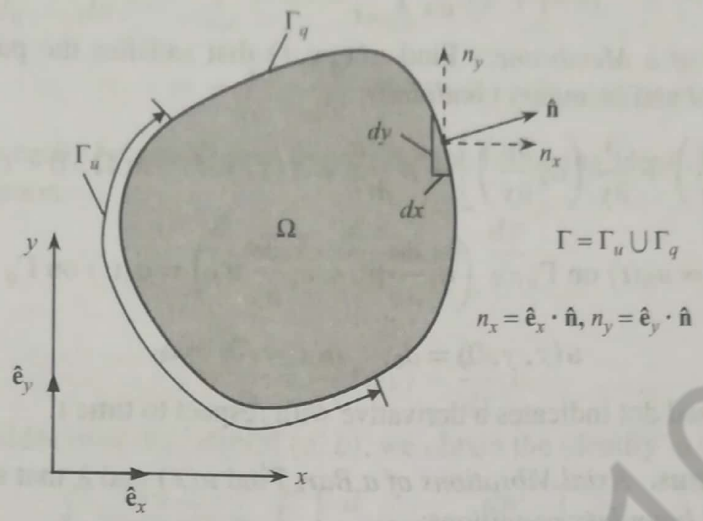
$$u(0) = u_0, \quad \left(\frac{du}{dx}\right)_{x=0} = d_0$$

$$\left[\frac{d}{dx}\left(b\frac{d^2u}{dx^2}\right)\right]_{x=L} = m_0, \quad \left(b\frac{d^2u}{dx^2}\right)_{x=L} = v_0 \tag{2.2.14b}$$

Steady Heat Conduction in a Two-Dimensional Region and *Transverse Deflections of a Membrane* [Fig. 2.2.3(b)]: Find $u(x, y)$ that satisfies the second-order partial differential



(*a*) A one-dimensional domain



(*b*) A two-dimensional domain

Figure 2.2.3 (*a*) One-dimensional domain. (*b*) Two-dimensional domain.

equation and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]+cu=f \quad \text{in } \Omega \tag{2.2.15a}$$

$$u=u_0 \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial x}n_y\right)=q_0 \text{ on } \Gamma_q \tag{2.2.15b}$$

where (n_x, n_y) are the direction cosines of the unit normal vector $\hat{\mathbf{n}}$ to the boundary Γ_q.

Initial Value Problems. *A General First-Order Equation:* Find $u(t)$ that satisfies the first-order differential equation and *initial conditions*:

$$a\frac{du}{dt}+cu=f \quad \text{for} \quad 0<t\le T \tag{2.2.16a}$$

$$u(0)=u_0 \tag{2.2.16b}$$

A General Second-Order Equation: Find $u(t)$ that satisfies the second-order differential equation and *initial conditions*:

$$a\frac{du}{dt}+b\frac{d^2u}{dt^2}+cu=f \quad \text{for} \quad 0<t\le T \tag{2.2.17a}$$

$$u(0)=u_0, \left(b\frac{du}{dt}\right)_{t=0}=v_0 \tag{2.2.17b}$$

Boundary and Initial Value Problems. *Unsteady Heat Transfer in a Rod:* Find $u(x,t)$ that satisfies the partial differential equation and *initial* and *boundary conditions*:

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\rho\frac{\partial u}{\partial t}=f(x,t) \quad \text{for} \quad 0<x<L,\ 0<t\le T \tag{2.2.18a}$$

$$u(0,t)=d_0(t), \quad \left(a\frac{du}{dx}\right)_{x=L}=q_0(t), \quad u(x,0)=u_0(x) \tag{2.2.18b}$$

Unsteady Motion of a Membrane: Find $u(x,y,t)$ that satisfies the partial differential equation and *initial* and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]+\rho\frac{\partial^2 u}{\partial t^2}=f(x,y,t) \quad \text{in } \Omega,\ 0<t\le T \tag{2.2.19a}$$

$$u=u_0(t) \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y\right)=q_0(t) \text{ on } \Gamma_q \tag{2.2.19b}$$

$$u(x,y,0)=d_0, \quad \dot{u}(x,y,0)=v_0 \tag{2.2.19c}$$

where the superposed dot indicates a derivative with respect to time t.

Eigenvalue Problems. *Axial Vibrations of a Bar:* Find $u(x)$ and λ that satisfy the differential equation and *boundary conditions*:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)-\lambda u=0 \quad \text{for} \quad 0<x<L \tag{2.2.20a}$$

$$u(0)=0, \quad \left(a\frac{du}{dx}\right)_{x=L}=0 \tag{2.2.20b}$$

Transverse Vibrations of a Membrane: Find $u(x,y)$ and λ that satisfy the partial differential equation and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]-\lambda u=0 \quad \text{in } \Omega \tag{2.2.21a}$$

$$u=0 \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y\right)=0 \text{ on } \Gamma_q \tag{2.2.21b}$$

The values of λ are called *eigenvalues,* and the associated functions u are called *eigenfunctions*.

The set of specified functions and parameters (e.g., a, b, c, ρ, f, u_0, d_0, q_0, v_0, and so on) are called the *data* of the problem. Differential equations in which the right-hand side f is zero are called *homogeneous differential equations,* and boundary (initial) conditions in which the specified data is zero are called homogeneous boundary (initial) conditions. The *exact solution* of a differential equation is the function that identically satisfies the differential equation at every point of the domain and for all times $t>0$, and satisfies the specified boundary and/or initial conditions.

Finite Element Formulation Starting from Governing Differential Equations

As discussed in Chapter 1, the finite element method has been applied to a wide variety of problems in various fields of application. This has been possible because of the common features in the mathematical formulation of these seemingly different problems, e.g. many problems of engineering analysis can be represented by partial differential equations. In some cases, the same *type* of partial differential equation (e.g. the two-dimensional Laplace/Poisson equation) can represent a large number of physical problems (e.g. ground water seepage, torsion of bars, and heat flow). Thus, to gain a proper perspective of the method of finite elements, we would like to present it in this chapter as a method for finding an approximate solution to differential equations.

The Weighted Residual (WR) method is a powerful way of finding approximate solutions to differential equations. In particular, The Galerkin Weighted Residual formulation is the most popular from the finite element point of view. Piece-wise trial function approximation of the weak form of the Galerkin weighted residual technique forms the basis of the finite element method. In what follows, we will first introduce the general weighted residual technique and the Galerkin form of the weighted residual technique, using a set of trial functions, each of which is valid over the entire solution domain. We will then introduce the weak form of the same. Finally, we will present the piece-wise trial function approximation concept, wherein each of the trial functions used is valid only over a small part of the domain. This leads us to the formulation of the finite element method.

2.1 Weighted Residual Method—Use of a Single Continuous Trial Function

Let us consider a general problem of engineering analysis described in the form of a differential equation (to be valid within a particular domain Ω), while satisfying the prescribed boundary conditions on the boundary Γ. Our scheme of finding approximate solution to differential equations consists of the following steps:

- Assume a guess (or trial) solution to the problem. For example, for a one-dimensional problem, we may choose a trial solution as

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots \tag{2.1}$$

- In general, the function so assumed will satisfy neither the differential equation within the domain (Ω) nor the boundary conditions (on Γ). By substituting the assumed function in the differential equation and the boundary conditions of the problem, find the error in satisfying these (we will call these "domain residual" and "boundary residual").
- Determine the unknown parameters (c_0, c_1, c_2, ...) in the assumed trial function in such a way as to make these residuals as low as possible.

In the process, if we can make the domain and boundary residuals identical to zero everywhere, we will get the exact solution to the problem itself. In general, we expect to get a reasonably accurate solution to the problem at hand. **In the context of the finite element method, we will limit our discussion to trial solutions that satisfy the applicable boundary conditions and hence, only domain residual remains.** The choice of trial solutions that implicitly satisfy the differential equation but not the boundary conditions (thus resulting in nonzero boundary residual) leads to the "boundary element method". As a detailed discussion on the boundary element method is beyond the scope of this text, the interested reader may refer to standard texts for details of this technique.

We will use the following simple example to illustrate the above method of finding approximate solutions to differential equations.

Example 2.1. Consider a uniform rod subjected to a uniform axial load as illustrated in Figure 2.1. It can be readily shown that the deformation of the bar is governed by the differential equation

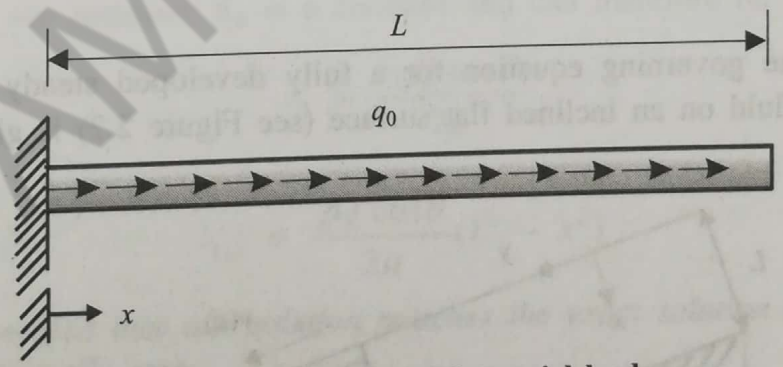


Fig. 2.1 Rod subjected to axial load.

$$AE\frac{d^2u}{dx^2} + q_0 = 0 \tag{2.2}$$

with the boundary conditions $u(0) = 0$, $\left.\frac{du}{dx}\right|_{x=L} = 0$.

Let us now find an approximate solution to this problem using the method just discussed.

Step 1: *Assume a trial or guess solution.* Let

$$u(x) \approx \hat{u}(x) = c_0 + c_1 x + c_2 x^2 \tag{2.3}$$

where the constants c_0, c_1, c_2 are yet to be determined. In order to satisfy the first boundary condition that $\hat{u}(0) = 0$, we have $c_0 = 0$. To satisfy the second boundary condition, we have $c_1 = -2c_2L$. Thus we now have, for our trial solution,

$$\hat{u}(x) = c_2(x^2 - 2Lx) \tag{2.4}$$

Since the trial solution contains only one free parameter c_2, it is often referred to as a "one-parameter solution".

