UNIT I

2 MARKS

29. Write the potential energy for beam of span "L" simply supported at ends, subjected to a concentrated lead 'p' midspan. Assume EI constant. (Nov/Dec 2008)

Potential energy for beam is given by

$$\pi = u - H$$

Strain energy $u = \frac{EI\pi^4}{4\ell^3}\left(a_1^2 + 81a_2^2\right)$

Workdone by enteral force, H = P (a1 +a2)

$$\pi = \frac{EI\pi^4}{4\ell^3}\left(a_1^2 + 82a_2^2\right) - p\left(a_1 - a_2\right)$$

30. Distinguish between 1D bar element and 1D beam element (Nov/Dec 2009)

	1D Bar element	1D Beam element
(i)	The element has axial deformation	The element has transverse and rotation deformation
(ii)	The stiffness matrix	The stiffness matrix has 4× 4

31. What is Galerkin method of approximation? (Nov/Dec 2009)

In this method the trail function N_i (x) itself is considered as the weighing functions that is

$$W_i = N_i(x)$$

$$\int_0 N_i(x)R(x : a_1 a_2 .....a_n)dx = 0$$

$$i = 1, 2, 3.........n$$

32. Classify boundary conditions. (NOV / DEC 2011)

a. Displacement Based Boundary conditions
b. Stress based boundary conditions (or) (a) Primary and (b) Secondary boundary conditions.

33. Name the weighted residual methods (NOV / DEC 2011)

a) Point collocation method.
b) Subdomain collocation method

c) Least squared method
d) Galerekin's method

34. What is the limitation of using a finite difference method? (MAY / JUNE 2010)

(i) Used to solve heat transfer, fluid mechanics of structured problems

(ii) Suitable for two dimensional regions with boundaries parallel to co-ordinate axies.

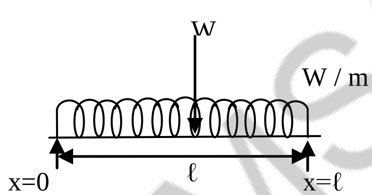
(iii) Difficult to use when regions have curved or irregular boundaries

35. List the various methods of solving boundary value problems. (MAY / JUNE 2010)

Finite difference method
Finite element method

16 MARKS

11 Determine the expression for deflection and bending moment in a simply supported beam subjected to uniformly distributed load over entire span. Find the deflection and moment at midspan and compase with race solution using Rayleigh. Ritz method. (Nov /Dec 2008)



Given

To find

1. Deflection and bending moment at midspan.

2. Compare with exact solutions.

Solution:

We know that, for simply supported beam, the Fourier series

is the approximating functions.

To make this series more simple let us consider only two terms.

Deflection,

Where a1, a2 are ritz parameters.

We know that,

Total potential energy of the beam, π = U– H

Where U = strain energy

H = workdone by enteral force.

The strain energy u, of the beam due to bending is given by

$$U=\frac{EI}{2}\int_0^1\left(\frac{d^2y}{dx^2}\right)^2 dx.$$

$$\frac{dy}{dx}=a_1\cos\frac{\pi x}{\ell}\times(\pi/\ell)+a_2\cos\frac{3\pi x}{\ell}(3\pi/\ell)$$

$$\frac{d^2y}{dx^2}=\left[\frac{-a_1\pi^2}{\ell^2}\sin\frac{\pi x}{\ell}=\frac{a_2 9\pi^2}{\ell^2}\sin^0\frac{3\pi x}{\ell}\right]$$

$$U=\frac{EI}{2}\int_0^1\left[-\frac{a_1\pi^2}{\ell^2}\sin\frac{\pi x}{\ell}=\frac{a_2 9\pi^2}{\ell^2}\sin\frac{3\pi x}{\ell}\right]^2 dx$$

$$U=\frac{EI\pi^4}{2\ell^4}\int_0^1\left[a_1^2\sin^2\frac{\pi x}{\ell}+81a_2^2\sin^2\frac{3\pi x}{\ell}+81a_1a_2\sin\frac{\pi x}{\ell}\sin\frac{3\pi x}{\ell}\right]dx$$

$$\left[(a+b)^2=a^2+b^2+2ab\right]$$

$$U=\frac{EI\pi^4}{2\ell^4}\int_0^1\left[a_1^2\sin\frac{\pi x}{\ell}+81a_2^2\sin^2\frac{3\pi x}{\ell}+18a_1a_2\sin\frac{\pi x}{\ell}\sin\frac{3\pi x}{\ell}\right]dx$$

$$\int_0^1 a_1^2\sin^2\frac{\pi x}{\ell}dx=a_1^2\int_0^1\frac{1}{2}\left(1-\cos\frac{2\pi x}{\ell}\right)dx$$

$$=\frac{a_1^2}{2}\int_0^1\left(1-\cos\frac{2\pi x}{\ell}\right)dx$$

$$=\frac{a_1^2}{2}\left[(x)_0^1-\left(\frac{\sin\frac{2\pi x}{\ell}}{\sin\frac{2\pi}{\ell}}\right)_0^1\right]$$

$$=\frac{a_1^2}{2}\left[1-\frac{1}{2\pi}(0-0)\right]$$

$$=\frac{a_1^2\ell}{2}$$

$$\int_0^1 a_1^2\sin^2\frac{\pi x}{\ell}dx=\frac{a_1^2\ell}{2}$$

Similarly,

$$\int_0^1 81a_2^2 \sin^2 \frac{3\pi x}{\ell} = 81a_2^2 \int_0^1 \frac{1}{2}\left(1 - \cos\frac{6\pi x}{\ell}\right)dx$$

$$\left[\sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= \frac{81a_2^2}{2}\left((x)_0^1 - \left(\frac{\sin\frac{6\pi x}{\ell}}{\frac{6\pi}{\ell}} \right)_0^1 \right)$$

$$= \frac{81a_2^2}{2}\left(\ell - \frac{1}{6\pi}(\sin 6\pi - \sin 0) \right)$$

$$= \frac{81a_2^2}{2}(\ell - 0)$$

$$\int_0^1 81a_2^2 \sin^2 \frac{3\pi x}{\ell} dx = \frac{81a_2^2 k\ell}{2}$$

$$\int_0^1 81a_1a_2 \sin\frac{\pi x}{\ell} \sin\frac{3\pi x}{\ell} = 18a_1a_2 \int_0^1 \sin\frac{\pi x}{\ell}\sin\frac{3\pi x}{\ell}$$

$$= 9a_1a_2(0-0) = 0$$

$$[\sin 2\pi = 0,\ \sin 4\pi = 0,\ \sin 0 = 0]$$

$$\int_0^1 18a_1a_2 \sin\frac{\pi x}{\ell}\sin\frac{3\pi x}{\ell} = 0$$

$$\therefore U = \frac{EI\pi^4}{2\ell^4}\left[\frac{a_1^2\ell}{2} + \frac{81a_2^2\ell}{2} + 0\right]$$

Strain energy $\quad U = \frac{EI\pi^4}{2\ell^4}\left(a_1^2 + 81a_2^2\right)$

We know that,

Work done by external force,

$$H = \int_0^l wydx = \int_0^l w\left(a_1 \sin\frac{\pi x}{\ell} + a_2 \sin\frac{3\pi x}{\ell}\right)dx$$

$$= w\int_0^l \left(a_1 \sin\frac{\pi x}{\ell} + a_2 \sin\frac{3\pi x}{\ell}\right)dx$$

$$= w\left(a_1\int_0^l \sin\frac{\pi x}{\ell}dx + a_2\int_0^l \sin\frac{3\pi x}{2}dx\right)$$

$$= w\left(\frac{-a_1\ell}{\pi}\left(\frac{\pi x}{\ell}\right)_0^l - \frac{a_2\ell}{3\pi}\left(\cos\frac{3\pi x}{\ell}\right)_0^\ell\right)$$

$$= w\left(\frac{-a_1\ell}{\pi}((-1)-1) = \frac{a_2\ell}{3\pi}(-1-1)\right)$$

$$= w\left(\frac{2a_1\ell}{\pi} + \frac{2a_2\ell}{3\pi}\right)$$

$$= \frac{2w\ell}{\pi}\left(a_1 + \frac{a_2}{3}\right)$$

$$H = \frac{2w\ell}{\pi}\left(a_1 + \frac{a_2}{3}\right)$$

So, $\pi = U - H$

$$= \frac{EI\pi^4}{\ell^4}\left(a_1^2 + 81a^2\right) = \frac{2w\ell}{\pi}\left(a_1 + \frac{a_2}{3}\right)$$

For stationary value of π, the following condition must be satisfied.

$$\frac{\partial\pi}{\partial a_1} = 0 \text{ and } \frac{\partial\pi}{\partial a_2} = 0$$

$$\frac{\partial\pi}{\partial a_1} = \frac{EI\pi^4}{4\ell^3}(2a_1) - \frac{2w\ell}{\pi} = 0$$

$$\frac{EI\pi^4}{4\ell^3} \times 2a_1 = \frac{2w\ell}{\pi}$$

$$a_1 = \frac{4w\ell^4}{EI\pi^5}$$

Similarly,

$$\frac{\partial\pi}{\partial a_2} = \frac{EI\pi^4}{4\ell^3}(162a_2) - \frac{2w\ell}{\pi}(1/3) = 0$$

$$a_2 = \frac{4w\ell^4}{243EI\pi^5}$$

$$y = a_1 \sin\frac{\pi x}{\ell} + a_2 \sin\frac{3\pi x}{\ell}$$

Substituting a_1 and a_2 values

$$y = \frac{4w\ell^4}{EI\pi^5}\sin\frac{\pi x}{\ell} + \frac{4w\ell^4}{243EI\pi^5}\sin\frac{3\pi x}{\ell}$$

We know that, maximum deflection accurse at x = l/2

$$y_{max} = \frac{4w\ell^4}{EI\pi^5}\sin\frac{\pi(l/2)}{\ell} + \frac{4w\ell^4}{243EI\pi^5}\sin\frac{3(l/2)}{1}$$

$$y_{max} = \frac{4w\ell^4}{EI\pi^5}\left(1-\frac{1}{243}\right)$$

$$y_{max} = 0.0130\frac{w\ell^4}{EI}$$

12 (i) What is constitute relationship? Express the constitute relations for a linear elastic isotropic material including initial stress and strain. (Nov/Dec 2009)

It gives the relationship the stress and strain. The stress can be given as

$$\{\sigma\} = [D]\ \{e\}$$

Where [D] = stress strain relationship matrix (or) constitutive matrix

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} (1-\mu) & \mu & \mu & 0 & 0 & 0 \\ \mu & (1-\mu) & \mu & 0 & 0 & 0 \\ \mu & \mu & (1-\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{1-2\mu}{2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{1-2\mu}{2}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

12 (ii) Consider the differential equation $\left(d^2y/dx^2\right) + 400x^2 = 0$ for $0 \le x \le 1$ **subject to boundary conditions y(0) = 0; y(1) = 0. The functional corresponding to this problem, to be extremized is given by**

$$I = \int_0^1 \left\{-0.5(dy/dx)^2 + 400x^2y\right\}$$

Find the solution of the problem using Rayleigh – Ritz method by considering a two – term solution as

$y(x) = c_1x(1-x) + c_2x^2(1-x)$ **(Nov/Dec 2009)**

$$Y(x) = C_1 x(1-x) - C_2 x^2(1-x)$$

$$= C_1(x - x^2) - C_2(x^2 - x) \qquad ----(A)$$

$$\frac{dy}{dx} = C_1(1-2x) + C_2(2x - 3x^2)$$

$$= C_1(1-2x) + C_2 x(2-3x)$$

$$\left[\frac{dy}{dx}\right]^2 = \left[C_1(1-2x) + C_2 x(2-3x)\right]^2$$

$$= C_1^2\left(1 - 4x + 4x^2\right) + C_2^2 x^2\left(4 - 12x + 9x^2\right)$$
$$+2C_1C_2x(1-2x)(2-3x)$$

$$= C_1^2(1 - 4x + 4x^2) + C_2^2 x^2(4 - 12x + 9x^2)$$
$$+2C_1C_2x(1-2x)(2-3x)$$

$$= C_1^2\left(1 - 4x - 4x^2\right) + C_2^2 x^2(4 - 12x + 9x^2)$$
$$+2C_1C_2x(2 - 7x + 6x^2)$$

But we know that

$$I = \int_0^1 \left[-0.5\left(\frac{dy}{dx}\right)^2 + 400x^2\gamma\right]$$

$$= \frac{-1}{2}\int_0^1 \left(\frac{dy}{dx}\right)^2 + 400\int_0^1 x^2 y$$

On sub the values of Y and $\frac{dy}{dx}$

$$I = \frac{1}{2}\left[\int_0^1 C_1^2(1 - 4x + 4x^2) + C_2^2 x^2(4 - 12x + 9x^2 + 2C_1C_2x(2 - 7x + 6x^2)\right]$$
$$+400\left[\int_0^1 x^2 C_1 x(1-x) - C_2^2 x^2(1-x)\right]$$

On sub integral value

$$I=\frac{1}{2}\left[\int_0^1 C_1^2\left(1-4x-4x^2\right)dx-\int_0^1 C_2^2x^2(4-12x-9x^2)dx+2C_1C_2\int_0^1 x(2-7x-6x^2)dx\right]$$

$$+400\left[\int_0^1 x^2C_1x(1-x)or+\int_0^1 C_2x^2(1-x)dx\right]$$

$$=\frac{1}{2}\left[C_1^2\left(x-\frac{4x^2}{2}+\frac{4x^3}{3}\right)_0^1+C_2^2\left(4x^2-\frac{12x^3}{3}+\frac{9x^4}{4}\right)_0^1+2C_1C_2\left(\frac{x^2}{2}\left(2x-\frac{7x^2}{2}+\frac{6x^3}{3}\right)\right)\right]$$

$$+400\left[C_1\left(\frac{x^4}{4}\left(x-\frac{x^2}{2}\right)\right)_0^1+C_2\frac{x^3}{3}\left(x-\frac{x^2}{2}\right)_0^1\right]$$

$$=\frac{1}{2}\left[C_1^2\left(1-\frac{4}{2}+\frac{4}{3}\right)+C_2^2\times\left(\frac{4}{3}-\frac{12}{4}+\frac{9}{5}\right)+2C_1C_2\left(\frac{1}{2}\left(2-\frac{7}{2}-\frac{6}{3}\right)\right)+400\left[C_1\frac{1}{4}\left(1-\frac{1}{2}\right)\right]+C_2\times\frac{1}{3}\left(1-\frac{1}{2}\right)\right]$$

$$=\frac{1}{2}\left\{\left[\frac{C_1^2}{3}+\frac{2}{15}C_2^2+\frac{1}{3}C_1C_2\right]+400\left[\frac{C_1}{20}+\frac{C_2}{30}\right]\right\}$$

$$=\frac{C_1^2}{6}-\frac{1}{15}C_2^2-\frac{1}{6}C_1C_2+20C_1+\frac{40}{3}C_2$$

On differentiation with respect to C1 and C2

$$\frac{\partial I}{\partial C_1}=0$$

$$\frac{\partial}{\partial C_1}[\frac{-1}{6}C_1^2-\frac{1}{15}C_2^2-\frac{1}{6}C_1C_2-20C_1-\frac{40}{3}C_2=0$$

$$\frac{-1}{6}\times 2C_1-\frac{1}{6}C_2-20=0$$

$$\frac{-C_1}{3}-\frac{C_2}{6}+20=0$$

$$\frac{\partial I}{\partial C_2}=0 \qquad ---------(1)$$

$$\frac{\partial I}{\partial C_2}\left[\frac{-1}{6}C_1^2-\frac{1}{15}C_2^2-\frac{1}{6}C_1C_2+20C_1+\frac{40}{3}C_2\right]$$

$$\frac{-2C_2}{15}-\frac{1}{6}C_1+\frac{40}{3}=0 \qquad ---------(2)$$

By solving (1) and (2)