Step 2: *Find the domain residual.* Substituting in the governing differential equation

$$R_d = AE\frac{d^2\hat{u}}{dx^2} + q_0 = AE(2c_2) + q_0 \tag{2.5}$$

Step 3: *Minimise the residual.* Since there is one residual to be minimised and one parameter to be determined, we can readily solve for the undetermined coefficient by setting the residual to zero, i.e., $R_d = 0$, yielding

$$c_2 = \frac{-q_0}{2AE} \tag{2.6}$$

Thus our final solution is

$$\hat{u}(x) = \left(\frac{q_0}{2AE}\right)(2xL - x^2) \tag{2.7}$$

For this simple example, since we could make the residual identically zero everywhere, our final solution tallies with the exact solution.

Example 2.2. The governing equation for a fully developed steady laminar flow of a Newtonian viscous fluid on an inclined flat surface (see Figure 2.2) is given by

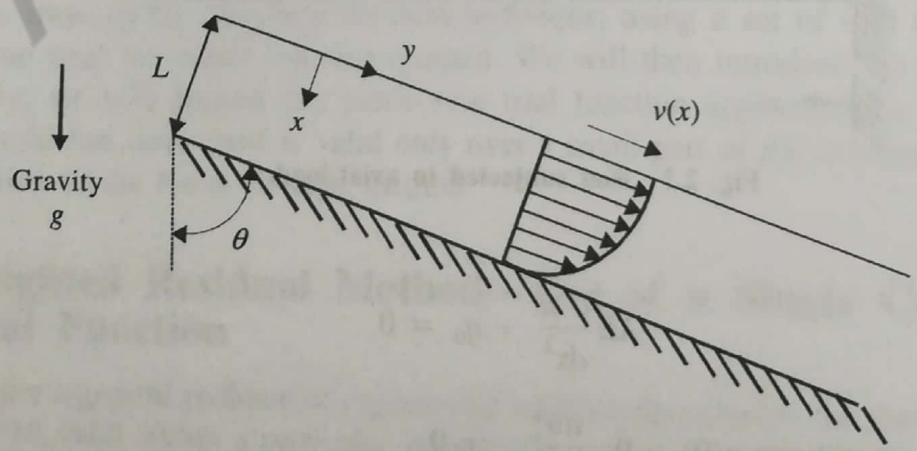


Fig. 2.2 Laminar flow on an inclined surface.

$$\mu \frac{d^2v}{dx^2} + \rho g \cos\theta = 0 \tag{2.8}$$

where

μ = coefficient of viscosity,
v = fluid velocity,
ρ = density,
g = acceleration due to gravity,
θ = angle between the inclined surface and the vertical.

The boundary conditions are given by

$$\left.\frac{dv}{dx}\right|_{x=0} = 0 \qquad \text{(zero shear stress)} \tag{2.9}$$

$$v(L) = 0 \qquad \text{(no slip)}$$

Let us find the velocity distribution $v(x)$ using the weighted residual method.

Step 1: *Assume a trial solution.* Let

$$v(x) \approx \hat{v}(x) = c_0 + c_1 x + c_2 x^2 \tag{2.10}$$

Hence,

$$\frac{d\hat{v}}{dx} = c_1 + 2c_2 x \tag{2.11}$$

From the boundary conditions, $c_1 = 0$, $c_0 = -c_2 L^2$. Therefore,

$$\hat{v}(x) = c_2(x^2 - L^2) \tag{2.12}$$

Step 2: *Find the domain residual*

$$R_d = \mu(2c_2) + \rho g \cos\theta \tag{2.13}$$

Step 3: *Minimise the residual.* R_d is a constant and can therefore be set to zero. Hence,

$$c_2 = \frac{-\rho g \cos\theta}{2\mu} \tag{2.14}$$

Therefore,

$$\hat{v}_{(x)} = \frac{\rho g \cos\theta}{2\mu}(L^2 - x^2) \tag{2.15}$$

It is readily verified that our solution matches the exact solution as we can make the domain residual identically zero.

Example 2.3. Consider the problem of a cantilever beam under uniformly distributed load q_0 as shown in Figure 2.3. The governing differential equation is given by

$$EI \frac{d^4v}{dx^4} - q_0 = 0 \tag{2.16}$$

and the boundary conditions are given by

$$v(0) = 0, \qquad \frac{dv}{dx}(0) = 0$$

$$\frac{d^2v}{dx^2}(L) = 0, \qquad \frac{d^3v}{dx^3}(L) = 0 \tag{2.17}$$

where the first two boundary conditions enforce zero displacement and slope at the fixed end and the last two conditions prescribe zero bending moment and shear force at the free end.

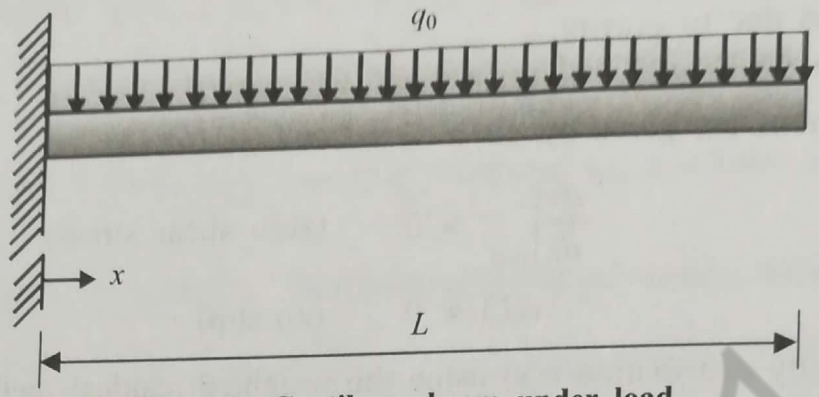


Fig. 2.3 Cantilever beam under load.

Step 1: *Assume a trial or guess solution.* We observe that it is not easy to select a trial function that satisfies all the boundary conditions. Let us choose $\hat{v}(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$.

From the boundary conditions that $\hat{v}(0) = 0$ and $d\hat{v}/dx\ (0) = 0$, we have $c_0 = 0 = c_1$. In order to satisfy the boundary conditions at $x = L$, we should have

$$c_2 = -3c_3L - 6c_4L^2, \qquad c_3 = -4c_4L \tag{2.18}$$

Substituting and rearranging the terms in the trial solution, we get

$$\hat{v}(x) = c_4(x^4 - 4Lx^3 + 6x^2L^2) \tag{2.19}$$

We thus observe that finding trial solution functions that satisfy all the boundary conditions could, in general, be cumbersome.

Step 2: *Find the domain residual.* Substituting in the differential equation, we get the domain residual as

$$R_d(x) = 24EIc_4 - q_0 \tag{2.20}$$

Step 3: *Minimise the residual.* Since there is one residual to be minimised and one parameter to be determined, we can readily solve for the undetermined coefficient c_4 by setting the residual to zero, i.e., $R_d = 0$, yielding thereby

$$c_4 = q_0/(24EI) \tag{2.21}$$

Thus our trial solution is

$$\hat{v}(x) = q_0/(24EI)\ [x^4 - 4Lx^3 + 6x^2L^2]$$

which can be readily verified to be the exact solution itself. This is to be expected since we were able to make the residual identically zero within the entire domain.

Example 2.4. Let us consider the example of a simply supported beam under uniformly distributed load as shown in Figure 2.4. The governing differential equation and the boundary conditions are given by

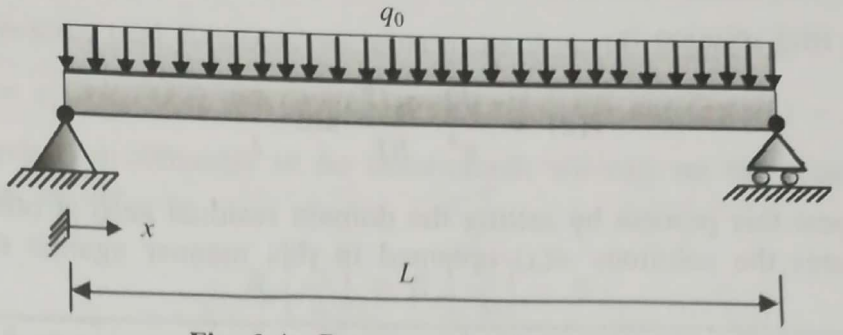


Fig. 2.4 Beam on simple supports.

$$EI\frac{d^4v}{dx^4} - q_0 = 0$$

$$v(0) = 0, \qquad \frac{d^2v}{dx^2}(0) = 0 \tag{2.22}$$

$$v(L) = 0, \qquad \frac{d^2v}{dx^2}(L) = 0$$

Step 1: *Assume a trial solution.* We can use the approach of Example 2.3 and find a polynomial trial solution satisfying all boundary conditions. However, in view of the special boundary conditions of simple supports, we can make the process simple by choosing trigonometric functions. Let

$$v(x) \approx \hat{v}(x) = c_1 \sin(\pi x/L) \tag{2.23}$$

This one-parameter trial solution satisfies all boundary conditions.

Step 2: *Find the domain residual.* Substituting the trial solution $\hat{v}(x)$ in the governing differential equation, the domain residual is obtained as

$$R_d = c_1(\pi/L)^4(EI) \sin(\pi x/L) - q_0 \tag{2.24}$$

Step 3: *Minimise the residual.* We observe that, unlike in the previous examples, the domain residual is now varying from point to point within the domain ($0 < x < L$). Since we have only one coefficient to be determined, we can set the residual zero only at any one point of our choice within the domain if we follow the approach of previous examples. This technique is called the **point collocation technique**, wherein we set the residual (in general, a function of x) to zero at chosen points within the domain—the number of points being equal to the number of coefficients in the trial function that need to be determined. We will illustrate this procedure now. In this procedure, however, there is a danger that the residual might be unduly large at some other points within the domain. We may thus want to "minimise" the residual in an overall sense over the entire domain rather than setting it identically zero at only few selected points. This will be discussed in the next section.

Solution by point collocation. Let us make $R_d = 0$ at $x = L/4$, i.e.

$$c_1(\pi/L)^4 (EI) \sin(\pi/4) - q_0 = 0 \tag{2.25}$$

yielding thereby

$$c_1 = \frac{\sqrt{2}}{\pi^4}\frac{q_0L^4}{EI} \tag{2.26}$$

and the resulting trial solution is

$$\hat{v}(x) = \frac{\sqrt{2}}{\pi^4} \frac{q_0 L^4}{EI} \sin \frac{\pi x}{L} \tag{2.27}$$

We may repeat this process by setting the domain residual zero at other axial locations. Figure 2.5 compares the solutions $\hat{v}(x)$ obtained in this manner against the classical exact

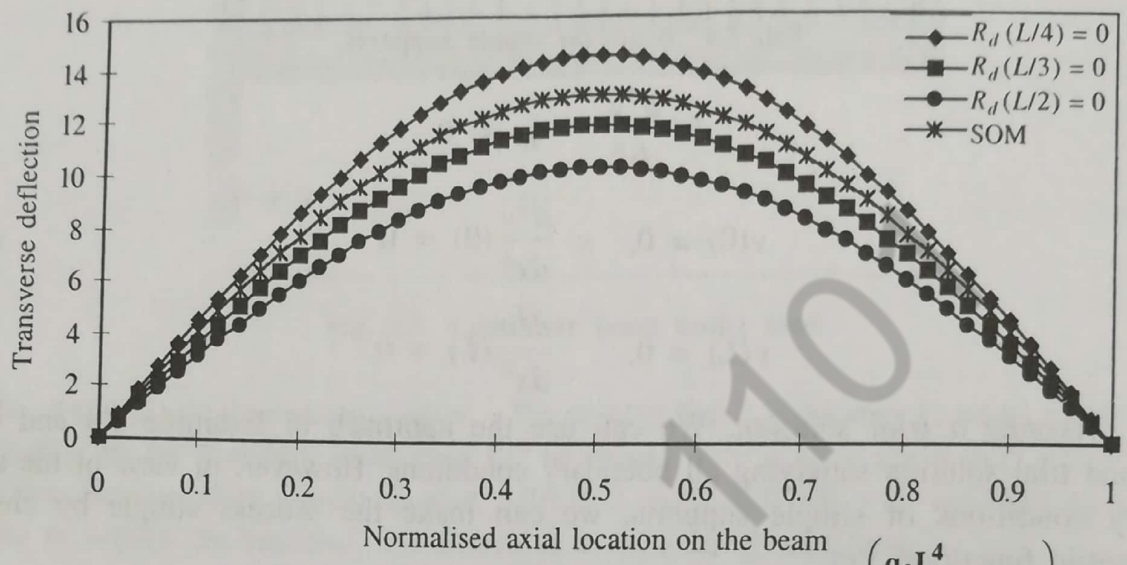


Fig. 2.5 Simply supported beam deflection. $\left(\frac{q_0 L^4}{EI} = 1000\right)$

solution. Figure 2.6 shows a plot of the domain residual in each case. In this way, we are able to generate different approximate solutions to the problem but each of these solutions deviates appreciably from the exact solution. We can improve our trial solution by adding one more term to the Sine series. Since the problem at hand is symmetric about $x = L/2$, we modify our trial solution as follows:

$$v(x) \approx \hat{v}(x) = c_1 \sin(\pi x/L) + c_3 \sin(3\pi x/L) \tag{2.28}$$

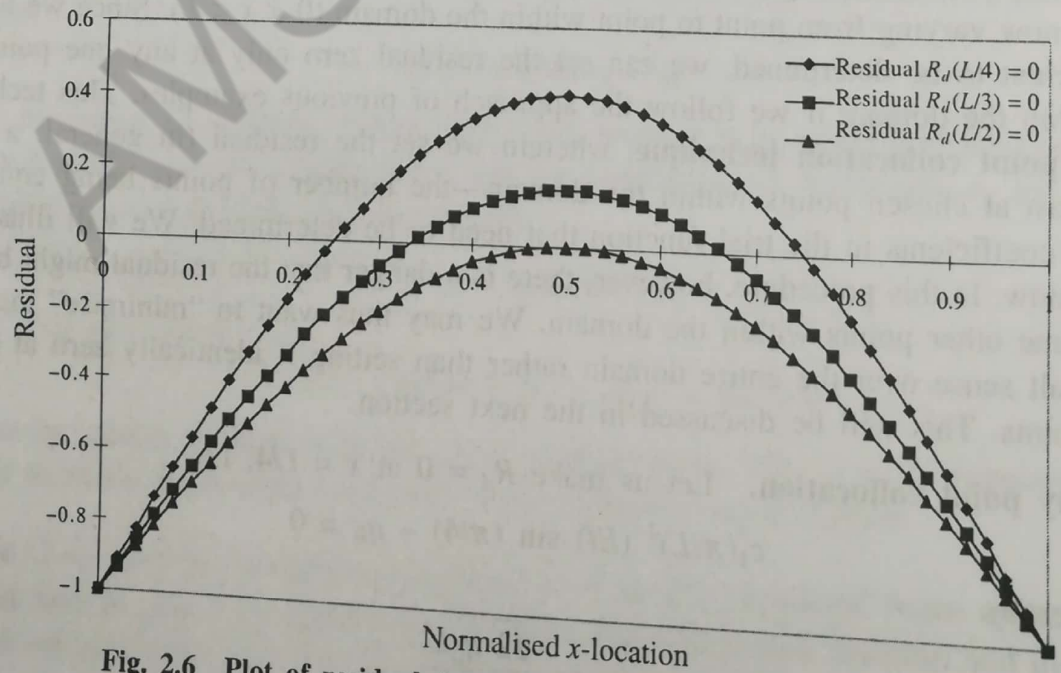


Fig. 2.6 Plot of residual-simply supported beam (Example 2.4).

With this two-term trial function, we obtain the domain residual as

$$R_d = c_1(\pi/L)^4(EI)\ \sin\ (\pi x/L) + c_3(3\pi/L)^4(EI)\ \sin\ (3\pi x/L) - q_0 \tag{2.29}$$

Since we have two constants to be determined, we can set the residual zero at two selected points. Let us set

$$R_d\left(\frac{L}{4}\right) = R_d\left(\frac{L}{3}\right) = 0 \tag{2.30}$$

i.e.,

$$c_1\left(\frac{\pi}{L}\right)^4 EI\left(\frac{1}{\sqrt{2}}\right) + c_3\left(\frac{3\pi}{L}\right)^4 EI\left(\frac{1}{\sqrt{2}}\right) - q_0 = 0 \tag{2.31}$$

$$c_1\left(\frac{\pi}{L}\right)^4 EI\left(\frac{\sqrt{3}}{2}\right) + c_3\left(\frac{3\pi}{L}\right)^4 EI(0) - q_0 = 0 \tag{2.32}$$

Solving these equations, we obtain

$$c_1 = \frac{2}{\sqrt{3}\,\pi^4}\frac{q_0L^4}{EI} \tag{2.33}$$

$$c_3 = 0.00003289\ \frac{q_0L^4}{EI} \tag{2.34}$$

Figure 2.7 compares the solution $\hat{v}(x)$ obtained in this manner against the classical exact solution. Figure 2.8 shows a plot of the domain residual. We observe that this approximates the solution perhaps better than the earlier one-parameter solution, but it still deviates appreciably from the exact solution.

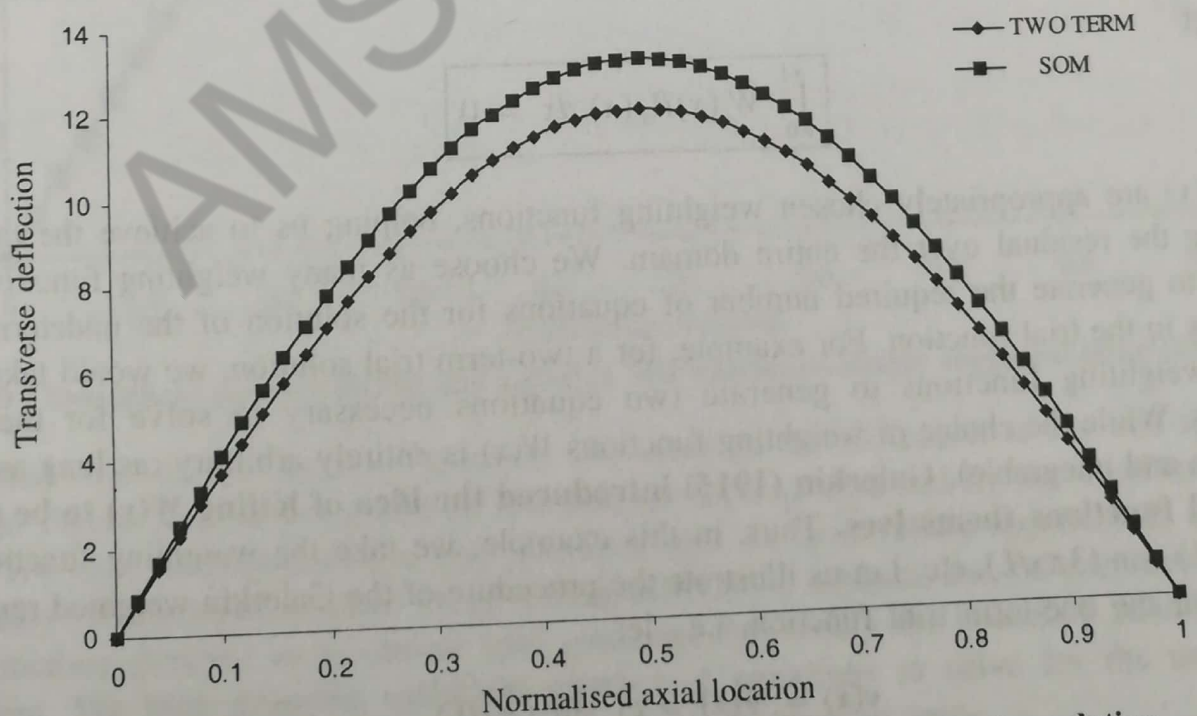


Fig. 2.7 Simply supported beam deflection—Two-parameter solution.