We get C1 = 26.66

C2 = 66.66

Sub the value of C1 and C2 in (A)

$$Y=\frac{80}{3}x(1-x)+\frac{200}{3}x^2(1-x)$$

13 (i) A physical phenomenon is governed by the differential equation $\left(d^2w/dx^2\right)-10x^2=5$ for $0\le x\le L$. The boundary conditions are given by $w(0) = w(1) = 0$. By taking a two – term trail solution as

$\omega(x) = C_1 f_1(x) + C_2 f_2(x)$ **with** $f_1(x) = x(x-1)$ & $f_2(x) = x^2(x-1)$ **find the solution of the problem using the Galerkin method. (Nov/Dec 2009)**

$$\omega(x) = C_1 f_1(x) + C_2 f_2(x)$$
$$= C_1 x(x-1) + C_2 x^2 (x-1)$$
$$\frac{d\omega}{dx} = C_1(2x-1) + C_2(3x^2 - 2x)$$

$$\frac{d^2\omega}{dx^2} = 2C_1 + 6C_2 x - 2C_2$$

But the given equation is

$$\frac{d^2\omega}{dx^2} - 10x^2 - 5 = 0$$
$$2C_1 + 6C_2 x - 2C_2 - 10x^2 - 5 = R$$

On integrating

$$\int_0^1 f_1(x)\left[2C_1 + 6C_2 x - 2C_2 - 10x^2 - 5\right]dx = 0$$
$$\int_0^1 \left(x^2 - x\right)\left(2C_1 + 6C_2 x - 2C_2 - 10x^2 - 5\right)dx = 0$$
$$\int_0^1 (x^2 - x)\left(2C_1 + 6C_2 x - 2C_2 - 10x^2 - 5\right)dx = 0$$
$$\int_0^1 \left(2C_1 x^2 + 6C_2 x^3 - 2C_2 x^2 - 10x^4 - 5x^2 - 2C_1 x - 6C_2 x^2 + 2C_2 x + 10x^3 + 5x\right)dx = 0$$

On solving

$$\frac{2C_1}{3} + \frac{6C_2}{4} - \frac{2C_2}{3} - \frac{10}{5} - \frac{2C_1}{2} - \frac{6C_2}{3} + \frac{2C_2}{2} + \frac{10}{4} + \frac{5}{2} = 0$$
$$\frac{2C_1}{3} + \frac{3C_2}{2} - \frac{2}{3}C_2 - 2 - \frac{5}{3} - C_1 - 2C_2 + C_2 + \frac{5}{2} - \frac{5}{2}$$
$$\frac{4C_1 + 5C_2 - 12 - 10 - 6C_2 + 15 + 15}{6} = 0$$
$$-2C_1 - C_2 + 8 = 0$$
$$C_1 + 0.5C_2 = 8 \qquad ----(1)$$

On integrating

$$\int_0^1 f_2(x)(2C_1 - 6C_2 - 2C_2 - 10x^2 - 5)dx = 0$$

$$f_2(x) = x^2(x-1)$$

$$\int_0^1 \left(x^3 - x^2\right)\left(2C_1 - 6C_2x - 2C_2 - 10x^2 - 5\right)dx = 0$$

$$\int_0^1 \left(2C_1x^3 - 6C_2x^4 - 2C_2x^3 - 10x^5 - 5x^3 - 2C_1x_2 - 6C_2x_3 - 2C_2x^2 - 10x^4 - 5x^2\right)dx = 0$$

On solving

$$\frac{2C_1}{4} - \frac{6C_2}{5} - \frac{2C_2}{4} - \frac{10}{6} - \frac{5}{4}$$
$$-\frac{2C_1}{3} - \frac{6C_2}{4} - \frac{2C_2}{3} - \frac{10}{5} - \frac{5}{3} = 0$$
$$\frac{C_1}{2} - \frac{6C_2}{5} - \frac{C_2}{2} - \frac{5}{3} - \frac{5}{4}$$
$$-\frac{2C_1}{3} - \frac{3C_2}{2} - \frac{2C_2}{3} - 2 - \frac{5}{3} = 0$$
$$C_1 + 0.8\ C_2 = 4.5 \qquad -----(2)$$

On solving (1) and (2)

$$C_1 + 0.5C_2 = 4$$
$$\underline{C_1 - 0.8C_2 = 4.5}$$
$$-0.3C_2 = -0.5$$
$$C_2 = 1.66$$
$$C_1 = 3.16$$

$$\omega(x) = 3.16\left(x^2 - x\right) + 1.66\left(x^3 - x^2\right)$$

13 (ii) Solve the following system of equations using Gauss elimination method. (Nov/Dec 2009)

$$x_1 + 3x_2 + 2x_3 = 13$$
$$-2x_1 + x_2 - x_3 = -3$$
$$-5x_1 + x_2 - 3x_3 = 6$$

Step 1: Write in matrix form

Step 2: Choose the matrix form

$$\begin{pmatrix} 1 & 3 & 2 & 13 \\ -2 & 1 & -1 & -3 \\ -5 & 1 & 3 & 6 \end{pmatrix}$$

Step 2: Choose the pivot element

$$\begin{pmatrix} 1 & 3 & 2 & 13 \\ -2 & 1 & -1 & -3 \\ -5 & 1 & 3 & 6 \end{pmatrix}$$

Step 3: Make other component equal to zero

$$\begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 7 & 3 & 23 \\ 0 & 16 & 13 & 71 \end{pmatrix} \begin{matrix} R_2 = R_2 + 2R_1 \\ R_3 = R_3 + 5R_1 \\ \end{matrix}$$

Step 4: neglect 1st row and 1st column

$$u\begin{pmatrix} 7 & 3 & 23 \\ 16 & 13 & 71 \end{pmatrix}$$

Step 5: Choose the pivot element, move the pivot row to the top, so swap the

$$\begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 16 & 13 & 71 \\ 0 & 7 & 3 & 23 \end{pmatrix}$$

R_2 is divided by 16

$$\begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & \frac{13}{16} & \frac{71}{16} \\ 0 & 7 & 3 & 23 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & \frac{13}{16} & \frac{71}{16} \\ 0 & 0 & \frac{-37}{8} & \frac{213}{4} \end{pmatrix} R_3 = R_3 - 7R_2$$

The above matrix can be written in

$$x + 3y - 2x = 13$$
$$y - \frac{13}{16}z = \frac{71}{16}$$
$$\frac{-39}{8}z = \frac{213}{4}$$

On solving the above equation we get

X = 1

Y = 2

Z = 3

14. The following differential equation is available for a physical phenomenon

$\frac{d^2y}{dx^2} + 50 = 0, \quad 0 \le x \le 10$ **Trail function is y = a x (10- x)**

Boundary conditions are y(0) = 0; y(10) = 0

Find the value of the parameters a1 by the following methods.

(i) Point collocation (ii) Subdomain collocation (iii) Least squares (iv) Galerkin (NOV / DEC 2011)

$$\frac{d^2y}{dx^2} + 50 = 0; \quad 0 \le x \le 10$$
$$y(0) = 0$$
$$y(10) = 0$$

Trail function $y = a_1 x (10 - x)$

Find a_1

Verify the trail function satisfies the boundary conduction (or) Not

Trail function y = a1 x x(10 - x)

x = 0; y = 0

x = 10; y = 0

it satisfies the boundary condition

(i) Point collocation method

$$y = a_1 x(10 - x)$$
$$= a_1(10x - x^2)$$
$$\frac{dy}{dx} = a_1(10 - 2x)$$
$$\frac{d^2y}{dx^2} = -2a_1$$

Substituting the $\frac{d^2y}{dx^3}$ value in given differential equation

Residual $R = 2a_1 + 50$

Residual are set to zero

$R = 2a_1 + 50$

$-2a_1 = 50; a_1 = 25$

(ii) Sub domain collocation method

$$\int_0^{10} R dx = 0; \quad \int_0^{10} (-2a_1 + 50) dx = 0$$

$$\int_0^{10} (-2a_1 dx + 50 dx) = 0 = \left[-2a_1 x + 50x\right]_0^{10} = 0$$

$$-2a_1(10) + 50(10) - (0) = 0$$

$$-2a_1 = -500$$

$$a_1 = 25$$

(iii) Least squares method

$$\ell = \int_0^{10} R^2 dx$$

But $\dfrac{\partial I}{\partial a_1} = \displaystyle\int_0^{10} R \frac{\partial R}{\partial a_1} dx$

But $R = -2a_1 + 50 \dfrac{\partial R}{\partial a_1} = -2$

Substitute R and $\dfrac{\partial R}{\partial a_1} = -2$

Substitute R and $\dfrac{\partial R}{\partial a_1}$ in $\dfrac{\partial I}{\partial a_1}$

$$\frac{\partial I}{\partial a_1} = \int_0^{10} (-2a_1 + 50)(-2) dx$$

$$\frac{\partial I}{\partial a_1} = 0;$$

So $\displaystyle\int_0^{10} (-2a_1 + 50)(-2) dx = 0$

$$\int_0^{10} (+4a_1 - 100) dx = 0$$

$$\left[4a_1 x - 100x\right]_0^{10} = 0$$

$$4a_1(10) - 100(10) - (0) = 0$$

$$40a_1 - 1000 = 0$$

$$a_1 = 1000 / 40 = 25;$$

$$a_1 = 25$$

(iv) Galerkin's method

Hence Trail function itself is the weighting function (w_i)

$$\int_0^{10} W_i R dx = 0$$

$$\int_0^{10} a_1 x(10-x)(-2a_1+50)dx = 0$$

$$a_1\int_0^{10} \left[x(10-x)\times(-2a_1+50)\right]dx = 0$$

$$a_1\int_0^{10} (10x-x^2)(-2a_1+50)dx = 0$$

The above equation simplifies to

$$-1000a_1 + 25{,}000 + 666.66a_1 - 16{,}666.66 = 0$$
$$-333.33a_1 = -83333.33$$
$$a_1 = 25$$

The nature of parameter a1 = 25 for all the methods.

15. Explain the general procedures of finite element analysis (NOV / DEC 2011)

Generally the displacements (or) stiffness method is more desirable because its formation is simpler in most structural analysis problems.

Step 1:

Discretization of structure

The art of sub dividing a structure into a convenient number of smaller elements is known as discretization

a) One dimensional element
b) Two dimensional element
c) Three dimensional element
d) Axisymmetric Element

Step 2:

Selection of displacement of Nodes and Elements

The numbering process decides the size of stiffness matrix and while numbering the nodes, the following condition should be satisfied.

$$\begin{Bmatrix}\text{maximum}\\ \text{Node Number}\end{Bmatrix} - \begin{Bmatrix}\text{minimum}\\ \text{Node Number}\end{Bmatrix} = \text{Minimum}$$

Step 3:

Selection of displacement function (or) Interpolation function

Choose a displacement within each element, polynomial or Linear, quadratic and cubic form are used as displacement function to formulate the finite element equation and perform differentiation and integration.

Step 4:

Define the material behaviour using strain,

Displacement and stress strain Relationship.

$$\rho = \frac{du}{dx}$$

$\sigma = E\rho$

Step 5:

Derivation of Element stiffness matrix and Equation

$$\left\{F^e\right\} = \left[R^e\right]\left\{u^e\right\}$$

Where e = element

{F} = Vector of Elements Nodal forces

K = Element stiffness matrix

{u} = Element Displacement vector

Step 6:

Assemble the Element equation to obtain the global (or) Total Equation

By using a method of superposition ie Direct stiffness method.

[F] = [K] {u}

[F] = Global force vector

[K] = Global Stiffness matrix

{u} = Global displacement vector

Step 7:

Applying boundary conditions

The general equation is modified to account for the boundary conditions of the problem.

Step 8:-

Solution of the unknown Displacements

A set of simultaneous algebraic equation are formed and the unknown displacements {u} are calculated.

Step 9:

Computation of the element strains and stresses from the nodal Displacements {u}

Step 10:

Interpret the Results (post processing), Displaying Result in a Graphical form.

16. Discuss the following methods to solve the given differential equation:

$$\mathbf{EI}\frac{d^2y}{dx^2} - M(x) = 0$$

With the boundary condition $y(0) = 0$ and $y(H) = 0$

(i) Variational method (ii) Collocation method. (MAY / JUNE 2010)

$EI\frac{d^2y}{dx^2} - \mu(2) = 0$ (1)

Boundary condition $y(\mu) = 0$ $y(0) = 0$ $y(\mu) = 0$

1. Collocation method

Step 1: Verify the trial function satisfies the boundary condition or not

Assume the trial function

$$Y = a_1(x - x^4)$$

When $x = 0$ $y = 0$

$Y = a_1(x - x^4)$

$\frac{dy}{dx} = a_1(1 - 4x^3)$

$\frac{d^2y}{dx^2} = a_1(0 - 12x^2)$

$\frac{d^2y}{dx^2} = -12a_1x^2$

Substitute $\frac{d^2y}{dx^2}$ value in equation (1)

$R = -12a_1x^2EI - M(x)$

Residuals are set to zero

$R = -12a_1x^2EI - M(x) = 0$ .. 2

$$R = -12a_1\left(\frac{1}{2}\right)^2 EI - M\left(\frac{1}{2}\right) = 0$$

$$-3EIa_1 - \frac{M}{2} = 0$$

$$-3EIa_1 = \frac{M}{2}$$

$$a_1 = -\frac{M}{2\times 3EI}$$

$$a_1 = -\frac{M}{6EI}$$

Hence the trial function is

$$Y = \frac{M}{6EI}\left(x - x^4\right)$$

2. Variational method

$$\int_0^1 W_i R dx = 0 \qquad ..1$$

The trial functions is

$$Y = W_i = a_1 x\left(x-1\right) \qquad ..(2)$$

Substitute in (1)

$$\int_0^1 a_1 x\left(x-1\right)x - 12a_1 x^2 EI - M\left(x\right)dx = 0$$

$$-a_1\int_0^1 \left(x^2-1\right)\left(12a_1 x^2 EI - Mx\right)dx = 0$$

$$-a_1\int_0^1 \left[12a_1 x^4 EI - 12a_1 x^2 EI - Mx^3 - Mx\right]dr = 0$$

$$12a_1\left(\frac{x^5}{5}\right)_0^1 EI - 12a_1 EI\left(\frac{x^3}{3}\right)_0^1 - M\left(\frac{x^4}{4}\right)_0^1 - M\left(\frac{x^2}{2}\right)_0^1 = 0$$

$$\frac{12}{5}a_1 EI - 4a_1 EI - \frac{M}{4} - \frac{M}{2} = 0$$

$$\left(4891EI - 169a_1 EI\right) = 10M - 20M = 0$$

$$28a_1EI - 10M = 0$$

$$28a_1EI = -10M$$

$$a_1 = \frac{-10M}{28EI}$$

$$a_1 = \frac{-5M}{14EI} \qquad ..3$$

Sub a_1 in 2

approximate solution is

$$Y = \frac{-5}{14}\frac{M}{EI}\left(x^2 - 1\right)$$

Compare the result with exact solution

$$\frac{d^2y}{dx^2} = \frac{Mx}{EI}(\text{given})$$

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2} - \frac{M(x)}{EI} - C_1$$

$$Y = \int \frac{dy}{dx} = \frac{M(a)}{EI} + C_1x + C_2 \qquad ..4$$

Apply boundary condition in 4

$X = 0 \quad Y = 0$

$$C_2 = 0$$

$X = H \qquad Y = 0$

$$0 = \frac{M(x)}{EI} - C_1H$$

$$C_2 = -\frac{M(x)}{EIH}$$

Sub C_1 and C_2 in 4

$$Y = \frac{M(x)}{EI} - \frac{M(x)}{EIH} + 0$$

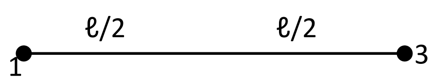
UNIT II

2 MARKS

33. Why do you mean by higher order elements? (Nov/Dec 2008)

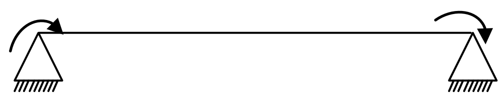
When formulating the truss element and the one dimension heat conduction b/n the line elements having a single degree of freedom at each of two nodes are considered. While quite approximate for the problems considered.

$$u = a_0 + a_1 n + a_2 n^2$$



ℓ^2

34. Write the stiffness matrix for the simple beam element given below. (Nov/Dec 2008)



The stiffness matrix [k] for the above element is given by

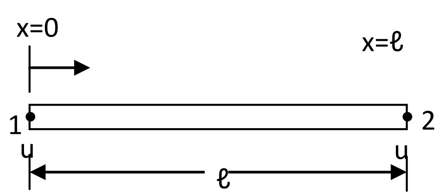
$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & 6L & 4L^2 \end{bmatrix}$$

Where E = young's modulus

I = moment of the inertia

L = length of the beam.