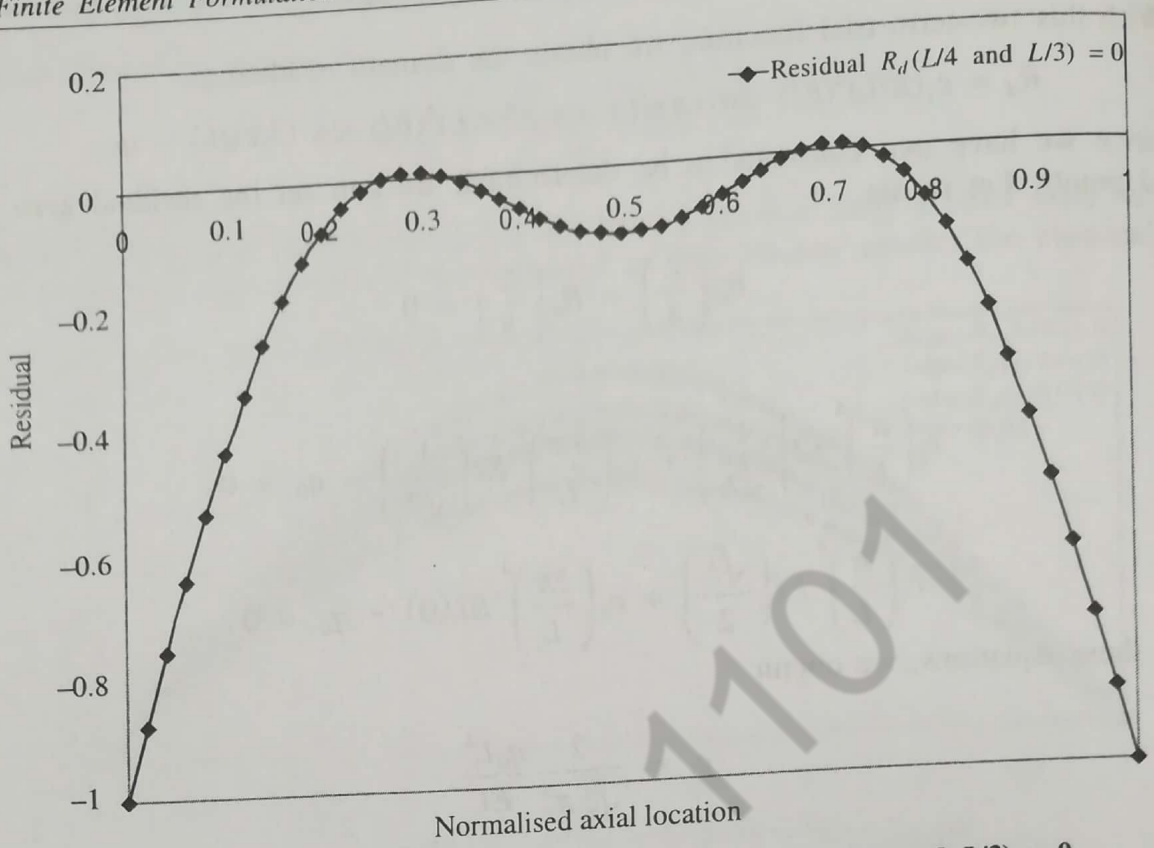


Fig. 2.8 Domain residual—Two-parameter solution $R_d(L/4$ and $L/3) = 0$.

Rather than trying to set the residual exactly zero at a few selected points and having no control over the residual at all the other points in the domain, we will now try to minimise the residual in an overall sense. This technique is called the *weighted residual technique*.

Solution by weighted residual technique. For the problem at hand, we will formulate this technique as

$$\boxed{\int_0^L W_i(x) R_d(x)\, dx = 0} \tag{2.35}$$

where $W_i(x)$ are appropriately chosen weighting functions, helping us to achieve the task of minimising the residual over the entire domain. We choose as many weighting functions as necessary to generate the required number of equations for the solution of the undetermined coefficients in the trial function. For example, for a two-term trial solution, we would take two different weighting functions to generate two equations necessary to solve for the two coefficients. While the choice of weighting functions $W(x)$ is entirely arbitrary (as long as they are nonzero and integrable), **Galerkin (1915) introduced the idea of letting $W(x)$ to be same as the trial functions themselves.** Thus, in this example, we take the weighting function to be $\sin(\pi x/L)$, $\sin(3\pi x/L)$, etc. Let us illustrate the procedure of the Galerkin weighted residual technique for the one-term trial function, i.e., let

$$v(x) \approx \hat{v}(x) = c_1 \sin(\pi x/L) \tag{2.36}$$

2.2 The General Weighted Residual (WR) Statement

Having understood the basic technique and successfully solved a few problems, we can no attempt to write down the general weighted residual statement.

For the unknown field variable u, we assume an approximate solution of the form

$$\boxed{u \approx \hat{u} = \phi + \sum_{i=1}^{n} c_i N_i} \tag{2.57}$$

where c_i are the independent coefficients to be determined by the WR process and the functions ϕ and N_i are preselected such that $\hat{u}$ satisfies all the prescribed boundary conditions. Let $R_d\ (x, y, z)$ be the domain residual. The WR statement can then be written as

$$\boxed{\int_\Omega W_i R_d \, d\Omega = 0 \quad \text{for } i = 1, 2, \ldots, n} \tag{2.58}$$

where $W_i = N_i$.

It is observed that this criterion is implicitly satisfied by the exact solution *for any and every weighting function* since R_d is trivially zero everywhere. Thus the principal idea of

weighted residual approach is that, if we are able to satisfy this criterion for a sufficiently large number of independent weighting functions, then it is likely that the assumed solution will be reasonably close to the exact solution. More strongly, if we have taken a series representation (e.g. polynomial, trigonometric) for the trial function (and, therefore, W), we expect that our result will get better as we include more terms in the series. Thus we expect good convergence properties.

Example 2.7. To show the application of the Galerkin technique to a more complex situation, consider the simply supported rectangular plate subjected to uniform load as shown in Figure 2.11. The governing differential equation is given by

$$\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - q_0 = 0 \tag{2.59}$$

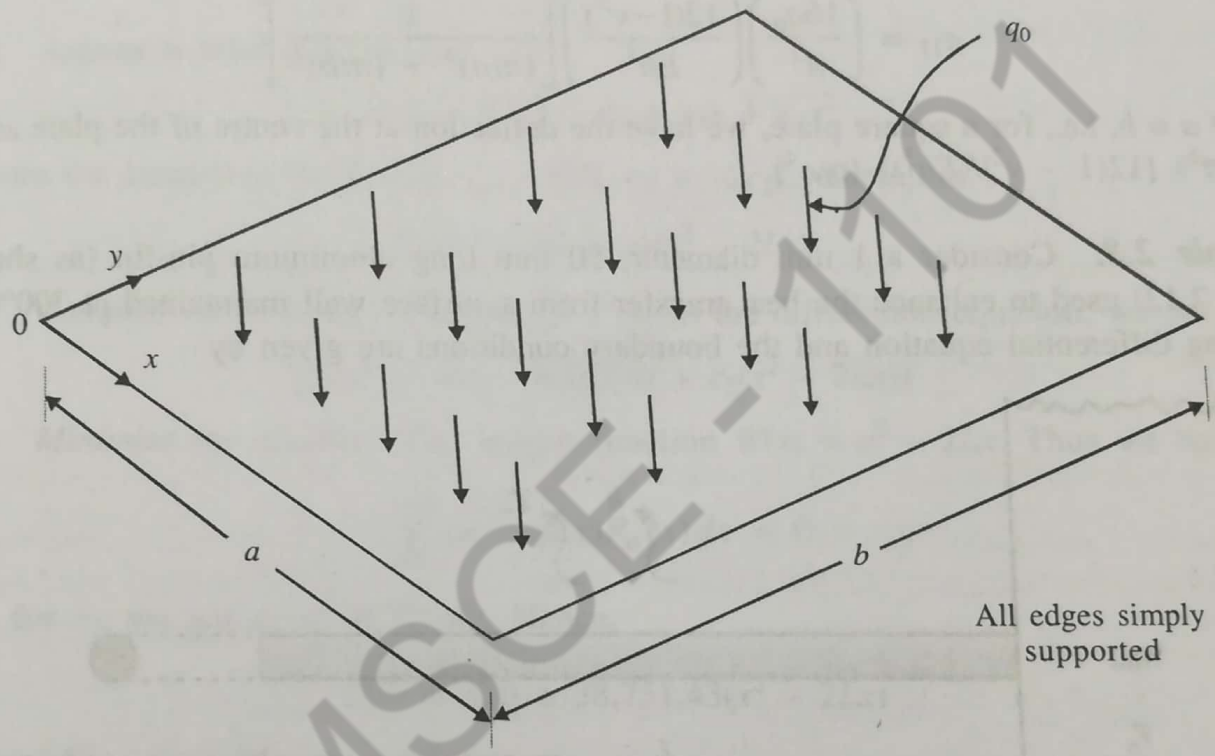


Fig. 2.11 A simply supported plate.

and the boundary conditions by

$$w(0, y) = 0 = w(a, y); \qquad \partial^2 w/\partial x^2 = 0 \text{ for } x = 0 \text{ and } a$$
$$w(x, 0) = 0 = w(x, b) \qquad \partial^2 w/\partial y^2 = 0 \text{ for } y = 0 \text{ and } b$$

Step 1: *Assume a trial or guess solution.* In view of the boundary conditions for simple supports, we will choose a trigonometric function for trial solution.

Let us choose a one-term trial function given by

$$w(x, y) \approx \hat{w}(x, y) = c_{11} \sin(\pi x/a) \sin(\pi y/b) \tag{2.60}$$

It is easy to see that our trial function satisfies all the boundary conditions of the problem.

Step 2: *Find the domain residual.* The domain residual is obtained by substituting $\hat{w}(x, y)$ in the differential equation as follows:

$$R_d = \frac{Eh^3}{12(1-\nu^2)} c_{11} \left[\left(\frac{\pi}{a} \right)^2 + \left(\frac{\pi}{b} \right)^2 \right]^2 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} - q_0 \tag{2.61}$$

Step 3: *Minimise the residual.* The weighting function is $W(x, y) = \sin(\pi x/a) \sin(\pi y/b)$. The weighted residual statement will now be

$$\int_0^b \int_0^a \left(\sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right) \left\{ \frac{Eh^3}{12(1-\nu^2)} c_{11} \left[(\pi/a)^2 + (\pi/b)^2 \right]^2 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} - q_0 \right\} dx\, dy = 0 \tag{2.62}$$

Solving for c_{11}, we get

$$c_{11} = \left(\frac{16 q_0}{\pi^2} \right) \left(\frac{12(1-\nu^2)}{Eh^3} \right) \left[\frac{1}{(\pi/a)^2 + (\pi/b)^2} \right]^2 \tag{2.63}$$

If $a = b$, i.e., for a square plate, we have the deflection at the centre of the plate as given by $(4/\pi^6)\ [12(1 - \nu^2)/(Eh^3)]\ (q_0 a^4)$.

Example 2.8. Consider a 1 mm diameter, 50 mm long aluminium pin-fin (as shown in Figure 2.12) used to enhance the heat transfer from a surface wall maintained at 300°C. The governing differential equation and the boundary conditions are given by

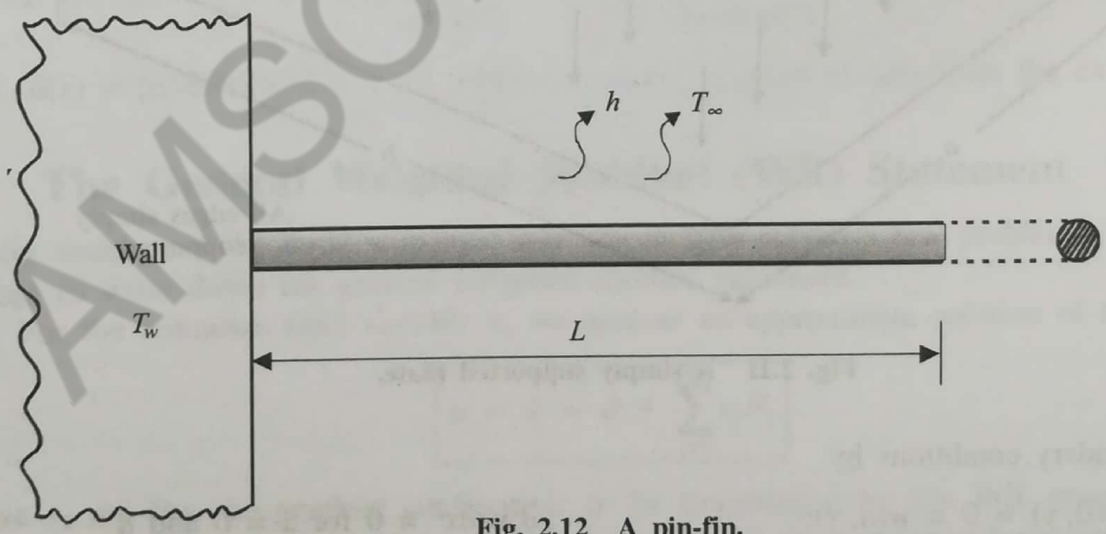


Fig. 2.12 A pin-fin.

$$k \frac{d^2T}{dx^2} = \frac{Ph}{A_c} (T - T_\infty)$$

$$T(0) = T_w = 300°C$$

$$\frac{dT}{dx}(L) = 0 \qquad \text{(insulated tip)} \tag{2.64}$$

where

k = coefficient of thermal conductivity,
P = perimeter,
A_c = cross-sectional area,
h = convective heat transfer coefficient,
T_w = wall temperature,
T_∞ = ambient temperature.

Let k = 200 W/m/°C for aluminium, h = 20 W/m^2°C, T_∞ = 30°C. Estimate the temperature distribution in the fin using the Galerkin weighted residual method.

With the given numerical values, we have

$$\frac{d^2T}{dx^2} = 400(T - 30) \tag{2.65}$$

with $T(0) = 300°C$, $\frac{dT}{dx}(L) = 0$.

Step 1: *Assume a trial solution.* Let

$$T(x) \approx \hat{T}(x) = c_0 + c_1x + c_2x^2$$

From the boundary conditions, $c_0 = 300$, $c_1 = -2c_2L$. Therefore,

$$\hat{T}(x) = 300 + c_2(x^2 - 2Lx) \tag{2.66}$$

Step 2: *Compute the residual.* Substituting $\hat{T}(x)$ in the differential equation, we get

$$R_d(x) = 2c_2 - 400[270 + c_2(x^2 - 2Lx)] \tag{2.67}$$

Step 3: *Minimise the residual.* The weight function $W(x) = x^2 - 2Lx$. Thus we have

$$\int_0^L (x^2 - 2Lx)R_d(x)\,dx = 0 \tag{2.68}$$

Solving for c_2, we get c_2 = 38,751.43. Hence,

$$\hat{T}(x) = 300 + 38{,}751.43(x^2 - 2Lx) \tag{2.69}$$

We can readily obtain the exact solution as

$$T(x)\big|_{\text{exact}} = T_\infty + \left(\frac{T_w - T_\infty}{\cosh(mL)}\right)\cosh[m(L - x)]$$

where

$$m^2 = \frac{hP}{kA_c} \tag{2.70}$$

We observe that the exact solution is an exponential function. The approximate solution obtained just now is quadratic. We can improve our approximation by taking higher degree terms in the polynomial trial function. Let

$$\hat{T}(x) = c_0 + c_1x + c_2x^2 + c_3x^3 \tag{2.71}$$

From the boundary conditions, $c_0 = 300°C$.

$$c_1 = -2c_2L - 3c_3L^2 \tag{2.72}$$

Thus,

$$\hat{T}(x) = 300 + c_2(x^2 - 2Lx) + c_3(x^3 - 3L^2x) \tag{2.73}$$

The weighted residual equations can be developed using the two weighting functions $(x^2 - 2Lx)$ and $(x^3 - 3L^2x)$. We can get the solution as

$$c_2 = 48{,}860.4, \qquad c_3 = -109{,}229 \tag{2.74}$$

Thus,

$$\hat{T}(x) = 300 + 48{,}860.4(x^2 - 2Lx) - 109{,}229(x^3 - 3L^2x) \tag{2.75}$$

We can further improve upon our solution by taking one more term in the series, i.e.

$$\hat{T}(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 \tag{2.76}$$

From the boundary conditions we have

$$c_0 = 300, \qquad c_1 = -2c_2L - 3c_3L^2 - 4c_4L^3 \tag{2.77}$$

Thus,

$$\hat{T}(x) = 300 + c_2(x^2 - 2Lx) + c_3(x^3 - 3L^2x) + c_4(x^4 - 4L^3x) \tag{2.78}$$

We can develop the three weighted residual equations using the weighting functions $(x^2 - 2Lx)$, $(x^3 - 3L^2x)$ and $(x^4 - 4L^3x)$. We can get the solution as

$$c_2 = 53{,}702.8, \qquad c_3 = -255{,}668, \qquad c_4 = 1.32 \times 10^6$$

Therefore,

$$\hat{T}(x) = 300 + 53{,}702.8\,(x^2 - 2Lx) - 255{,}668\,(x^3 - 3L^2x) + 1.32 \times 10^6\,(x^4 - 4L^3x) \tag{2.79}$$

Table 2.1 compares the various approximate solutions with the exact solution. It is observed that the accuracy of the Galerkin approximate solutions can be systematically improved by taking more and more terms in the series solution. However, the mathematical

Table 2.1 Comparison of Various Solutions for Temperature in a Fin

Axial location	Quadratic solution	Cubic solution	Quartic solution	Exact solution
0	300	300	300	300
0.005	281.59	280.87	280.75	280.75
0.01	265.12	264.11	264.00	264.02
0.015	250.59	249.62	249.60	249.62
0.02	238.00	237.33	237.39	237.43
0.025	227.34	227.16	227.27	227.31
0.03	218.62	219.02	219.12	219.16
0.035	211.84	212.83	212.86	212.91
0.04	207.00	208.51	208.43	208.49
0.045	204.09	205.98	205.79	205.85
0.05	203.12	205.16	204.91	204.97

Chapter 2

MATHEMATICAL PRELIMINARIES, INTEGRAL FORMULATIONS, AND VARIATIONAL METHODS

2.1 GENERAL INTRODUCTION

2.1.1 Variational Principles and Methods

This chapter is devoted to a review of some mathematical preliminaries that prove to be useful in the sequel and to a study of integral formulations and more commonly used variational methods such as the Ritz, Galerkin, collocation, and least-squares methods. Since the finite element method can be viewed as an elementwise application of a variational method (see Section 1.4), it is useful to learn how variational methods work. We begin with a discussion of the general meaning of the phrases "variational methods" and "variational formulations" used in the context of finite element formulations.

The phrase "direct variational methods" refers to methods that make use of variational principles, such as the principles of virtual work and the principle of minimum total potential energy in solid and structural mechanics, to determine approximate solutions of problems [see Oden and Reddy (1983) and Reddy (2002)]. In the classical sense, a *variational principle* has to do with finding the extremum (i.e., minimum or maximum) or stationary values of a functional with respect to the variables of the problem. The functional includes all the intrinsic features of the problem, such as the governing equations, boundary and/or initial conditions, and constraint conditions, if any. In solid and structural mechanics problems, the functional represents the total energy of the system, and in other problems, it is simply an integral representation of the governing equations.

Variational principles have always played an important role in mechanics (see the references at the end of the chapter). First, many problems of mechanics are posed in terms of finding the extremum (i.e., minima or maxima) and thus, by their nature, can be formulated in terms of variational statements. Second, there are problems that can be formulated by other means, such as the conservation laws (as illustrated in Chapter 1), but these can also be formulated by means of variational principles. Third, variational formulations form a powerful basis for obtaining approximate solutions to practical problems, many of which

are intractable otherwise. The principle of minimum total potential energy, for example, can be regarded as a substitute to the equations of equilibrium of an elastic body as well as a basis for the development of displacement finite element models that can be used to determine approximate displacement and stress fields in the body [Reddy (2002)]. Variational formulations can also serve to unify diverse fields, suggest new theories, and provide a powerful means to study the existence and uniqueness of solutions to problems. Similarly, Hamilton's principle [see Reddy (2002)] can be used in lieu of the equations governing dynamical systems, and the variational forms presented by Biot (1972) replace certain equations in linear continuum thermodynamics.