35. Write the shape functions for a ID, 2 nodes element. (Nov/Dec 2008)



For one dimensional, two nodes bar element

Displacement functions, u = N1 u1 + N2 u2

Where, shape functions, $N_1 = \frac{\ell - x}{\ell}$

Shape function, N2 = x/ℓ

36. State the properties of stiffness matrix. (Nov/Dec 2009)

1. It is a symmetric matrix

2. Sum of elements in any column must be equal to zero.

3. It is an unstable element

37. Why polynomials are generally used as shape functions? (NOV / DEC 2011)

a) It is easy to formulates and computerize the finite element equation
b) It is easy to perform Differentiation and integration
c) The accuracy of the results can be improved by increasing the order of the polynomial .

38. State the assumption made while finding the forces in a truss. (NOV / DEC 2011)

a) All the members are pin jointed
b) The truss is loaded only at the joints
c) The self weights of the members are neglected unless stated.

39. What is higher order element? (NOV / DEC 2011)

When the finite element approximation is gradually refining the mesh (Increasing the no of Element) in comparing the solution then it is called Higher order elements.

40. What is meant by dynamics analysis? (NOV / DEC 2011)

It is an analysis to find out the response of a system (temperature, displacement) as a function of time with External distribances. The state should be insteady (or) transient Ex: Vibration analysis

41. Determine the element mass matrix for one – dimensional, dynamic structural analysis problems. Assume the two – node, linear element. (NOV / DEC 2011)

General force equation

$$[F] = [K]\{u\}$$

$$[K] = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[F] = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$ from 2 Node one dimensional element

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

So $$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

42. Write down the stiffness matrix equation for one dimensional heat conduction element. (NOV / DEC 2011)

[K] [T] = {f}

Where [K] = conduction stiffness matrix

[T] = vector of unknown temperatures

{f} = Thermal load vector

Or

$$[K_c] = \frac{AR}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

A = area of element in m^2.

K_c = stiffness matrix for Heat conduction

k = thermal conductivity of the element W/mK

ℓ = length of the element in m

43. Why polynomials are generally used a shape function

It is used due to following reasons

1. Differentiation and integration are quite easy

2. Accuracy of results can be improved by increasing the order of polynomial

3. Easy to formulade and computering the finited element equation.

44. List the properties of the global stiffness matrix. (MAY / JUNE 2010)

(i) It is a symmetrix matrix

(ii) Sum if element in any column must be equal to zero.

(iii) It is unstable element.

45. List the characteristics of shape functions. (MAY / JUNE 2010)

(i) It has unit value at one node point and zero value at other nodal points

(ii) Sum of shape function is equal to one.

46. What do you mean by the terms: c^0, c^1 and c^n continuity? (MAY / JUNE 2010)

C^0- governing differential equation is quesiharmonic, ϕ has to be continues

C^1 – governing differential equation is biharmonic, ϕ as well as daivative has to be continess inside and between element

C^n – governing differential equations is polynomial

47. Name a few boundary conditions involved in any heat transfer analysis. (MAY / JUNE 2010)

1. Heat flux boundary conditions

2. Forced and natural boundary conditions

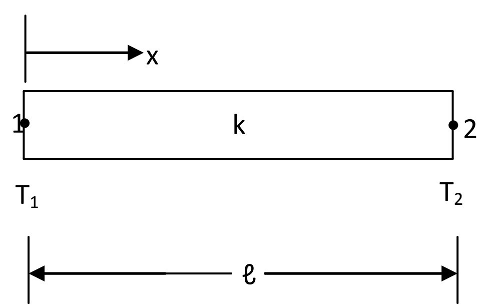
48. What is shape function? (Nov/Dec 2009)

In FEM, field variables with an element are generally expressed by the following approximation relation

$$\phi(x,y) = N_1(x,y)\phi_1 + N_2(x_3 y)\phi_2 + N_3(x,y)\phi_3$$
$$\phi_1\phi_2\phi_3 \Rightarrow \text{field variables at the nodes}$$
$$N_1N_2N_3 \Rightarrow \text{shape functions}$$

16 marks

11. Derivation of stiffness matrix for the one dimensional heat conduction. (Nov /Dec 2008)



We know that,

Stiffness matrix $[k]=\int_v [B]^T [D][B] dv$

In one dimensional element,

Temperature function, $T = N_1T_1+N_2T_2$

Where,

$$N_1 = \frac{\ell - x}{\ell}$$

$$N_2 = \frac{x}{\ell}$$

We know that,

Strain – displacement matrix,

$$[B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

$$[B] = \begin{bmatrix} \frac{-1}{\ell} & \frac{1}{\ell} \end{bmatrix}$$

$$[B]^T = \begin{Bmatrix} \frac{-1}{\ell} \\ \frac{1}{\ell} \end{Bmatrix}$$

One dimensional heat condition problems,

[D]=[K]=k= Thermal conductivity of the material

Substitute [B], $[B]^T$ and [D] values in stiffness matrix equation

$$\left.\begin{array}{r}\text{stiffness matrix for}\\ \text{heat conduction}\end{array}\right\}[K_C] = \int_0^{\ell} \begin{Bmatrix} \frac{-1}{\ell} \\ \frac{1}{\ell} \end{Bmatrix} \times k \times \begin{bmatrix} \frac{-1}{\ell} & \frac{1}{\ell} \end{bmatrix} dv$$

$$= \int_0^{\ell} \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} k\ dv$$

[Matrix multiplication $(2\times1)\times(1\times2) = (2\times2)$]

$$= \int_0^{\ell} \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} k\ A\ dx$$

$$= Ak \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} \int_0^{\ell} dx$$

$$= Ak \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} [x]_0^{\ell}$$

$$= Ak \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} (\ell - 0)$$

$$= Ak\ell \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix}$$

$$= \frac{Ak\ell}{\ell^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

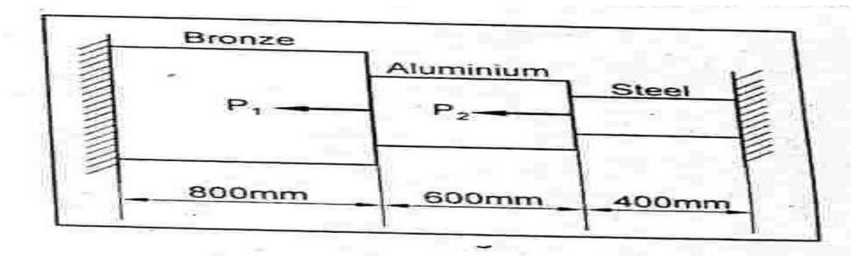
$$[K_C] = \frac{Ak}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Where, A = Area of the element, m^2.

k = Thermal conductivity of the element, W/mk

ℓ = Length of the element, m

12. The stepped bar shown in Fig 1 is subjected to an increase in temperature, Δ T = 80° C. Determine the displacements, element stress and support reactions. (Nov/Dec 2009)



$\alpha_{Al} = 23\times10^{-6}\,\ell°C$

$\alpha_{S} = 11.7\times10^{-6}\,\ell°C$

$P_1 = 60kN$

$P_2 = 75kN$

$\Delta T = 80°C$

Bronze	Aluminium	Steel
A = 2400 mm^2	1200 mm^2	600 mm^2
E=83 GPa	70 GPa	200 GPa

$P_1 = 60kN$

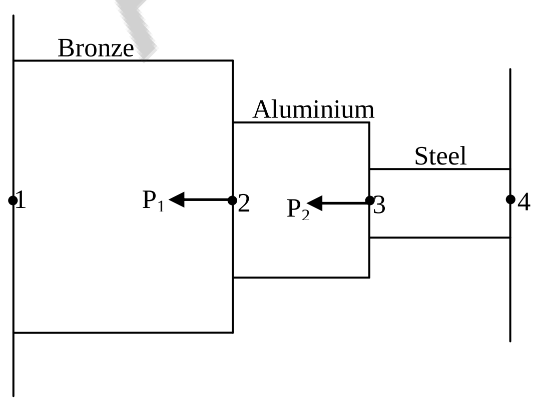
$P_2 = -75kN$

$\Delta T = -80°C$

$\ell_1 = 800mm$

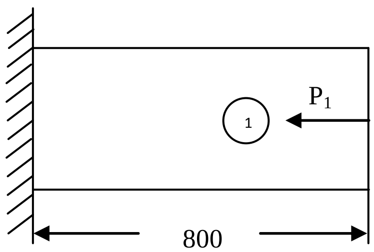
$\ell_2 = 600mm$

$\ell_3 = 400mm$



For element 1

$$\frac{\Delta_1 E_2}{\ell_1} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$



$$A_1 = 2400 \text{mm}^2$$
$$E_1 = 83 \text{GPa} = 83 \times 10^9 \text{N/m}^2$$
$$E = 83 \times 10^3 \text{N/mm}^2$$
$$\alpha_1 = 18.9 \times^{-6} \ell^2 \text{C}$$
$$\frac{2400 \times 83 \times 10^3}{800} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$
$$249 \times 10^3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad ---(1)$$

Element 2

$$A_2 = 2400 \text{mm}^2$$
$$\alpha_2 = 23 \times 10^{-6} \ell° \text{C}$$
$$\ell_2 = 600 \text{mm}$$
$$E_2 = 700 \text{GPa} = 70 \times 10^9 \text{N/m}$$
$$= 70 \times 10^3 \text{N/mn}^2$$
$$\frac{A_2 E_2}{\ell_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$
$$\frac{1200 \times 70 \times 10^3}{600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$
$$10^3 \begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \quad -----(2)$$

For element 3

$A_3 = 600\,mm^2$

$\alpha_3 = 11.7 \times 10^{-6}$

$E_3 = 200 \times 10^3\,N/mmm^2$

$\ell_3 = 400$

$$\frac{600 \times 200 \times 10^3}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$10^3\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} \qquad -----(3)$$

Assembling the equn (1) (2) and (3)

$$10^3\begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 249+140 & -140 & 0 \\ 0 & -140 & 140+300 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$10^3\begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 389 & -140 & 0 \\ 0 & -140 & 440 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \qquad ----(4)$$

We know that load vector $[F] = EA\alpha T\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

For element 1

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = E_1 A_1 \alpha_1 \Delta T\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= 83 \times 10^3 \times 2400 \times 189 \times 10^{-6} \times 80 \times \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= 10^3\begin{Bmatrix} -301.19 \\ 301.19 \end{Bmatrix} \qquad ---(5)$$

For element 2

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = 70 \times 10^3 \times 1200 \times 23 \times 10^6 \times 80 \times \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= 10^3\begin{Bmatrix} -154.5 \\ 154.5 \end{Bmatrix} \qquad ----(6)$$

For element 3

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = 200 \times 10^3 \times 600 \times 11.7 \times 10^{-6} \times 80 \times \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= 10^3 \begin{Bmatrix} -112.3 \\ 112.3 \end{Bmatrix} \qquad ----(7)$$

On assembling equan 5, 6, and 7

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = 10^3 \begin{Bmatrix} -301.1 \\ 301.1 - 154.5 \\ 154.5 - 112.3 \\ 1123 \end{Bmatrix}$$

$$= 10^3 \begin{Bmatrix} -301.1 \\ 146.6 \\ 42.2 \\ 112.3 \end{Bmatrix}$$

Axial load is acting at node 2 and node 3

At node 2 = 60×10^3 N

Node 3 = 75×10^3 N

$\therefore$ the load is a acting towards left we put negative.