2.1.2 Variational Formulations

The classical use of the phrase "variational formulations" refers to the construction of a functional (whose meaning will be made clear shortly) or a variational principle that is equivalent to the governing equations of the problem. The modern use of the phrase refers to the formulation in which the governing equations are translated into equivalent weighted-integral statements that are not necessarily equivalent to a variational principle. Even those problems that do not admit variational principles in the classical sense (e.g., the Navier–Stokes equations governing the flow of viscous or inviscid fluids) can now be formulated using weighted-integral statements.

The importance of variational formulations of physical laws, in the modern or general sense of the phrase, goes far beyond its use as simply an alternate to other formulations [Oden and Reddy (1983)]. In fact, variational forms of the laws of continuum physics may be the only natural and rigorously correct way to think of them. While all sufficiently smooth fields lead to meaningful variational forms, the converse is not true: There exist physical phenomena which can be adequately modeled mathematically only in a variational setting; they are nonsensical when viewed locally.

The starting point for the discussion of the finite element method is differential equations governing the physical phenomena under study. As such, we shall first discuss why integral statements of the differential equations are needed.

2.1.3 Need for Weighted-Integral Statements

In almost all approximate methods used to determine the solution of differential and/or integral equations, we seek a solution in the form

$$u(\mathbf{x}) \approx U_N(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}) \tag{2.1.1}$$

where u represents the solution of a particular differential equation and associated boundary conditions, and U_N is its approximation that is represented as a linear combination of unknown parameters c_j and known functions ϕ_j of position $\mathbf{x}$ in the domain Ω on which the problem is posed. We shall shortly discuss the conditions on ϕ_j. The approximate solution U_N is completely known only when c_j are known. Thus, we must find a means to determine c_j such that U_N satisfies the equations governing u. If somehow we can find U_N that satisfies the differential equation at every point $\mathbf{x}$ of the domain Ω

and conditions on the boundary Γ of Ω, then $U_N(\mathbf{x}) = u(\mathbf{x})$, which is the exact solution of the problem. Of course, approximate methods are not about problems for which exact solutions can be determined by some methods of mathematical analysis; the role of approximate methods is to find an approximate solution of problems that do not admit analytical solutions. When the exact solution cannot be determined, the alternative is to find a solution U_N that satisfies the governing equations in an approximate way. In the process of satisfying the governing equations approximately, we obtain (not accidentally but by planning) N algebraic relations among the N parameters $c_1, c_2, \cdots, c_N$. A detailed discussion of these ideas is given in the next few paragraphs in connection with a specific problem.

Consider the problem of solving the differential equation

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] + c(x)u = f(x) \quad \text{for } 0 < x < L \tag{2.1.2a}$$

subjected to the boundary conditions

$$u(0) = u_0, \quad \left[a(x)\frac{du}{dx}\right]_{x=1} = Q_0 \tag{2.1.2b}$$

where $a(x)$, $c(x)$, and $f(x)$ are known functions, u_0 and Q_0 are known parameters, and $u(x)$ is the function to be determined. The set $a(x)$, $c(x)$, $f(x)$, u_0, and Q_0 is called the problem *data*. An example of the above problem is given by the heat transfer in an uninsulated rod (see Example 1.2.2): here u denotes the temperature (θ), $f(x)$ is the internal heat generation per unit length (Ag), $a(x)$ is the thermal resistance (kA), $c = \beta P$, u_0 is the specified temperature (θ_0), and Q_0 is the specified heat.

We seek an approximate solution over the entire domain $\Omega = (0, L)$ in the form

$$U_N \equiv \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x) \tag{2.1.3}$$

where the c_j are coefficients to be determined and $\phi_j(x)$ and $\phi_0(x)$ are functions chosen such that the specified boundary conditions of the problem are satisfied by the N-parameter approximate solution U_N. Note that the particular form in (2.1.3) has two parts: one containing the unknowns ($\sum c_j \phi_j$) that is termed the homogeneous part and the other is the nonhomogeneous part (ϕ_0) that has the sole purpose of satisfying the specified boundary conditions of the problem. Since ϕ_0 satisfies the boundary conditions, the sum $\sum c_j \phi_j$ must satisfy, for arbitrary c_j, the homogeneous form of the boundary conditions ($Bu = \hat{u}$ is said to be a nonhomogeneous boundary condition when $\hat{u} \neq 0$, and it is termed a homogeneous boundary condition when $\hat{u} = 0$; here B denotes some operator). Thus, in the present case, the actual boundary conditions are both nonhomogeneous ($B = 1$ and $\hat{u} = u_0$ at $x = 0$, and $B = a(x)(d/dx)$ and $\hat{u} = Q_0$ at $x = L$). The particular form (2.1.3) is convenient in selecting ϕ_0 and ϕ_j. Thus, ϕ_0 and ϕ_j satisfy the conditions

$$B\phi_0 = \hat{u}, \quad B\phi_j = 0 \text{ for all } j = 1, 2, \cdots, n \tag{2.1.4}$$

To be more specific, let $L = 1$, $u_0 = 1$, $Q_0 = 0$, $a(x) = x$, $c(x) = 1$, $f(x) = 0$, and $N = 2$. Then we choose the approximate solution in the form

$$U_2 = c_1\phi_1 + c_2\phi_2 + \phi_0 \text{ with } \phi_0 = 1,\ \phi_1(x) = x^2 - 2x,\ \phi_2 = x^3 - 3x$$

that satisfies the boundary conditions (2.1.2*b*) of the problem for any values of c_1 and c_2 because

$$\phi_0(0)=1, \quad \left(x\frac{d\phi_0}{dx}\right)_{x=1}=0; \quad \phi_j(0)=0, \quad \left(\frac{d\phi_j}{dx}\right)_{x=1}=0 \text{ for } j=1,2 \tag{2.1.5}$$

To make U_2 satisfy the differential equation (2.1.2*a*), we must have

$$-\frac{dU_2}{dx}-x\frac{d^2U_2}{dx^2}+U_2=-2c_1(x-1)-3c_2(x^2-1)-2c_1x-6c_2x^2$$
$$+c_1(x^2-2x)+c_2(x^3-3x)+1=0 \tag{2.1.6}$$

Since this expression must be zero for any value of x, the coefficients of the various powers of x must be zero:

$$1+2c_1+3c_2=0$$
$$-(6c_1+3c_2)=0$$
$$c_1-9c_2=0$$
$$c_2=0$$

The above relations are inconsistent; hence, there is *no solution* to the equations. On the other hand, we can require the approximate solution U_N to satisfy the differential equation (2.1.2*a*) in the weighted-integral sense,

$$\int_0^1 w(x)R\,dx=0 \tag{2.1.7}$$

where R denotes the left side of the equality in (2.1.6) and is called the *residual*,

$$R\equiv-\frac{dU_N}{dx}-x\frac{d^2U_N}{dx^2}+U_N$$

and $w(x)$ is called a *weight function*. From (2.1.7), we obtain as many linearly independent equations as there are linearly independent functions for $w(x)$. The number of linearly independent choices of w must be restricted to $N=2$ so that we have exactly the same number of equations as the number of unknown coefficients, c_j. For example, in the present example, if we take $w=1$ and $w=x$, we obtain

$$0=\int_0^1 1\cdot R\,dx=(1+2c_1+3c_2)+\frac{1}{2}(-6c_1-3c_2)+\frac{1}{3}(c_1-9c_2)+\frac{1}{4}c_2$$
$$0=\int_0^1 x\cdot R\,dx=\frac{1}{2}(1+2c_1+3c_2)+\frac{1}{3}(-6c_1-3c_2)+\frac{1}{4}(c_1-9c_2)+\frac{1}{5}c_2$$

or

$$\frac{2}{3}c_1+\frac{5}{4}c_2=1, \quad \frac{3}{4}c_1+\frac{31}{20}c_2=\frac{1}{2} \tag{2.1.8}$$

which provides two linearly independent equations for c_1 and c_2 (whose solution is $c_1=\frac{222}{23}$ and $c_2=-\frac{100}{23}$).

The above discussion clearly demonstrates the need for weighted-integral statements of the type in (2.1.7); they provide the means for obtaining as many algebraic equations

as there are unknown coefficients in the approximate solution. This chapter deals with the construction of different types of integral statements used in different variational methods. The variational methods differ from each other in the choice of the weight function w and/or the integral statement used, which in turn dictates the choice of the approximation functions ϕ_j. In the finite element method, a given domain is viewed as an assemblage of subdomains (i.e., finite elements), and an approximate solution is sought over each subdomain in the same way as in variational methods. Therefore, it is informative to study variational methods before we study the finite element method.

Our goal in this chapter is to illustrate the basic steps in the weighted-integral formulations and associated approximations of boundary value, eigenvalue, and initial value problems. Toward this goal, we first introduce the necessary terminology and notation.

Boundary Value Problems. *Steady State Heat Transfer in a Fin and Axial Deformation of a Bar* [Fig. 2.2.3(*a*)]: Find $u(x)$ that satisfies the second-order differential equation and *boundary conditions*:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu=f \quad \text{for} \quad 0<x<L \tag{2.2.13a}$$

$$u(0)=u_0, \quad \left(a\frac{du}{dx}\right)_{x=L}=q_0 \tag{2.2.13b}$$

Bending of Elastic Beams under Transverse Load: Find $u(x)$ that satisfies the fourth-order differential equation and *boundary conditions*:

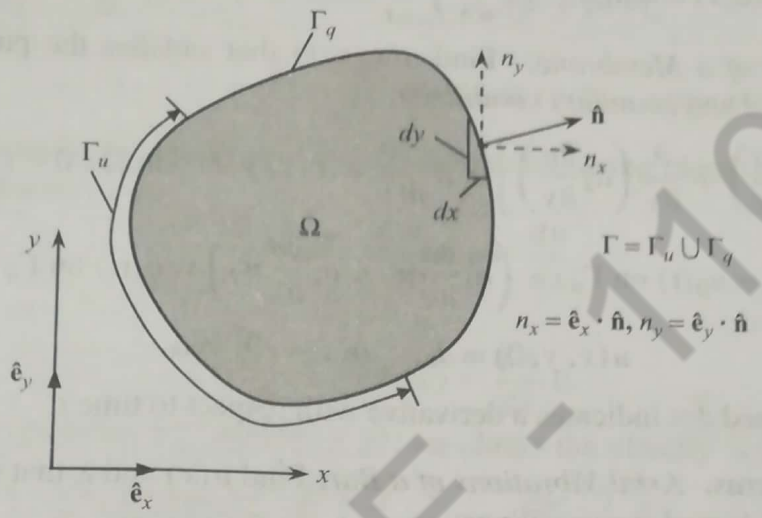
$$\frac{d^2}{dx^2}\left(b\frac{d^2u}{dx^2}\right)+cu=f \quad \text{for} \quad 0<x<L \tag{2.2.14a}$$

$$u(0)=u_0, \quad \left(\frac{du}{dx}\right)_{x=0}=d_0$$

$$\left[\frac{d}{dx}\left(b\frac{d^2u}{dx^2}\right)\right]_{x=L}=m_0, \quad \left(b\frac{d^2u}{dx^2}\right)_{x=L}=v_0 \tag{2.2.14b}$$

Steady Heat Conduction in a Two-Dimensional Region and *Transverse Deflections of a Membrane* [Fig. 2.2.3(*b*)]: Find $u(x, y)$ that satisfies the second-order partial differential

(*a*) A one-dimensional domain



(*b*) A two-dimensional domain

Figure 2.2.3 (*a*) One-dimensional domain. (*b*) Two-dimensional domain.

equation and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]+cu=f \quad \text{in } \Omega \tag{2.2.15a}$$

$$u=u_0 \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial x}n_y\right)=q_0 \text{ on } \Gamma_q \tag{2.2.15b}$$

where (n_x, n_y) are the direction cosines of the unit normal vector $\hat{\mathbf{n}}$ to the boundary Γ_q.

potential Π_p. Based on the PSTP, we set up the necessary equations to solve for the unknown coefficients. We will now discuss the details of the Rayleigh–Ritz method, which is popularly used in structural mechanics.

3.3.1 Rayleigh–Ritz Method

The Rayleigh–Ritz (R–R) method consists of three basic steps:

Step 1: *Assume a displacement field.* Let the displacement field be given by $\{\phi(x) + \sum c_i N_i\}$, $i = 1, 2, \ldots, n$, where N_i are the shape functions and c_i are the as yet undetermined coefficients. It is observed that the assumed displacement field should satisfy both the essential boundary conditions of the problem and internal compatibility, i.e. there should be no kinks, voids, etc. within the structure.

Step 2: *Evaluation of the total potential.* For the system under consideration, evaluate the total potential Π_p consistent with the assumed displacement field in Step 1 above.

Step 3: *Set up and solve the system of equations.* By virtue of the PSTP, the total potential will be stationary with respect to small variations in the displacement field. The variations in the displacement field in our case are attained by small variations in the coefficients c_i. Thus we have

$$\frac{\partial \Pi_p}{\partial c_i} = 0, \qquad i = 1, 2, \ldots, n \tag{3.41}$$

which will yield the necessary equations to be solved for the coefficients c_i.

This method was first formulated by Rayleigh (1877) and later refined and generalised by Ritz (1908). Rayleigh worked with just one term, viz., $c_1 N_1$ while Ritz extended the technique to an n-term approximation. However, both of them used shape functions N which were single composite functions valid over the entire domain of the problem. Further extension of their technique by using piece-wise defined shape functions N_i will lead us to the variational formulation of the finite element method.

We will now illustrate the basic scheme of R–R method with some simple example problems.

Example 3.1. *A bar under uniform load.* Consider a bar clamped at one end and left free at the other end and subjected to a uniform axial load q_0 as shown in Figure 3.4. The governing differential equation is given by

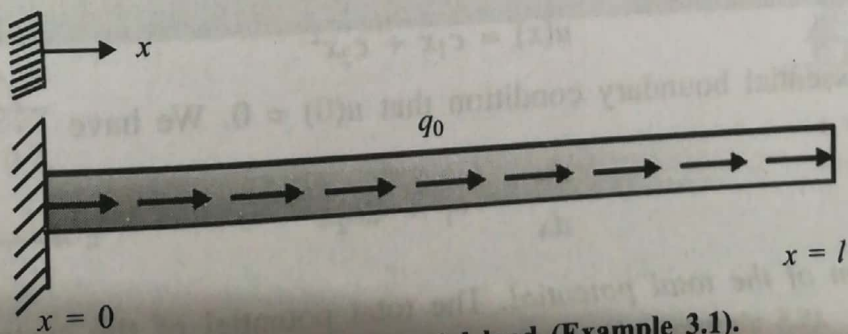


Fig. 3.4 Rod under axial load (Example 3.1).

$$AE\frac{d^2u}{dx^2} + q = 0 \tag{3.42}$$

with the boundary conditions $u(0) = 0$; $\left.\frac{du}{dx}\right|_{x=L} = 0$.

We discussed the solution using the Galerkin weighted residual method in Example 3.1. We will now illustrate the solution using the R–R method. For a general deformation $u(x)$,

$$\text{Strain energy stored in the bar, } U = \int_0^L \left[\frac{1}{2}AE\left(\frac{du}{dx}\right)^2\right] dx \tag{3.43}$$

$$\text{Potential of the external forces, } V = -\int_0^L q_0 u\, dx - P_L u(L) \tag{3.44}$$

where, for the sake of generality, a tip load $P_L = AE\left.\frac{du}{dx}\right|_{x=L}$ has been included in the potential of external forces. In view of the second boundary condition, the expression for the potential of external forces reduces (for this example) to the following:

$$V = -\int_0^L q_0 u\, dx \tag{3.45}$$

Thus we need to find $u(x)$ that minimises the total potential of the system given by

$$\Pi_P = \int_0^L \left[\frac{1}{2}AE\left(\frac{du}{dx}\right)^2 - q_0 u\right] dx \tag{3.46}$$

subject to the essential boundary condition that $u(0) = 0$. It is to be observed that the force boundary condition at the free end ($du/dx\,|_{x=L} = 0$) has been incorporated in the expression for the functional itself and thus the trial function assumed need only satisfy the essential boundary condition. Also, we observe that the continuity demanded on $u(x)$ is lower (i.e. need to be differentiable only once) compared to the differential equation form.

Let us now solve for the displacement field using the R–R method.

Step 1: *Assume a displacement field.* Let us assume that

$$u(x) \approx c_1 x + c_2 x^2 \tag{3.47}$$

This satisfies the essential boundary condition that $u(0) = 0$. We have

$$\frac{du}{dx} = c_1 + 2c_2 x \tag{3.48}$$

Step 2: *Evaluation of the total potential.* The total potential of the system is given as

$$\Pi_p = \int_0^L \left[\frac{AE}{2}(c_1 + 2c_2 x)^2 - q_0(c_1 x + c_2 x^2) \right] dx$$

$$= \frac{AE}{2}\left[c_1^2 L + \frac{4c_2^2}{3L^3} + 2c_1 c_2 L^2 \right] - q_0 \frac{c_1 L^2}{2} - q_0 c_2 \frac{L^3}{3} \tag{3.49}$$

Step 3: *Set up and solve the system of equations.* From the principle of stationary total potential (PSTP), we have

$$\frac{\partial \Pi_p}{\partial c_i} = 0, \qquad i = 1, 2 \tag{3.50}$$

Therefore,

$$\frac{\partial \Pi_p}{\partial c_i} = 0 \Rightarrow \frac{AE}{2}(2c_1 L + 2c_2 L^2) - \frac{q_0 L^2}{2} = 0$$

$$\frac{\partial \Pi_p}{\partial c_2} = 0 \Rightarrow \frac{AE}{2}(8c_2 L^3/3 + 2c_1 L^2) - \frac{q_0 L^3}{3} = 0 \tag{3.51}$$

Solving, we obtain

$$c_1 = \frac{q_0 L}{AE}, \qquad c_2 = -\frac{q_0}{2AE}$$

Thus,

$$u(x) = \frac{q_0}{AE} x(L - x/2) = \frac{q_0}{2AE}(2Lx - x^2) \tag{3.52}$$

which is observed to be the same as that obtained in Example 2.1.

Example 3.2. *A simply supported beam under uniform load.* Consider a simply supported beam under uniformly distributed load q_0 as shown in Figure 3.5. For a deformation $v(x)$, we have

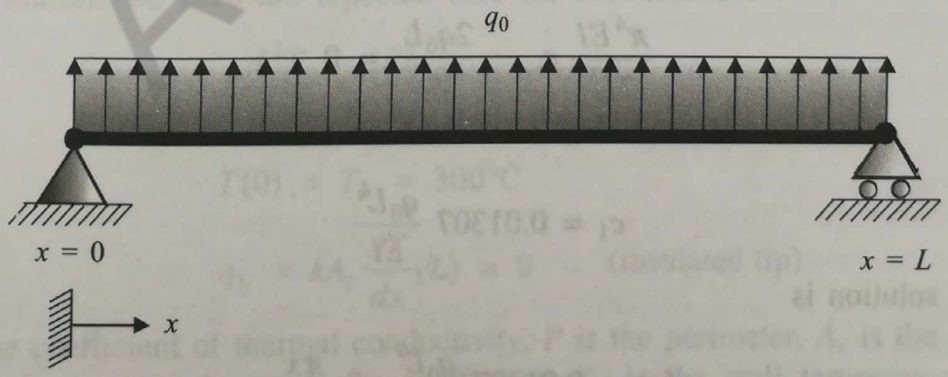


Fig. 3.5 Simply supported beam under load (Example 3.2).