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = 10^3 \begin{Bmatrix} -301.2 \\ 146.6 - 60 \\ 42.2 - 75 \\ 1123 \end{Bmatrix}$$

$$= 10^3 \begin{Bmatrix} -301.2 \\ 86.6 \\ -32.7 \\ 112.3 \end{Bmatrix} \qquad ---(8)$$

Sub in equn (8) in (4)

$$10^3 \begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 389 & -140 & 0 \\ 0 & -140 & 440 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -301.2 \\ 86.6 \\ -32.76 \\ 112.3 \end{Bmatrix}$$

$\therefore u_l = u_u = 0$ neglect 1st row 1st column and 4th row 4th column

Column of the matrix

$$\begin{bmatrix} 389 & -140 \\ -140 & 440 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 86.6 \\ -32.7 \end{Bmatrix}$$

$$389u_2 - 140u_3 = 86.6 \qquad ----(9)$$
$$-140u_2 + 440u_3 = -32.7 \qquad ----(10)$$

Multiple 9 by 3.143

$$1222.63u_2 - 440u_3 = 272.18$$
$$-140u_2 + 440u_3 = 32.7$$
$$1082.6u_2 = 239.48$$
$$u_2 = 0.2212$$

On sub u_2 in (10)\

$$-140(0.2212) - 440\,u_3 = -32.4$$
$$u_3 = -0.00345 \text{ mm}$$

2. Stress

We know that Thermal stress $\sigma = E\dfrac{du}{dx} - E\alpha\Delta T$

Element 1

$$\sigma_1 = \frac{E_1(u_2 - u_1)}{\ell_1} = E_1\alpha_1\Delta T$$
$$= \frac{83\times10^3(0.2212-0)}{800} - 83\times10^3\times18.9\times10^{-6}\times80$$
$$= -102.54 N/mm^2$$

Element 2

$$\sigma_2 = \frac{E_2(u_3 - u_2)}{\ell_2} - E_2\alpha_2\Delta T$$
$$= \frac{70\times10^3(-0.00345-0.2212)}{600} - 70\times10^3\times23\times10^{-6}\times80$$
$$= -155.0\,N/mm^2$$

Element 3

$$\sigma_3 = \frac{E_3(u_4 - u_3)}{\ell_3} - E_3\alpha_3\Delta T$$
$$= \frac{200\times10^3(0-0.0034)}{400} - 200\times10^3\times11.7\times10^{-6}\times80$$
$$= -185.47 N/mm^2$$

3. Reaction force

{R} = [k] {u} – [F]

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 10^3 \begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 249+140 & -140 & 0 \\ 0 & -140 & 140+300 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$= \begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 249+140 & -140 & 0 \\ 0 & -140 & 140+300 & +300 \\ 0 & 0 & -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.22 \\ 0.003 \\ 0 \end{Bmatrix} - 10^3 \begin{Bmatrix} -301.1 \\ 86.6 \\ -32.7 \\ 112.3 \end{Bmatrix}$$

$$= 10^3 \begin{Bmatrix} -55.07 \\ 86.56 \\ -32.48 \\ 1.03 \end{Bmatrix} - 10^3 \begin{Bmatrix} 301.1 \\ 86.6 \\ -32.7 \\ 112.3 \end{Bmatrix}$$

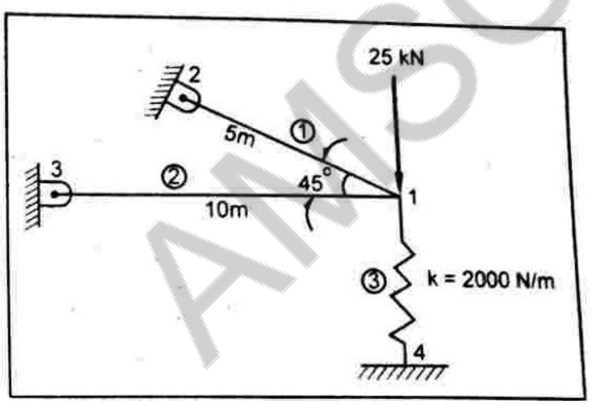
$$= 10^3 \begin{Bmatrix} 246.13 \\ 0 \\ 0 \\ -111.3 \end{Bmatrix}$$

$$R_1 = 246.13 \times 10^3 N$$

$$R_4 = -111.3 \times 10^3 N$$

$$R_2 = R_3 = 0$$

13. Consider a two – bar truss supported a spring shown in fig. 2. Both bars have E = 210 GPa and A = 5.0 × 10^{-4}. Bar one has a length of 5 m and bar two has a length of 10 m. The spring stiffness is k = 2k N/m. Determine the horizontal and vertical displacements at the joint 1 and stress in each bar. (Nov/Dec 2009)



$$[K] \frac{AE}{\ell} \begin{bmatrix} \ell_1^2 & \ell_1 m_1 & -\ell_1^2 & -\ell_1 m_1 \\ \ell_1 m_1 & m_1^2 & -\ell_1 m_1 & -m_1^2 \\ -\ell_2^2 & -\ell_1 m_1 & -\ell_1^2 & \ell_1 m_m \\ \ell_1 m_1 & -m_1^2 & \ell_1 m_1 & m_1^2 \end{bmatrix}$$

Element 1

$$\ell_{e1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(500 - 0)^2 + (0 - 0)^2}$$

$$\ell_{e1} = 5000\,\text{mm}$$

$$m_1 = \frac{y_2 - y_1}{\ell_1}$$

$$= \frac{0 - 0}{5000}$$

$$m_1 = 0$$

$$\ell_1 = \frac{x_2 - x_1}{\ell_{e1}} = \frac{5000}{5000} = 1$$

Element 2

$$\ell_{e2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$
$$= \sqrt{10000^2}$$
$$= 10000\,\text{mm}$$
$$\ell_2 = \frac{x_3 - x_1}{\ell_{e2}} = \frac{10.000}{10.000}$$
$$\ell_2 = 1$$
$$m_2 = \frac{y_2 - y_1}{\ell_{e2}}$$
$$= \frac{10.000}{10.000}$$
$$m_2 = 1$$
$$[K] = \frac{5\times10^{-4}\times210\times10^{9}}{5}\times10^{3}\begin{bmatrix}1 & -1 & -1 & 1\\ -1 & 1 & 1 & -1\\ -1 & 1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}$$
$$= 105\times10^{5}\begin{bmatrix}1 & -1 & -1 & 1\\ -1 & 1 & 1 & -1\\ -1 & 1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}$$

Since $\theta^2 = 180°$

Cos θ = -1

$S_i\,\theta_a$ = +0

$$[K_2] = 105\times5\times10^{3}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$
$$\theta_3 = 270 \;\; \cos\theta_3 = 0 \;\; \sin\theta_3 = 1$$
$$[K_3] - k\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix} = 200\times10^{3}\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}$$

We know that $\{F\} = \begin{bmatrix}F_1\\ F_2\end{bmatrix} = \begin{Bmatrix}0\\ -25000\end{Bmatrix}$

$$\{F\} = [K]\{u\}$$

On solving we get

$$u_1 = -1.724 \times 10^{-3}\,m$$
$$u_2 = -3.45 \times 10^{-3}\,m$$

$$\sigma_1 = \frac{E}{\ell_{e1}}\left[-0.707 - 0.707 - 0.7070.0.707\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

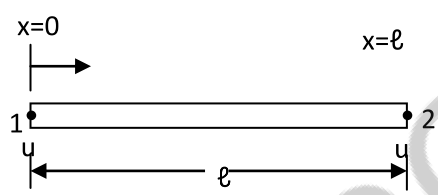
$$= \frac{210 \times 10^3}{5}\left[0.707, -0.707, -0.707, 0.707\right]\begin{Bmatrix} -1.724 \times 10^{-3} \\ -3.45 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

By solving σ_1 = 51.2 MPa

$$\sigma_2 = \frac{E}{\ell_2}\left[10 - 10\right]\begin{Bmatrix} -1.724 \times 10^{-3} \\ -3.458 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma_2 = 36.2\text{MPa}$$

14. (i) Derive the stiffness matrix and finite element equation for one dimensional bar. (NOV / DEC 2011)



Consider one dimensional bar element with nodes 1 and 2 as shown above.

u_1 and u_2 are the nodal displacement

- Stiffness matrix $[K] = \int_v [B]^T [D][B]\,dv$
- Displacements functions, $u = N_1 u_1 + N_2 u_2$

$$N_1 = \frac{\ell - x}{\ell}$$
$$N_2 = \frac{x}{\ell}$$

- Strain displacement matrix $[B] = \left[\frac{dN_1}{dx} \frac{dN_2}{dx}\right]$

$$[B] = \left[\frac{-1}{\ell} \frac{1}{\ell}\right]$$

$$[B]^T = \begin{Bmatrix} \frac{-1}{\ell} \\ \frac{1}{\ell} \end{Bmatrix}$$

- For one dimensional problems

[D] = [E] = e = young' s modulus

$$\text{So } [K] = \int_0^\ell \begin{Bmatrix} \frac{-1}{\ell} \\ \frac{1}{\ell} \end{Bmatrix} \times E \times \begin{bmatrix} \frac{-1}{\ell} \frac{1}{\ell} \end{bmatrix} dv = \int_0^\ell \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} E\, dv$$

$$= \int_0^\ell \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} EA \ dx$$

$$\Rightarrow AE \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} \int_0^\ell dx \Rightarrow AE \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} [x]_0^\ell$$

$$\Rightarrow AE \begin{bmatrix} \frac{1}{\ell^2} & \frac{-1}{\ell^2} \\ \frac{-1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} (\ell - 0) = \frac{AE\ell}{\ell^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Or $[K] = \frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ stiffness matrix for 1D bar element.

General force equation

$$[F] = [K] = \{u\}$$

$$[K] = \frac{A - E}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[F] = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Finite element equation from 1d bar element.

15. (ii) Discuss procedure using the commercial package (P, C, programs) available today for solving problems of FEM. Take a structural problem to explain the same. (NOV / DEC 2011)

Commercial package p.c programs

(a) Pre – processors

It is used to create geometric model of the given problem. Cartesian cylindrical and spherical co – ordinates systems are available, Finite element model generates nodes, elements (linear, quadratic) and connection of Elements called mesh.

Once the finite elements are developed the following data needs to be supplied.

1. Geometric constants

2. Material constants

3. Loads

4. Constants and times

5. Analysis of file

(b) Post processor

The results will be displayed depending on the type of data and the method of presentation

(c) Analysis of post – processing data

It provides an approximate solution to converge to the exact solution.

Eg: SAP, NASTRAN, ADINA, NISA, ANSYS and ABAQUS

EX; "To determine the maximum deflection, shear force and bending moment diagram for the given beam using ANSYS software"

Elements 6 – Beam – 4

Material property – E = 2.1×10^5 N/mm^2.

r =0.3

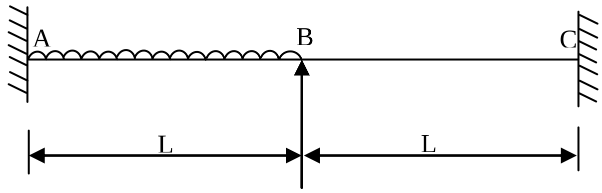
Real constant – Dia – 50 mm

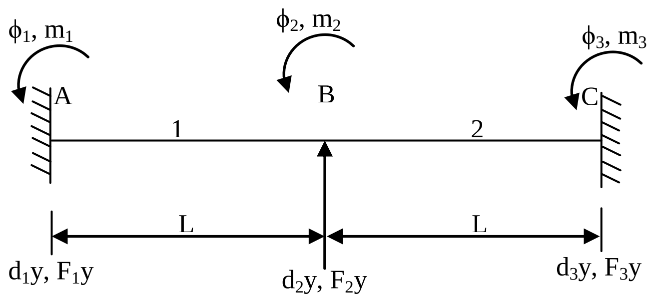
Area = 1962 mm^2

$I_{xx} = I_{yy} = 306163 mm^4$

- Preference – structural – Ok
- Pre-processor – Element type details
- Real constants – Geometry details
- Materials property – Properties details
- Modelling – Create key points, line
- Meshing – Mesh the element
- Load – Apply the load
- Solution – Solve – current –LS
- Post processor – Plot all results
- Shear fence and BM – create table and solve and see the result by general post processor

16. A fixed beam length of 2L m carries α uniformly distributed load of α w (in N/m) which run over a length of 'L' m from the fixed end, as shown in fig. L Calculate the rotation at point B using FEA.. (NOV / DEC 2011)





- To find Rotation at point B, ϕ_2 for element (1) (Nodes, 1, 2, 3, 4 ie $d_{1y}, \phi_1, d_{2y}, \phi_2$)
- Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix} \quad ---(1)$$

For uniformly distributed load

$$F_{1y} = \frac{-wL}{2}; m_1 = \frac{-wL^2}{12}; F_{2y} = \frac{-wL}{2}; m_2 = \frac{wL^2}{12}$$

$$(1) \Rightarrow \frac{EI}{L^2}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} \frac{-wL}{2} \\ \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix} \quad ---(2)$$

- For elements (2) Nodes 3, 4, 5, 6 is $d_{2y}, \phi_2, d_{3y}, \phi_3$)
- Finite element Equation

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} F_{2y} \\ m_2 \\ F_{3y} \\ m_3 \end{Bmatrix} \quad ---(3)$$

- There is no load moment on Element (2)

So $F_{2y} = F_{3y} = 0; m_2 = m_3 = 0;$

$$\Rightarrow \frac{EI}{L^2}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad ---(4)$$

Assemble the finite elements ie, assemble the finite element equation (2) and (4) which is equation No. (5)

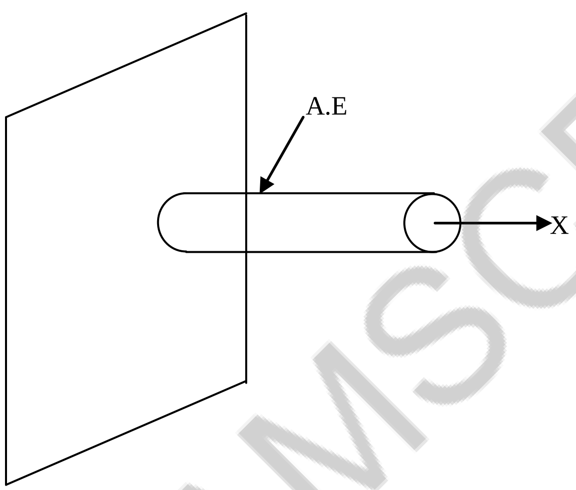
Apply the boundary conditions

(i) A is fixed d_{1y} and ϕ_1 =0

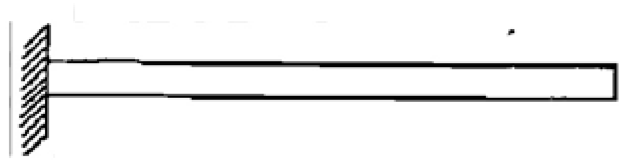
(ii) At B Displacement d_{2y} =0

(iii) C is fixed. So displacement d_{3y} and rotation ϕ_3 = 0

17. (a) Using two equal – length finite elements, determine the natural frequencies of the solid circular shaft fixed at one end shown in fig. (NOV / DEC 2011)



Consider a uniform cross – section bar (Figure) of length L of a material whose Young's modulus and density are given by E and ρ. Estimate the natural frequencies of axial vibration of the are using both consistent and lumped mass matrices.



Using just one element for the entire road (i.e l = L) and using lumped mass matrix, we have

$$\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \omega_{jump}^2 \rho A \ell \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad ---(1)$$

In view of the boundary condition at node 1 (u1 = 0), we have

$$\frac{AE}{L}u_2 = \omega_{jump}^2 \frac{\rho AL}{2} u_2 \qquad ---(2)$$

Hence

$$\omega_{jump} = \sqrt{\frac{2E}{\rho L^2}} = \frac{1.414}{L}\sqrt{\frac{E}{\rho}} \qquad --(3)$$

With one element and consistent mass matrix, we have

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \omega_{cons}^2 \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad ---(4)$$

And with $u_1 = 0$,

$$\frac{AE}{L}u_2 = \omega_{cons}^2 \left(\frac{\rho AL}{6}\right)(2u_2) \qquad ---(5)$$

Therefore,

$$\omega_{cons} = \sqrt{\frac{3E}{\rho L^2}} = \frac{1.732}{L}\sqrt{\frac{E}{\rho}} \qquad ---(6)$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-(\lambda/2) \end{vmatrix} = 0 \text{ yielding } \lambda 1 = 0.586, \lambda 2 = 3.414 \qquad -----(10)$$

Thus the natural frequencies are

$$\omega_1 = \frac{1.531}{L}\sqrt{\frac{E}{\rho}}, \omega_2 = \frac{3.695}{L}\sqrt{\frac{E}{\rho}} \qquad ---(11)$$

Results with consistent mass matrix

Using consistent mass matrices, we obtain

$$\frac{AE}{L/2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho AL\omega^2}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad ---(12)$$

Using u1 = 0 (boundary condition) and $\lambda = \frac{\omega^2 \rho L^2}{24E}$ for nontrivial solution, we obtain the equation

$$\begin{vmatrix} 2-4\lambda & -1-\lambda \\ -1-\lambda & 1-2\lambda \end{vmatrix} = 0 \qquad ---(13)$$

On solving, we get λ1 = 0.108 and λ2 = 1.32. Thus the natural frequencies are

$$\omega_1 = \frac{1.61}{L}\sqrt{\frac{E}{\rho}}, \omega_2 = \frac{5.63}{L}\sqrt{\frac{E}{\rho}} \qquad ---(14)$$

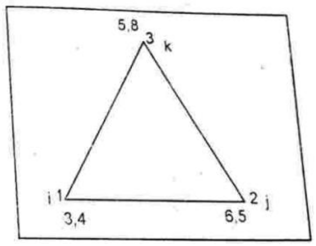
The exact solution can be readily verified to be

$$\omega_i = \frac{i\pi}{2L}\sqrt{\frac{E}{\rho}}, i = 1,3,5.......\infty \qquad ---(15)$$

$$\omega_1 = \frac{\pi}{2L}\sqrt{\frac{E}{\rho}}, \omega_2 = \frac{3\pi}{2L}\sqrt{\frac{E}{\rho}}, \omega_3 = \frac{5\pi}{2L}\sqrt{\frac{E}{\rho}}, ... \qquad ---(16)$$

18. The x, y co – ordinates of nodes i, j, and k an axisymmetric triangular element are given by (3, 4,) 6, 5) and (5, 8) cm respectively. The element displacement in cm vector is given as

q = [0.002, 0.001, 0.001, 0.004, -0.003, 0.007]^T. Determine the element strains. (NOV / DEC 2011)



We know that Strain displacement matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3` & z_3 \end{bmatrix}$$

Area

$$= \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 6 & 5 \\ 1 & 6 & 8 \end{vmatrix}$$

$$= \frac{1}{2}\left[(48-30) - 3(8-5) + 4(6-6)\right]$$

$$= 5\text{cm}^2$$

We know

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{3+6+5}{3}$$

$$r = 4.66$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{4+5+8}{3}$$

$$z = 5.66$$

$$\alpha_1 = r_2 z_2 - r_3 z_3 = 6\times 8 - 5\times 5$$

$$\alpha_1 = 23$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = 5\times 4 - 3\times 8$$

$$\alpha_2 = -4$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = 3\times 5 - 6\times 4$$

$$\alpha_3 = -9$$

$$\beta_1 = z_2 - z_3 = 5 - 8 \quad \beta_1 = -3$$

$$\beta_2 = z_3 - z_1 = 8 - 4 \quad \beta_2 = 4$$

$$\beta_3 = z_1 - z_2 = 4 - 5 \quad \beta_2 = -1$$

$$\gamma_1 = r_3 - r_2 = 5 - 6 \quad \gamma_1 = -1$$

$$\gamma_2 = r_1 - r_3 = 3 - 5 \quad \gamma_2 = -2$$

$$\gamma_3 = r_2 - r_1 = 6 - 3 \quad \gamma_3 = 3$$

We know that

$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{23}{4.66} - 3 + \frac{-1\times 5.66}{4.56} = 0.72$$

$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_3 z}{r} = \frac{-4}{4.66} + 4 + \frac{-2\times 5.66}{4.66} = 0.712$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-9}{4.66} - 1 + \frac{3\times 5.66}{4.66} = 0.712$$

$$[B]=\frac{1}{2\times 3}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0.72 & 0 & 0.712 & 0 & 0.712 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix}$$

Strain {e} = [B] {u}

$$\begin{Bmatrix} e_r \\ e_0 \\ e_z \\ \tau_{rz} \end{Bmatrix} = [B] \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0.72 & 0 & 0.712 & 0 & 0.712 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix} \begin{Bmatrix} 0.001 \\ 0.001 \\ 0.001 \\ 0.004 \\ -0.003 \\ 0.007 \end{Bmatrix}$$

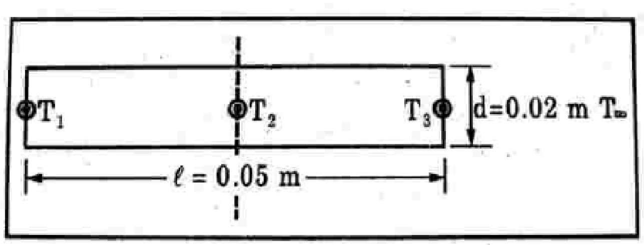
$$= \frac{1}{10}\begin{bmatrix} -0.006+0+0.004+0+0.003+0 \\ 0.00144+0+0.712+0-0.002136+0 \\ 0-0.001+0+0.008+0+0.021 \end{bmatrix}$$

$$= \begin{Bmatrix} 1\times 10^{-4} \\ -1.6\times 10^{-6} \\ 1.2\times 10^{-3} \\ -0.7\times 10^{-3} \end{Bmatrix}$$

19. A steel rod of diameter d = 2 cm, length t = 5 cm and thermal conductivity k = 50 W/m℃ is exposed at one end to a constant temperature of 320℃. The other end is in ambient air of temperature 20℃ with a convection co – efficient of h = 100 W/m^2℃. Determine the temperature at the midpoint of the rod using FEA. (NOV / DEC 2011)

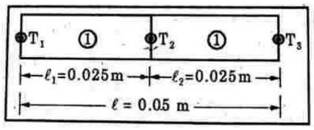
Given Data:

Diameter, d = 2 cm = 0.02 m
Length, ℓ = 5 cm = 0.05 m
Thermal conductivity, k = 50 W/m℃
One end temperature, T_1 = 320℃=273=593 K
Ambient air temperature, T_∞ = 20℃+273 = 293K
Convection co efficient, h = 100 W/m^2℃

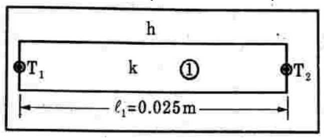


To find: Temperature at the midpoint of the rod (T_2)

Solution:- Discuss the rod into two equal size element



For element (Nodes 1, 2):



We know that, Finite element equation is

$$\left(\frac{Ak}{\ell_1}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}+\frac{hP\ell_1}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\right)\begin{Bmatrix}T_1\\T_2\end{Bmatrix}=\frac{QA\ell_1+PhT_\infty\ell_1}{2}\begin{Bmatrix}1\\1\end{Bmatrix} \quad --(1)$$

Where, P = Parimeter = π d = π × 0.02

$P = 0.0628$ m

A = Area $= \frac{\pi}{4}d^2 = \frac{\pi}{4}(0.02)^2$

$A = 3.14 \times 10^{-4}$ m^2.

Hear generation Q is not given, So neglect that terms, $\left(\frac{QA\ell_1}{2}\right)$

$$(1) \Rightarrow \left(\frac{Ak}{\ell_1}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}+\frac{hP\ell_1}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\right)\begin{Bmatrix}T_1\\T_2\end{Bmatrix}=\frac{PhT_\infty\ell_1}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

Substitute A , P, k, h, l_1 and T_∞ values,

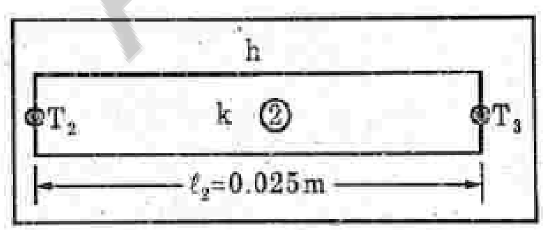
$$\left(\frac{3.14\times10^{-4}\times50}{0.025}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}+\frac{100\times0.0628\times0.025}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\right)\times\begin{Bmatrix}T_1\\T_2\end{Bmatrix}$$

$$=\frac{0.0628\times100\times293\times0.025}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\Rightarrow\left(0.628\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}+0.0262\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\right)\begin{Bmatrix}T_1\\T_2\end{Bmatrix}=23\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\left(\begin{bmatrix}0.628 & -0.628\\-0.628 & 0.628\end{bmatrix}+\begin{bmatrix}0.0524 & 0.0262\\0.0262 & 0.0524\end{bmatrix}\right)\begin{Bmatrix}T_1\\T_2\end{Bmatrix}=\begin{Bmatrix}23\\23\end{Bmatrix}$$

$$\Rightarrow\begin{bmatrix}0.6804 & -0.6018\\-0.6018 & 0.6804\end{bmatrix}\begin{Bmatrix}T_1\\T_2\end{Bmatrix}=\begin{Bmatrix}23\\23\end{Bmatrix} \quad ---(2)$$



For element 2: (Nodes 2, 3)

Since all the parameters in element (1) and (2) are same, the finite element equation becomes,

$$\Rightarrow\begin{bmatrix} 0.6804 & -0.6018 \\ -0.6018 & 0.6804 \end{bmatrix} \quad ---(3)$$

Assemble the finite element equation (2) and (3)

$$\begin{bmatrix} 0.6804 & -0.6017 & 0 \\ -0.6018 & 0.6804+0.6804 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 23 \\ 23+23 \\ 23 \end{Bmatrix}$$

$$\underset{[K]}{\begin{bmatrix} 0.6804 & -0.6017 & 0 \\ -0.6018 & 0.6804+0.6804 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix}}\underset{\{T\}}{\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}} = \underset{\{F\}}{\begin{Bmatrix} 23 \\ 46 \\ 23 \end{Bmatrix}} \quad ---(4)$$

To solve the above equation, the following steps to be followed.

Step 1: The first row and first column of the stiffness matrix [K] have been set equal to 0 except for the main diagonal, which has been set equal to 1.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.3608 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 23 \\ 46 \\ 23 \end{Bmatrix}$$

Step 2: The first row of the force matrix is replaced by the known temperature at node 1, i.e. T1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.3608 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 593 \\ 46 \\ 23 \end{Bmatrix}$$

Step 3: The second row, first column of stiffness matrix [K] value [From equation no. 4] is multiplied by known temperature at node 1, i.e, -0.6018 × 593 = -356.867. This value (as positive digit, i.e, 356.867) has been added to the second row of the force matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.3608 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 593 \\ 46+356.867 \\ 23 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.3608 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 593 \\ 402.867 \\ 23 \end{Bmatrix}$$

Solving equation (5),

$$1.3608T_2 - 0.6018T_3 = 402.867$$

$$-0.6018T_2 + 0.6804T_3 = 23 \quad ---(6)$$

Equation (6) × 1.1306,

$$1.5385T_2 - 0.6804T_3 = 455.485 \quad ---(8)$$

$$-0.6018T_2 + 0.6804T_3 = 23 \quad ---(7)$$

$$0.9367T_2 = 478.485$$

$$T_2 = 510.819\text{K}$$

Result: Temperature at the midpoint of the rod, T2 = 510.819 K

20. A wall of 0.6m thickness having thermal conductivity of 1.2 W/m. K the wall is to be insulated with a material of thickness 0.06 m having an average thermal conductivity of 0.3 W/m.K. The inner surface temperature is 1000°C and outside of the insulation is exposed to atmospheric air at 30°C with heat transfer co – efficient of 35 N/m²K. Calculate the nodal temperature using FEA. (NOV / DEC 2011)

Given:

Thickness of the wall, $l_1 = 0.6\text{m}$

Thermal conductivity of the wall, $k_1 = 1.2 W/mK$

Thickness of the insulation, $l_2 = 0.06$ m

Thermal conductivity of the insulation, $k_2 = 0.3$ W/ mK

Inner surface temperature, $T_1 = 1000°C+273=1273$ K

Atmosphere air temperature , $T_\infty = 30°C+273 = 303$ K

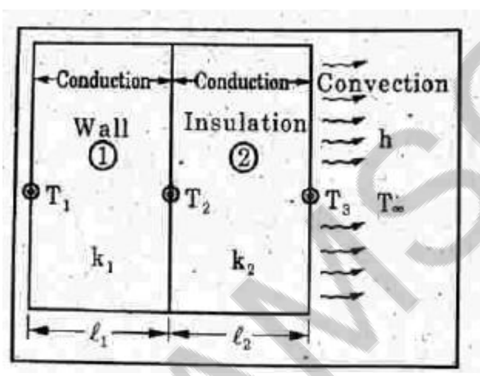
Heat transfer co efficient at outer side, $h = 35 W/m^2K$

To find: Nodal temperatures, (T2 and T3)

Solution:-

For element 1: (Nodes 1, 2)

Finite element equation is

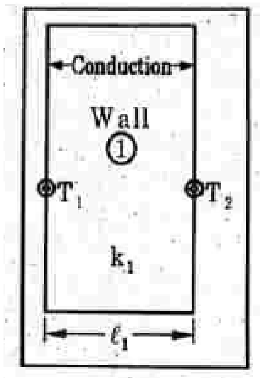


$$\frac{A_1 k_1}{\ell_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

For unit area, $A_1 = 1m^2$.

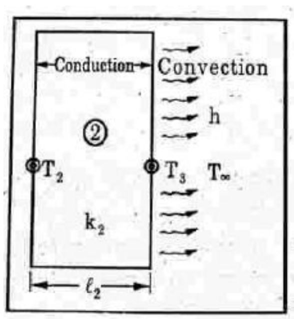
$$\frac{1.2}{0.6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad ---(1)$$



For element 2: (Nodes 2, 3)

This element is subjected to both conduction and convection. So, finite element equation is



$$\left(\frac{A_2 k_2}{\ell_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = hT_\infty A\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\left(\frac{1\times 0.3}{0.06}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + 35\times 1\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = 35\times 303\times 1\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\left(\begin{bmatrix}5 & -5\\ -5 & 5\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 35\end{bmatrix}\right)\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10.605\times 10^3\end{Bmatrix}$$

$$\begin{bmatrix}5 & -5\\ -5 & 40\end{bmatrix} = \begin{Bmatrix}0\\ 10.605\times 10^3\end{Bmatrix} \qquad ---(3)$$

Assemble the finite elements, i.e, assemble the finite element equation (1) and (2)

$$\begin{bmatrix}2 & -2 & 0\\ -2 & 2+5 & -5\\ 0 & -5 & 40\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\\ F_3\end{Bmatrix}$$

In this problem, there is no heat generation and there is no convection except from right end.

So, $\{F_1\} = \{F_2\} = 0$ & $\{F_3\} = 10.605 \times 10^3$

$$\Rightarrow \underset{[K]}{\underbrace{\begin{bmatrix} 2 & -2 & 0 \\ -2 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix}}} \underset{[T]}{\underbrace{\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}}} = \underset{\{F\}}{\underbrace{\begin{Bmatrix} 0 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}}} \qquad ---(3)$$

To solve the above equations, the following steps to be followed.