The strain energy

$$U = \int_0^L \frac{1}{2} EI \left(\frac{d^2 v}{dx^2} \right)^2 dx \tag{3.5}$$

The potential of the external forces is

$$V = -\int_0^L q_0 v \, dx \tag{3.5}$$

Thus we have the total potential

$$\Pi_p = \int_0^L \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 - q_0 v \right] dx \tag{3.5}$$

Step 1: *Assume a displacement field.* Let us assume $v(x) \approx c_1 \sin(\pi x/L)$. This satisfies the essential boundary conditions $v(0) = 0 = v(L)$. We have

$$\frac{d^2 v}{dx^2} = -c_1 \left(\frac{\pi}{L} \right)^2 \sin \frac{\pi x}{L} \tag{3.5}$$

Step 2: *Evaluation of the total potential.* The total potential of the system is given by

$$\Pi_p = \int_0^L \left[\frac{EI}{2} \left(-c_1 \left(\frac{\pi}{L} \right)^2 \sin \frac{\pi x}{L} \right)^2 dx - q_0 c_1 \sin \frac{\pi x}{L} \right] dx$$

$$= \frac{\pi^4 EI}{4 L^3} c_1^2 - \frac{2 q_0 L}{\pi} c_1 \tag{3.57}$$

Step 3: *Set up and solve the system of equations.* From the principle of stationary total potential, we have

$$\frac{\partial \Pi_p}{\partial c_1} = 0 \tag{3.58}$$

Therefore,

$$\frac{\pi^4 EI}{2L^3} c_1 - \frac{2 q_0 L}{\pi} = 0$$

i.e.

$$c_1 = 0.01307 \frac{q_0 L^4}{EI} \tag{3.59}$$

Thus the final solution is

$$v(x) = 0.01307 \frac{q_0 L^4}{EI} \sin \frac{\pi x}{L} \tag{3.60}$$

We observe that this solution is identical to the solution obtained from the Galerkin method (ref. Example 2.4). In the Galerkin method, the original Weighted Residual statement was equivalent to only the differential equation and, therefore, required that the trial solution assumed should satisfy all the boundary conditions of the problem (essential and natural). The R–R method is based on the functional in which the external forces, if any, applied at the boundaries are taken into account through the potential of the external forces. Thus the trial solution for the R–R method need only satisfy the essential boundary conditions. In general, if the differential equation form and the functional form of a given problem are both available, then the Galerkin method and the R–R method yield identical solutions when the problem involves only essential boundary conditions and when they both use the same shape functions.

Example 3.3. *Temperature distribution in a pin-fin.* Consider a 1 mm diameter, 50 mm long aluminium pin-fin as shown in Figure 3.6, used to enhance the heat transfer from a surface wall maintained at 300°C. Let k = 200 W/m/°C for aluminium; h = 20 W/m^2/°C, T_∞ = 30°C.

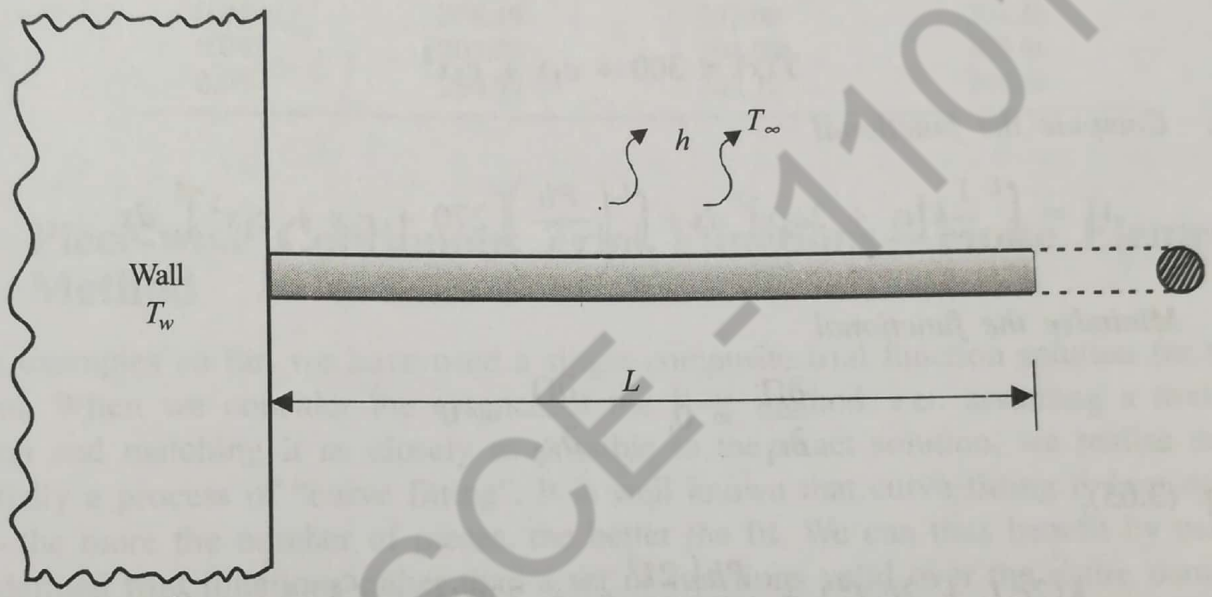


Fig. 3.6 A pin-fin (Example 3.3).

The temperature distribution in the fin was obtained using the Galerkin method in Example 2.8 based on the governing differential equation. The governing differential equation and the boundary conditions we used are repeated here for convenience:

$$k\frac{d^2T}{dx^2} = \frac{Ph}{A_c}(T - T_\infty)$$

$$T(0) = T_w = 300°C \tag{3.61}$$

$$q_L = kA_c\frac{dT}{dx}(L) = 0 \qquad \text{(insulated tip)}$$

where k is the coefficient of thermal conductivity, P is the perimeter, A_c is the cross-sectional area, h is the convective heat transfer coefficient, T_w is the wall temperature, and T_∞ the ambient temperature. The equivalent functional representation is given by

$$\Pi = \int_0^L \frac{1}{2} k \left(\frac{dT}{dx} \right)^2 dx + \int_0^L \frac{1}{2} \frac{Ph}{A_c} (T - T_\infty)^2 \, dx - q_L T_L \tag{3.62}$$

where, in general, the heat flux q_L is included. However, in view of the insulated tip boundary condition, the equivalent statement of the problem is to find $T(x)$ that minimises the functional

$$\Pi = \int_0^L \frac{1}{2} k \left(\frac{dT}{dx} \right)^2 dx + \int_0^L \frac{1}{2} \frac{Ph}{A_c} (T - T_\infty)^2 \, dx \tag{3.63}$$

subject to the boundary condition that $T(0) = T_w = 300°C$.

Step 1: *Assume a trial solution.* Let

$$T(x) \approx \hat{T}(x) = c_0 + c_1 x + c_2 x^2$$

From the boundary conditions, $c_0 = 300$. Therefore,

$$\hat{T}(x) = 300 + c_1 x + c_2 x^2 \tag{3.64}$$

Step 2: *Compute the functional*

$$\Pi = \int_0^L \frac{1}{2} k \left[c_1 + 2c_2 x \right]^2 dx + \int_0^L \left(\frac{Ph}{2A_c} \right) \left[270 + c_1 x + c_2 x^2 \right]^2 dx \tag{3.65}$$

Step 3: *Minimise the functional*

$$\frac{\partial \Pi}{\partial c_1} = 0, \qquad \frac{\partial \Pi}{\partial c_2} = 0 \tag{3.66}$$

From Eq. (3.65),

$$k(2c_1 L + 2c_2 L^2) + \frac{Ph}{A_c} \left(\frac{2L^3}{3} c_1 + 270L^2 + \frac{c_2}{2} L^4 \right) = 0$$

$$k \left(\frac{8}{3} c_2 L^3 + 2c_1 L^2 \right) + \frac{Ph}{A_c} \left(\frac{2}{5} L^5 c_2 + \frac{c_2}{2} L^4 + 180L^3 \right) = 0 \tag{3.67}$$

On substituting the numerical values and solving for the two coefficients c_1 and c_2, we get

$$c_1 = -3923.36, \qquad c_2 = 40{,}498.44 \tag{3.68}$$

Thus our approximate solution for temperature, based on the minimisation of the functional, is given as follows:

$$T(x) = 300 - 3923.36x + 40{,}498.44x^2 \tag{3.69}$$

The Galerkin WR solution obtained in Example 2.8 is reproduced here from Eq. (2.69) for ready reference:

$$T(x) = 300 + 38{,}751.43(x^2 - 2Lx) \tag{3.70}$$

The two approximations are compared with the exact solution in Table 3.1. Even though both trial solutions are quadratic, the boundary conditions are different. The R–R trial solution was $c_1x + c_2x^2$ (with two *independent* parameters c_1 and c_2) while the original Galerkin WR method used $c_2(x^2 - 2Lx)$ after satisfying both the boundary conditions.

Table 3.1 Comparison of Galerkin WR and R–R Solutions (Example 3.3)

Axial location	Exact solution	Galerkin WR solution	R–R solution
0	300	300	300
0.005	280.75	281.59	281.40
0.01	264.02	265.12	264.82
0.015	249.62	250.59	250.26
0.02	237.43	238.00	237.73
0.025	227.31	227.34	227.23
0.03	219.16	218.62	218.75
0.035	212.91	211.84	212.29
0.04	208.49	207.00	207.86
0.045	205.85	204.09	205.46
0.05	204.97	203.12	205.08

Table 3.1.1 Steps involved in the finite element analysis of a typical problem.

1. Discretization (or representation) of the given domain into a collection of preselected finite elements. (This step can be postponed until the finite element formulation of the equation is completed.)
   (*a*) Construct the finite element mesh of preselected elements.
   (*b*) Number the nodes and the elements.
   (*c*) Generate the geometric properties (e.g., coordinates and cross-sectional areas) needed for the problem.
2. Derivation of element equations for all typical elements in the mesh.
   (*a*) Construct the variational formulation of the given differential equation over the typical element.
   (*b*) Assume that a typical dependent variable u is of the form

$$u = \sum_{i=1}^{n} u_i \psi_i$$

   and substitute it into Step 2*a* to obtain element equations in the form

$$[K^e]\{u^e\} = \{F^e\}$$

   (*c*) Select, if already available in the literature, or derive element interpolation functions ψ_i and compute the element matrices.
3. Assembly of element equations to obtain the equations of the whole problem.
   (*a*) Identify the interelement continuity conditions among the primary variables (relationship between the local degrees of freedom and the global degrees of freedom—connectivity of elements) by relating element nodes to global nodes.
   (*b*) Identify the "equilibrium" conditions among the secondary variables (relationship between the local source or force components and the globally specified source components).
   (*c*) Assemble element equations using Steps 3*a* and 3*b*.
4. Imposition of the boundary conditions of the problem.
   (*a*) Identify the specified global primary degrees of freedom.
   (*b*) Identify the specified global secondary degrees of freedom (if not already done in Step 3*b*).
5. Solution of the assembled equations.
6. Postprocessing of the results.
   (*a*) Compute the gradient of the solution or other desired quantities from the primary degrees of freedom computed in Step 5.
   (*b*) Represent the results in tabular and/or graphical form.