Step 1: The first row and first column of the stiffness matrix [K] have been set equal to 0 except for the main diagonal which has been set equal to 1.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step 2: The first row of the force matrix is replaced by the known temperature at node 1, i.e, T_1.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1273 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step 3: The second row, first column of stiffness matrix [K] value [From equation no. 3) is multiplied by know, temperature at node 1, i.e, -2 × 1273 = -2546. This value (As positive digit i.e, 2546) has been added to the second row of the force matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1273 \\ 2546 \\ 10.605 \times 10^3 \end{Bmatrix} \qquad ---(4)$$

Solving equation (4),

$$7T_2 - 5T_3 = 2546 \qquad ---(5)$$
$$-5T_2 + 40T_3 = 10.605 \times 10^3 \qquad ---(6)$$

Equation (5) × 8

$$56T_2 - 40T_3 = 20.368 \times 10^3 \qquad ---(7)$$
$$-5T_2 + 40T_3 = 10.605 \times 10^3 \qquad ----(8)$$
$$51T_2 = 30.973 \times 10^3$$
$$T_2 = 607.313\text{K}$$

Substitute T_2 value in equation (5),

$$7 \times 607.313 - 5T_3 = 2546$$
$$T_3 = 341.03\text{K}$$

Result: Nodal temperatures:

$$T_1 = 1273K$$
$$T_2 = 607.313K$$
$$T_3 = 341.03K$$

Verification: Hest flow through composite wall is given by

$$Q = \frac{\Delta T_{overall}}{R}$$

[From HMT data book, C.P. Kothandaraman, page No. 43 & 44]

Where, $\Delta T = T_1 - T_\infty$

$$R = \frac{1}{\frac{1}{h_{in}A} + \frac{\ell_1}{k_1A} + \frac{\ell_2}{k_2A} + \frac{\ell_3}{k_3A} + \frac{1}{h_{out}A}}$$

$$Q = \frac{T_1 - T_\infty}{\frac{1}{h_{in}A} + \frac{\ell_1}{k_1A} + \frac{\ell_2}{k_2A} + \frac{\ell_3}{k_3A} + \frac{1}{h_{out}A}}$$

Heat transfer coefficient at inner side (h_i) and thickness (l_3)are not given. So, neglect that terms.

$$Q = \frac{1273 - 303}{\frac{0.6}{1.2} + \frac{0.06}{0.3} + \frac{1}{35}}$$

$$Q = 1331.37\, W/m^2$$

We know that, interface temperatures

$$Q = \frac{T_1 - T_\infty}{R} = \frac{T_1 - T_2}{R_1} = \frac{T_2 - T_3}{R_2} = \frac{T_3 - T_\infty}{R_{outer}} \qquad ----(8)$$

$$Q = \frac{T_1 - T_2}{R_1}$$

$$Q = \frac{T_1 - T_2}{\frac{\ell_1}{k_1A}}$$

$$1331.37 = \frac{1273 - T_2}{\frac{0.6}{1.2}}$$

$$T_2 = 607.31K$$

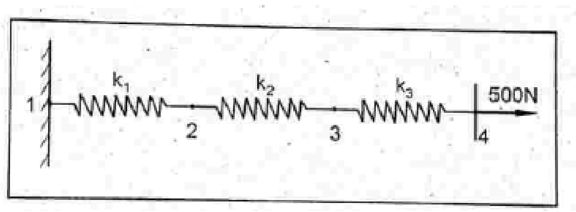
$$Q = \frac{T_2 - T_3}{R_2} = \frac{T_2 - T_3}{\frac{\ell_2}{k_2A}}$$

$$1331.37 = \frac{607.31 - T_3}{\frac{0.06}{0.3}}$$

$$T_3 = 341.03K$$

21 For the spring system shown in Figure 1, calculate the global stiffness matrix, displacements of nodes 2 and 3, the reaction forces at node 1 and 4. Also calculate the force is the spring 2. Assume, $k_1 = k_3 = 100\,N/m$

$k_2 = 200\,N/m$, $u_1 = u_4 = 0$ and $P = 500N$ (MAY / JUNE 2010)



$K_1 = K_3 = 100\,N/m$

$K_2 = 200\,N/m$

$U_1 = U_4 = 0$

$F_3 = 500N$

Finite element equation for spring element is

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Element 1

$$k_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$100\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad ..1$$

Element 2

$$k_2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$200\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{matrix} 2 \\ 3 \end{matrix}\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \qquad ..3$$

Element 3

$$k_3\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_3\\ F_4\end{Bmatrix}$$

$$\begin{matrix}3\\4\end{matrix}\begin{bmatrix}100 & -100\\ -100 & 100\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_3\\ F_4\end{Bmatrix} \qquad ..3$$

$$\begin{bmatrix}100 & -100 & 0 & 0\\ -100 & 100+200 & -200 & 0\\ 0 & -200 & 200+100 & -100\\ 0 & 0 & -100 & 100\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_2\\ F_3\\ F_4\end{Bmatrix}$$

$$\begin{bmatrix}100 & -100 & 0 & 0\\ -100 & 300 & -200 & 0\\ 0 & -200 & 300 & -100\\ 0 & 0 & -100 & 100\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_2\\ F_3\\ F_4\end{Bmatrix} \quad ..4$$

Applying boundary condition node 1 and 4 are fixed

$u_1 = u_4 = 0$

$F_1 = F_2 = F_4 = 0$

$F_3 = 500N$

$$\begin{bmatrix}100 & -100 & 0 & 0\\ -100 & 300 & -200 & 0\\ 0 & -200 & 300 & -100\\ 0 & 0 & -100 & 100\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ u_3\\ 0\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 500\\ 0\end{Bmatrix}$$

$\because\ U_1 = U_4 = 0$ neglect 1st row and 1st column and similarly 4th row and 4th column

$$\begin{bmatrix}300 & -200\\ -200 & 300\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}0\\ 500\end{Bmatrix}$$

$300u_2 - 200u_3 = 0 \qquad ..(5)$

$-200u_2 + 300u_3 = 500 \qquad ..(6)$

$(5)\times 2 \Rightarrow 600u_2 - 400u_3 = 0$

$(6)\times 3 \Rightarrow 600u_2 - 900u_3 = 15000$

$500u_3 = 1500$

$u_3 = 3m$

Sub u_3 in (5)

$300u_2 = 200\times3$

$= 600$

$u_2 = 2m$

Reaction forces

$\{R\} = [K]\{u\} - \{F\}$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix}$$

$$= \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 500 \\ 0 \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & -200 & -0 & -0 \\ 0 & -600 & -600 & -0 \\ 0 & -400 & -900 & -0 \\ 0 & -0 & -300 & -0 \end{bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 500 \\ 0 \end{Bmatrix}$$

$$= \begin{bmatrix} -200 \\ 0 \\ 500 \\ -300 \end{bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 500 \\ 0 \end{Bmatrix}$$

$$= \begin{bmatrix} -200 \\ 0 \\ 0 \\ -300 \end{bmatrix}$$

$R_1 = -200N$

$R_4 = -300N$

Solutions

Displacement $u_2 = 2n$

$u_3 = 3m$

Reaction force $R_1 = -200N$

$R_4 = -300N$

22. (a) Determine the joint displacements, the joint reactions, element forces and element stresses of the given truss elements. (MAY / JUNE 2010)

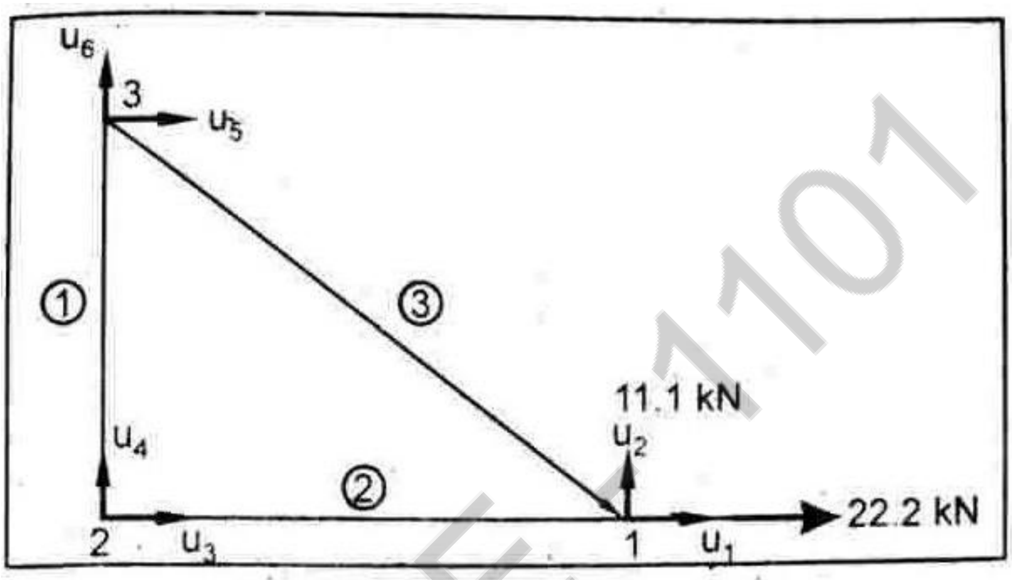


Fig. 2 Truss with applied load

Table 1: Element Property Data

Element	A cm^2	E N/m^2	L	Global Node Connection	α Degree
1	32.2	6.9e10	2.54	2 to 3	90
2	32.2	6.9e10	2.54	2 to 1	0
3	25.8	20.7e10	3.59	1 to 3	135

Co-ordinates are given below

$x_1 \quad y_1$

Node $1 = (2.54, 0)$*

$x_2 \quad y_2$

Node $2 = (0, 0)$

Node $3 = (0, 3.59)$

$x_3 \quad y_3$

Element (1)

$$le_1 = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$$

$$= \sqrt{(0 - 2.54)^2 - (0 - 0)^2}$$

$$= -2.54m$$

Direction cosine $l_1 = \frac{x_2 - x_1}{le_1} = \frac{0 - 2.54}{-2.54}$

$$l_1 = l_n$$

$$m_1 = \frac{y_2 - y_1}{le_1} = \frac{0 - 0}{-2.54}, \ m_1 = 0$$

Element (2)

$$le_2 = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}, \ le_2 = 3.59$$

$$= \sqrt{(0 - 0)^2 + (3.59 - 0)^2}, \quad l_2 = 0$$

$$m_2 = \frac{y_3 - y_2}{le_2} = \frac{3.59 - 0}{3.59}, \quad m_2 = 1$$

Element (3)

$$le_3 = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$

$$= \sqrt{(0 - 2.54)^2 + (3.59 - 0)^2}$$

$$= 4.39$$

$$l_3 = \frac{x_3 - x_1}{le_3} = \frac{0 - 2.54}{4.39} \qquad l_3 = -0.578$$

$$m_3 = \frac{y_3 - y_1}{le_3} = \frac{3.59 - 0}{4.39}, \quad m_3 = 0.818$$

Stiffness matrix

$$K = \frac{A_1E_1}{le_1}\begin{bmatrix} l_1^2 & l_1m_1 & -l_1^2 & -l_1m_1 \\ l_1m_1 & m_1^2 & -l_1m_1 & -m_1^2 \\ -l_1^2 & -l_1m_1 & l_1^2 & l_1m_1 \\ -l_1m_1 & -m_1^2 & l_1m_1 & m_1^2 \end{bmatrix}$$

Element (1)

$$K_1 = \frac{32.2\times6.9\times10^{10}}{-2.54}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element (2)

$$K_2 = \frac{38.7\times20.7\times10^{10}}{3.59}\begin{bmatrix} l_2^2 & l_2m_2 & -l_2^2 & -l_2m_2 \\ l_2m_2 & m_2^2 & -l_2m_2 & -m_2^2 \\ -l_2^2 & -l_2m_2 & l_2^2 & l_2m_2 \\ -l_2m_2 & -m_2^2 & l_2^2 & m_2^2 \end{bmatrix}$$

$$K_2 = 2.23\times10^{12}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Element (3)

$$K_3 = \frac{25.8\times20.7\times10^{10}}{4.39}\begin{bmatrix} l_3^2 & l_3m_3 & -l_3^2 & -l_3m_3 \\ l_3m_3 & m_3^2 & -l_3m_3 & -m_3^2 \\ -l_3^2 & -l_3m_3 & l_3^2 & l_3m_3 \\ -l_3m_3 & -m_3^2 & l_3m_3 & m_3^2 \end{bmatrix}$$

$$= 1.216\times10^{12}\begin{bmatrix} 0.334 & 0.473 & -0.334 & -0.473 \\ 0.473 & 0.669 & -0.473 & -0.669 \\ -0.334 & -0.473 & 0.334 & 0.473 \\ -0.473 & -0.669 & 0.473 & 0.669 \end{bmatrix}$$

Assembling all the stiffness matrix

0 -0.406	0 +0.575	0	0	-0.406	-0.575
0 +0.575	2.23 -0.813	0	-2.23	-0.575	-0.813

0	0	8.74 -0	0 - 0	-8.74	0
0	-2.23	0	0 +2.23	0	0
-0.406	-0.575	-8.74	0	8.74 +0.406	0.575
-0.575	-0.813	0	0	0 - 0.575	0 + 0.813

$$K = 1\times10^{12}\begin{bmatrix} 0.406 & 0.575 & 0 & 0 & -0.406 & -0.575 \\ 0.575 & 3.043 & 0 & -2.23 & -0.575 & -0.813 \\ 0 & 0 & 8.74 & 0 & -8.74 & 0 \\ 0 & -2.23 & 0 & 2.23 & 0 & 0 \\ -0.406 & -0.575 & -8.74 & 0 & 9.146 & 5.575 \\ -0.575 & -0.813 & 0 & 0 & 0.575 & 0.813 \end{bmatrix}$$

WK General Finite element equation is

$$\{F\} = [K]\{u\}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = 1\times10^{12}\begin{bmatrix} 0.406 & 0.575 & 0 & 0 & -0.406 & -0.575 \\ 0.575 & 3.043 & 0 & -2.23 & -0.575 & -0.813 \\ 0 & 0 & 8.74 & 0 & -8.74 & 0 \\ 0 & -2.23 & 0 & 2.23 & 0 & 0 \\ -0.406 & -0.575 & -8.74 & 0 & 9.146 & 5.575 \\ -0.575 & -0.813 & 0 & 0 & 0.575 & 0.813 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

Applying boundary condition

Node 2 & 3 is fixed

$$u_3 = u_4 = u_5 = u_6 = 0$$

At node 1 $F_2 = 11.1\text{kN}$ $\quad F_1 = 22.2\text{kN}$

$$= 1\times10^{12}\begin{bmatrix} 0.406 & 0.575 & 0 & 0 & -0.406 & -0.575 \\ 0.575 & 3.043 & 0 & -2.23 & -0.575 & -0.813 \\ 0 & 0 & 8.74 & 0 & -8.74 & 0 \\ 0 & -2.23 & 0 & 2.23 & 0 & 0 \\ -0.406 & -0.575 & -8.74 & 0 & 9.146 & 5.575 \\ -0.575 & -0.813 & 0 & 0 & 0.575 & 0.813 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 11.1\times10^{3} \\ 22.2\times10^{3} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\because U_3 = u_4 = u_5 = u_6 = 0$

delete 3, 4, 5, 6 row & column.

$$1\times10^{12}\begin{bmatrix} 0.406 & 0.575 \\ 0.575 & 3.043 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 11.1\times10^{3} \\ 22.2\times10^{3} \end{Bmatrix}$$

$$0.406\times10^{12}u_1 + 0.575\times10^{12}u_2 = 11.1\times10^{3} \quad \ldots(1)$$

$$0.575\times10^{12}u_1 + 3.043\times10^{12}u_2 = 22.2\times10^{3} \quad \ldots(2)$$

$(1)\times0.575$ $\qquad 0.233\times10^{12}u_1 + 0.330\times10^{12}u_2 = 6.38\times10^{3}$

$(2)\times0.406$ $\qquad 0.233\times10^{12}u_1 + 1.23\times10^{12}u_2 = 9.013\times10^{3}$

$$-0.905\times10^{12}u_2 = -2.633\times10^{3}$$

$$U_2 = 2.9\times10^{-09}\,m$$

Sub u_2 in (1)

$$0.406\times10^{12}u_1 + 1672.9 = 11.1\times10^{3}$$

$$u_1 = 2.32\times10^{-8}\,m$$

(2) Stress

$$\sigma = \frac{E}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

Element (1)

$$\sigma = \frac{6.9\times10^{10}}{2.54}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}-8\\-9\\0\\0\end{Bmatrix}$$

$$\sigma_1 = 2.716\times10^{10}\left[-2.32\times10^{-8}+0+0+0\right]$$

$$\sigma_1 = 630.2\,\text{N}/\text{m}^2$$

$$\sigma_1 = 0$$

Element (2)

$$\sigma_2 = \frac{2.07\times10^{10}}{3.59}\begin{bmatrix}0 & -1 & 0 & 1\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}$$

$$= 5.766\times10^{10}\begin{bmatrix}0 & -1 & 0 & 1\end{bmatrix}\begin{Bmatrix}2.32\times10^{-8}\\2.9\times10^{-9}\\0\\0\end{Bmatrix}$$

$$= 5.766\times10^{10}\begin{bmatrix}0 & -2.9\times10^{-9} & +0 & +0\end{bmatrix}$$

$$\sigma_2 = -167.2\,\text{N}/\text{m}^2$$

Element (3)

$$\sigma_3 = \frac{20.7\times10^{10}}{4.39}\begin{bmatrix}-l_3 & -m_3 & l_3 & m_3\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_5\\u_6\end{Bmatrix}$$

$$= 4.715\times10^{10}\begin{bmatrix}-0.578 & -0.818 & -0.578 & 0.819\end{bmatrix}\begin{Bmatrix}2.32\times10^{-8}\\2.9\times10^{-9}\\0\\0\end{Bmatrix}$$

$$= 4.715\times10^{10}\left[1.340\times10^{-8}-2.372\times10^{-9}-0-0\right]$$

$$= 4.715\times10^{10}\left[1.102\times10^{-8}\right]$$

$\sigma_3 = 519.9\,\text{N/m}^2$

(3) Reaction forces

$\{R\} = [K]\{u\} - \{F\}$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = 1\times10^{12} \begin{bmatrix} 0.406 & 0.575 & 0 & 0 & -0.406 & -0.575 \\ 0.575 & 3.043 & 0 & -2.23 & -0.575 & -0.813 \\ 0 & 0 & 8.74 & 0 & -8.74 & 0 \\ 0 & -2.23 & 0 & 2.23 & 0 & 0 \\ -0.406 & -0.575 & -8.74 & 0 & 9.146 & 5.575 \\ -0.575 & -0.813 & 0 & 0 & 0.575 & 0.813 \end{bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

$$R = 1\times10^{12}\left[0.416 \quad 0.575 \quad 0 \quad 0 \quad -0.406 \quad -0.575\right]$$

$$= \begin{Bmatrix} 2.32\times10^{-8} \\ 2.9\times10^{-9} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 11.1\times10^{3} \\ 22.2\times10^{3} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 1\times10^{12}\left[9.65\times10^{-9} - 1.66\times10^{-9} + 0 + 0 + 0 + 0\right] - 11.1\times10^{3}$$

$R_1 = -3110\text{N}$

$$R_2 = 1\times10^{12}\left[0.575 \quad 3.043 \quad 0 \quad -2.23 \quad -0.575 \quad -0.813\right]$$

$$\begin{Bmatrix} 2.32\times10^{-8} \\ 2.9\times10^{-9} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - 22.2\times10^{3}$$

$$= 1\times10^{-12}\left[1.334\times10^{-8} - 8.824\times10^{-9}\right]$$

$R_2 = 22164\text{N}$

$$R_3 = 1\times10^{-12}\left[1.334\times10^{-8} + 8.824\times10^{-9}\right]$$

$R_2 = 22164\text{N}$

$$R_3 = 1\times10^{12}\begin{bmatrix}0 & 0 & 8.74 & 0 & -8.74 & 0\end{bmatrix}\begin{Bmatrix}2.32\times10^{-8}\\2.9\times10^{-9}\\0\\0\\0\\0\end{Bmatrix}-0$$

$$=0$$

$$R_3 = 1\times10^{12}\begin{bmatrix}0 & -2.23 & 0 & 2.23 & 0 & 0\end{bmatrix}\begin{Bmatrix}2.32\times10^{-8}\\2.9\times10^{-9}\\0\\0\\0\\0\end{Bmatrix}-0$$

$$=1\times10^{12}\left[0-6.467\times10^{-19}\right]$$

$$R_4 = 64.67\text{N}$$

$$R_5 = \begin{bmatrix}-0.406 & -0.575 & -8.74 & 0 & 9.146 & 0.575\end{bmatrix}$$

$$\begin{Bmatrix}2.32\times10^{-8}\\2.9\times10^{-9}\\0\\0\\0\\0\end{Bmatrix}-0$$

$$=1\times10^{12}\left[-9.419\times10^{-19}-1.667\times10^{-9}-0+0+0+0\right]-0$$

$$R_5 = 11086\text{N}$$

$$R_6 = \begin{bmatrix}-0.575 & -0.813 & 0 & 0 & 0.575 & 0.813\end{bmatrix}\begin{Bmatrix}2.32\times10^{-8}\\2.9\times10^{-9}\\0\\0\\0\\0\end{Bmatrix}-0$$

$$=1\times10^{12}\left[-1.667\times10^{-9}-2.357\times10^{-9}+0+0+0+0\right]-0$$

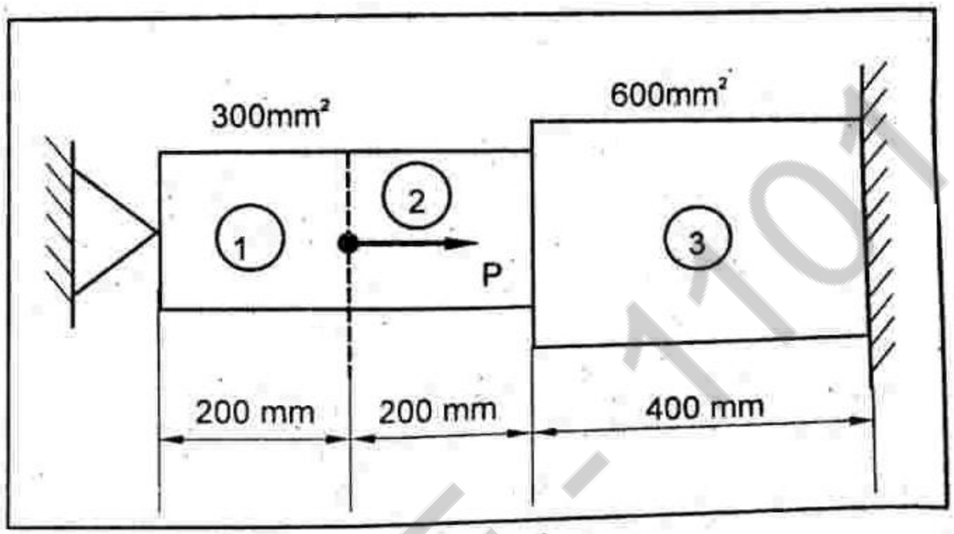
$$R_6 = -4024.7\text{N}$$

23 Consider the bar as shown in fig 3 calculate the following

(i) Nodal displacement

(ii) Element stresses

(iii) Support reactions (MAY / JUNE 2010)



Fig

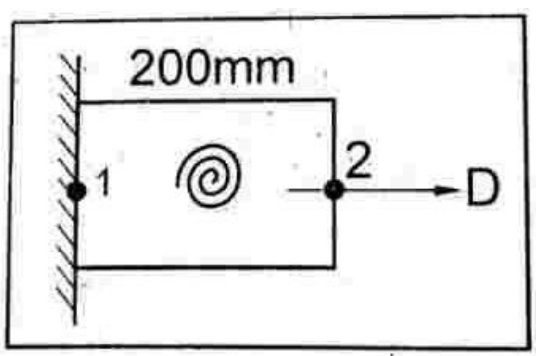
Given data

$A_2 = A_1 = 300mm^2$ $l_1 = l_2 = 200mm$ $E = 2\times10^3\ N/mm^2$

$A_3 = 600mm^2$ $l_3 = 400mm$ $p = 400kN$

Element (1)

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{A_1E}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



$= 3$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{A_1 E_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{300 \times 2 \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \ldots(1)$$

Element (2)

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{A_2 E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{300 \times 2 \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \ldots(2)$$

Element (3)

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = \frac{A_3 E_3}{l_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{600 \times 2 \times 10^5}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} \quad \ldots(3)$$

Assemble all the equations

$$1 \times 10^5 \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 3+3 & -3 & 0 \\ 0 & -3 & 3+3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \quad \ldots(4)$$

Apply boundary conditions

$u_1 = u_4 = 0$

$F_2 = 400 \times 10^3\,\text{N}$

$F_1 = F_3 = F_4 = 0$

$$1\times10^5\begin{bmatrix}3 & -3 & 0 & 0\\ -3 & 6 & -3 & 0\\ 0 & -3 & 6 & -3\\ 0 & 0 & -3 & 3\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ u_3\\ 0\end{Bmatrix}=\begin{Bmatrix}0\\ 400\times10^3\\ 0\\ 0\end{Bmatrix}$$

Neglect 1st & 4th row & column

$$1\times10^5\begin{bmatrix}6 & -3\\ -3 & 6\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}400\times10^3\\ 0\end{Bmatrix}$$

$$6\times10^5 u_2-3\times10^5 u_3=400\times10^3 \quad \ldots(1)$$

$$-3\times10^5 u_2-6\times10^5 u_3=0 \quad \ldots(2)$$

Multiply equation (2) by 2

$$6\times10^5 u_2-3\times10^5 u_3=400\times10^3$$

$$-6\times10^5 u_3-12\times10^5 u_3=0$$

On solving $9\times10^5 u_3=400\times10^3$

$$U_3=0.444\text{mm}$$

Substitute U_3 in equation (1)

$$6\times10^5 U_2-3\times10^5(0.444)=400\times10^3$$

$$6\times10^5 u_2=400\times10^3-3\times10^5(0.444)$$

$$U_2=0.888\text{mm}$$

(2) Stress

$$\sigma=E\frac{du}{dx}$$

Element (1)

$$\sigma_1=E_1\times\frac{u_2-u_1}{l_1}$$

$$=2\times10^5\times\frac{0.8888-0}{200}$$

$$\sigma_1=888.88\,\text{N/mm}^2$$

Element (2)

$$\sigma_2 = E_2 \times \frac{u_3 - u_2}{l_2}$$

$$= 2 \times 10^5 \times \frac{0.444 - 0.888}{200}$$

$\sigma_2 = 444.44\, N/mm^2$ (a compressive stress)

Element (3)

$$\sigma_3 = E_3 \times \frac{u_4 - u_3}{l_3}$$

$$= 2 \times 10^5 \times \frac{0 - 0.444}{400}$$

$$\sigma_3 = 222.22\, N/mm^2$$

(Compressive stress)

$$\{R\} = [K]\{u\} - \{F\}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{bmatrix} -3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.3888 \\ 0.444 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 400 \times 10^3 \\ 0 \\ 0 \end{Bmatrix}$$

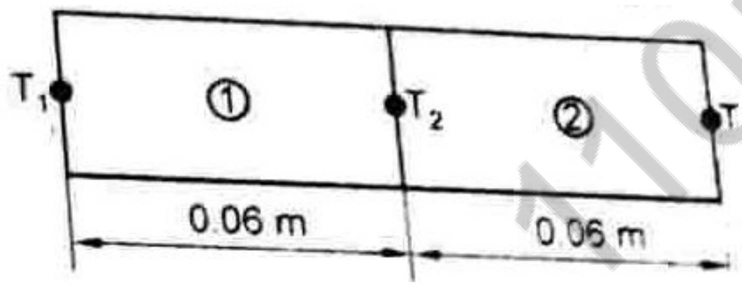
$$= 1 \times 10^5 \begin{bmatrix} 0 & -3 \times 0.888 & -0 & -0 \\ 0 & +6 \times 0.888 & -3 \times 0.444 & +0 \\ 0 & -3 \times 0.888 & +6 \times 0.444 & -0 \\ 0 & +0 & -3 \times 0.444 & -0 \end{bmatrix} - \begin{Bmatrix} 0 \\ 400 \times 10^3 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{bmatrix} -2.666 \\ 4 \\ 0 \\ -1.333 \end{bmatrix} - \begin{Bmatrix} 0 \\ 400 \times 10^3 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{Bmatrix} -2.66\times10^5 \\ 0 \\ 0 \\ -1.333\times10^5 \end{Bmatrix}$$

24. (b) Calculate the temperature distribution in a 1D fin with physical propaies. The fin is rectangular in shape and its 120mm long, 40 mm wide and 10 mm thick. Assume that convection heat loss occur from one end of the fin. Take k = 0.3 W/mmC. T = 20°C h = 1 × 10^{-3} × W/mm^{2}C (MAY / JUNE 2010)

Given data:



$l = 120\text{mm} = 0.12\text{m}$

$w = 40\text{mm} = 0.04\text{m}$

$t = 10\text{mm} = 0.01\text{m}$

$k = 0.3\,\text{W/mm°C}$

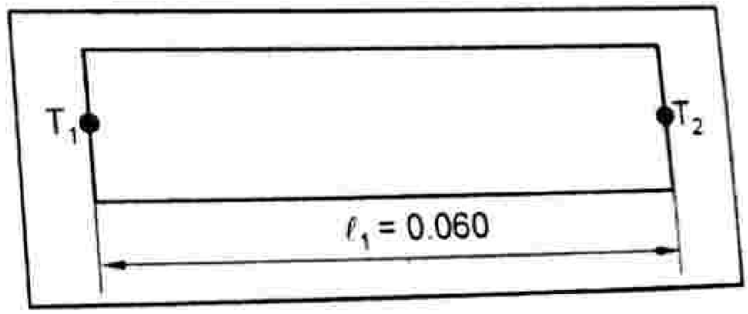
$= 0.3\times10^3\ \text{W/mm°C} = 300\,\text{W/mm°C}$

heat transfer coeff $h = 1\times10^{-3}\ \text{W/mm}^2\,°\text{C}$

$= 1\times10^{-3}\times10^6\ \text{w/m}^2\,°\text{C} = 1\times10^3\ \text{w/m}^2\,°\text{C}$

$T_\infty = 20°\text{C} = 20+273 = 293\text{K}$

$= T_1 = 120°\text{C} = 120+273 = 393\text{K}$



Element (1)

WKT

$$\left\{\frac{AK}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{hPl_1}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right\}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix}$$

$$=\frac{QAl_1+PhT_\infty l_1}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

$$P=\text{Perimeter}=2\left(w+t\right)$$

$$=2\left[0.04+0.01\right]=0.1m$$

$$A=\text{area}=\text{width}\times\text{thickness}$$

$$=0.04\times 0.01$$

$$A=4\times 10^{-4}\,m^2$$

$\because$ Q is not given neglect $\frac{QAl_1}{2}$

So equation (1) becomes

$$\left\{\frac{AK}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{hPl_1}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right\}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix}=\frac{PhT_\infty l_1}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

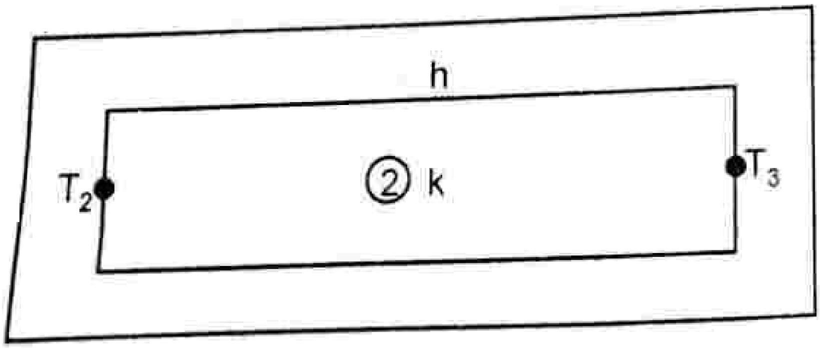
$$\left[\frac{4\times 10^{-4}\times 300}{0.06}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{1000\times 0.1\times 0.06}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right]\begin{Bmatrix}T_1\\ T_2\end{Bmatrix}$$

$$=\frac{0.1\times 1000\times 293\times 0.06}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

$$\left[\begin{bmatrix}2 & -2\\ 2 & -2\end{bmatrix}+\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right]\begin{Bmatrix}T_1\\ T_2\end{Bmatrix}=\begin{Bmatrix}879\\ 879\end{Bmatrix}$$

$$\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix}=\begin{Bmatrix}879\\ 879\end{Bmatrix}$$

Element (2)



$\because$ the parameters of element 1 and 2 are same, then the FE equation becomes

$$\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 879 \\ 879 \end{Bmatrix} \quad \ldots(2)$$

Assembling equation (1) and (2)

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4-4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 879 \\ 1958 \\ 879 \end{Bmatrix} \quad \ldots(3)$$

To solve the above equation the following steps to be followed.

1. Set the 1st row and column to be zero, expect for the main diagnol

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 879 \\ 1758 \\ 879 \end{Bmatrix}$$

2. 1st row of force matrix is replaced by the temperature at node (1)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 879 \\ 393+1758 \\ 879 \end{Bmatrix}$$

3. In 2nd row the 1st column of matrix is multiplied by a known temperature at node (1)

$$-1 \times 393 = -393$$

The value is added to 2nd row

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 879 \\ 393+1258 \\ 879 \end{Bmatrix} = \begin{Bmatrix} 393 \\ 2151 \\ 879 \end{Bmatrix} \quad \ldots(4)$$

On solving equation (4)

$$8T_2 - T_3 = 2151 \quad \ldots(5)$$

$-T_2 + 4T_3 = 879$

Multiple equation (6)

$8T_2 - 32T_3 = 7032$

$8T_2 - T_3 = 2151$

$31T_2 = 9183$

On sub we get

$T_3 = 296.23K$

$T_2 = 305.9K$

Unit III

2 MARKS

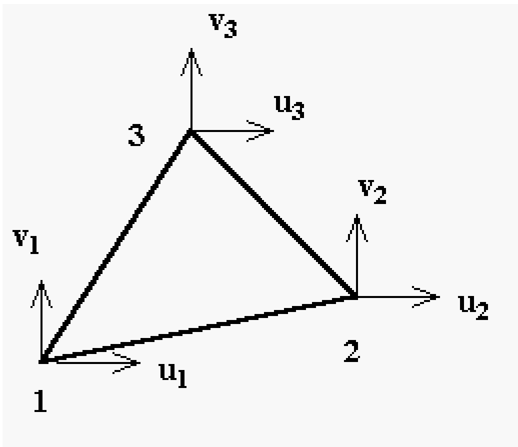
29. Explain the important properties of CST elements. (Nov/Dec 2008)

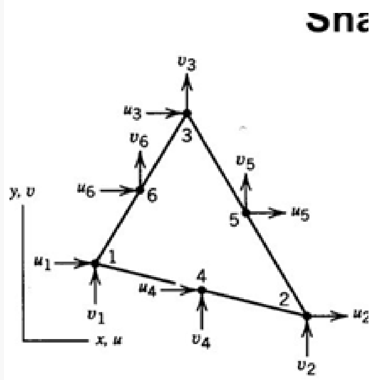
The strain components are constant throughout the volume of the element.

It has six unknown displacement degree of freedom.

30. What are CST and LST elements? (Nov/Dec 2009)

	Constant strain triangle (CST) element	**Linear strain triangle (LST)element**
(i)	It is 3 noded triangular element	It is a 6 nodal triangular element
(ii)	It has 6 degrees of freedom	It has 12 degrees of freedom





31. Distinguish between plane stress and plane strain problems. (Nov/Dec 2009)

Plane stress: It is defined as a state of stress in which the normal stress (σ) and shear is directly perpendicular to plane are assumed to be zero.

Plane strain: It is defined as a state of strain in which the strain normal to the XY plane and the shear strains are assumed to be zero.

32. Define plane strain analysis. (NOV / DEC 2011)

It is defined as an analysis in which plane strain to be a state of strain in which the strain normal to the xy plane and the shear strains are assumed to be zero.

33. Write down the nodal displacement equations for a two dimensional triangular elasticity element. (MAY / JUNE 2010)

Strain displacement matrix

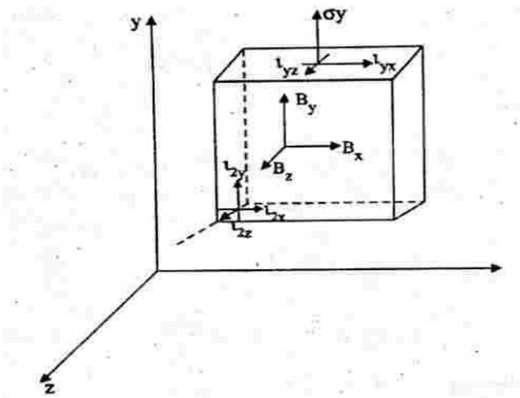
$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

16 MARKS

11. Derive the equation for equilibrium in case on the three dimensional stress system. (Nov /Dec 2008)

If a body is in equilibrium under specified static loads, the reactive forces and moments developed at the support points must balance the entrenally applied force and moment. In order words, the force and moment equilibrium equations for the overall body must be satish field.

Consider a three dimensional element as shown. It is subjected to normal stress $\sigma_x, \sigma_y, \sigma_z$ shear stress $\tau_{xy,} \tau_{yz}, \tau_{zx}$ and body forces B_x, B_y, B_z as shown.



Three dimensional stress element

Adding all the forces acting on the element in x direction

$\Sigma F_x = 0$

$$\left(\sigma_x + \frac{\partial \sigma_x}{\sigma_x} dx\right) dydz - \sigma_x dydz + \left[\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy\right] dndz = \tau_{ny} dndz$$

$$+\left[\tau_{yz} + \frac{\partial \tau_{xz}}{\partial z} dz\right] dndy - \tau \ln zdndy + B_n dn\, dy\, dz = 0$$

$$\sigma_x dy\ dz + \frac{\partial \sigma_x}{\partial_x} dx\ dy\ dz - \sigma_x dy\ dz$$

$$+\tau_{xy} dn\ dz + \frac{\partial z_{xy}}{\partial y} dy\ dx\ dz - \tau_{ny} dn\ dz\ + \tau_{nz} dn\ dy$$

$$+b_x dn\ dy\ dz = 0$$

$$\frac{\partial \sigma_x}{\partial x} dx\ dy\ dz + \frac{\partial \tau_{ny}}{dy} dn\ dy\ dz$$

$$+\frac{\partial \tau_{xz}}{\partial z} dn\ dy\ dz + B_x dn\ dy\ dz = 0$$

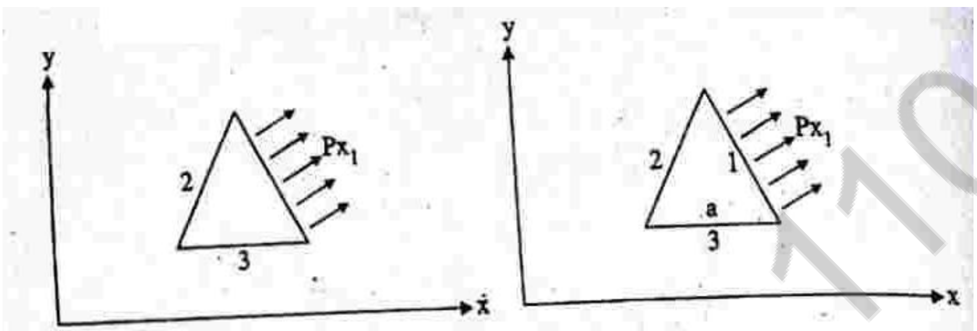
Divided by dx dy dz

$$\frac{\partial \sigma}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0$$

Similarly, adding all the forces acting on the element in the y and z directions.

$$\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+\frac{\partial\tau_{xy}}{\partial x}+B_y=0$$
$$\frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+B_z=0$$

12. Find the expression for nodal force vector in CST element shown in fig (1) subjected to a pressure P_{x1} on side1. (Nov /Dec 2008)



We know that,

$$\{F_s\}=+\int_0^t \begin{bmatrix} N_1P \\ 0 \\ N_2P \\ 0 \\ N_3P \\ 0 \end{bmatrix} dy$$

At x = a, y = y

$$N_1=\frac{P_1+q_1x+r_1y}{2A}$$
$$N_2=\frac{P_2+q_2x+r_2y}{2A}$$
$$N_3=\frac{P_3+q_3x+r_3y}{2A}$$
$$P_1=x_2y_3-y_2x_3$$
$$q_1=y_2-y_3$$
$$r_1=x_3-x_2$$

Substitute the co – ordinate

P1 = 0

q1 = 0

r1 =0

Substitute P_1, q_1, r_1 value is shape function N_2 thus N_1

$$N_2 = \frac{L(a-x)}{2A}$$
$$N_3 = \frac{Lx-ay}{2A}$$

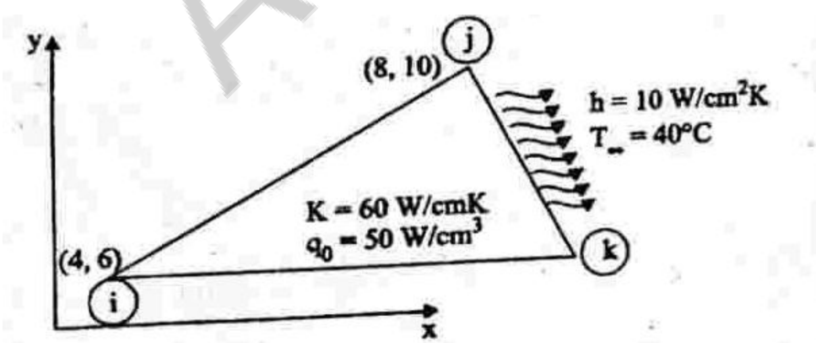
Substitute N1 , N2 and N3

$$\{F_s\} = \frac{t}{2\left(\frac{aL}{2}\right)}\begin{Bmatrix} a\left(\frac{L_2}{2}\right)P \\ 0 \\ 0 \\ 0 \\ \left(L^2 - \frac{L^2}{2}\right)ap \\ 0 \end{Bmatrix}$$

By simplifying we get

$$\{F_s\} = \begin{Bmatrix} F_s 1x \\ F_s 1y \\ F_s 2x \\ F_s 2y \\ F_s 3x \\ F_s 3y \end{Bmatrix} = \begin{Bmatrix} PLt/2 \\ 0 \\ 0 \\ 0 \\ PLt/2 \\ 0 \end{Bmatrix}$$

13. Compute element materials and vectors for the element is shown, when the edge K_j experience convection head loss. (Nov /Dec 2008)



For convection, we know the

$$[k_1]_e = \frac{60}{4\times 12}\begin{bmatrix} (2^2+4^2) & (4-32) & (-8+10) \\ (4-32) & (2^2-8^2) & (-8-32) \\ (-8+10) & (-8-32) & (-4^2+4^2) \end{bmatrix}$$

$$= \frac{60}{4\times 12}\begin{bmatrix} 20 & -28 & 8 \\ -28 & 68 & -40 \\ 8 & -40 & 32 \end{bmatrix}$$

$$[k_1]_e = \begin{bmatrix} 25 & -35 & 10 \\ -35 & 85 & -50 \\ 10 & -50 & 40 \end{bmatrix}$$

$$[k_2] = \frac{h_{kj}\ell_{kj}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[k_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.49 & 7.45 \\ 0 & 7.45 & 14.9 \end{bmatrix}$$

Vectors element using triangular element

$$[\vec{P}_1] = \frac{q_0 A_e}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \Rightarrow \frac{50\times 12}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$[\vec{P}_1] = \begin{Bmatrix} 200 \\ 200 \\ 200 \end{Bmatrix}$$

k = 60 w/ cmk;

h_{kj} = 10w/cm^2 k

q_0 = 50 w/ cm^2.

Element matrix,

$$A_e = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{matrix} 1 & 4 & 6 \\ 1 & 8 & 10 \\ 1 & 12 & 8 \end{matrix}$$

$$= \frac{1}{2}\left[(64-120) - 4(8-10) + 6(12-8)\right]$$

$$A = 12\text{cm}^2$$

Stiffness Matrix for conduction

$$[k_1]_e = \frac{kx}{4A_e}\begin{bmatrix} b_i^2 & b_{ij} & b_{ik} \\ b_{ij} & b_j^2 & b_j b_k \\ b_i b_k & b_j b_k & b_k^2 \end{bmatrix} + \frac{ky}{4A_e}\begin{bmatrix} c_i^2 & c_{ij} & c_{ik} \\ c_{ij} & c_j^2 & c_j c_k \\ c_i c_k & c_j c_k & c_k^2 \end{bmatrix}$$

Where

$$b_i = (y_2 - y_3) = (10-8) \Rightarrow 2$$
$$b_j = (y_3 - y_1) = (8-6) \Rightarrow 2$$
$$b_k = (y_1 - y_2) = (6-10) \Rightarrow -4$$
$$c_i = (x_3 - x_2) = (12-8) \Rightarrow 4$$
$$c_j = (x_1 - x_3) = (4-12) \Rightarrow -8$$
$$c_k = (x_2 - x_1) = (8-4) \Rightarrow 4$$

Similarly

$[\vec{P}_2] = 0$, since no boundary heat

$$[\vec{P}_3] = \frac{h_{kj} j_\infty \ell_{kj}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{10 \times 40 \times 4.47}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

$$[\vec{P}_3] = \begin{Bmatrix} 0 \\ 994 \\ 994 \end{Bmatrix}$$

Result

$$[k_1] = \begin{bmatrix} 25 & -35 & 10 \\ -35 & 85 & -50 \\ 10 & -50 & 40 \end{bmatrix} \text{For conduction}$$

$$[k_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.49 & 7.45 \\ 0 & 7.45 & 14.9 \end{bmatrix} \text{For conduction}$$

(ii) Vector element

$$[\vec{P}_1] = \begin{Bmatrix} 200 \\ 200 \\ 200 \end{Bmatrix}$$

$$[\vec{P}_2] = 0$$

$$[\vec{P}_3] = \begin{Bmatrix} 0 \\ 994 \\ 994 \end{Bmatrix}$$

14. For the constant strain triangular element in fig (i) assemble strain – displacement Matrix Take t = 20 mm and E = 2× 10^5 N/mm^2. (Nov /Dec 2008)

Given

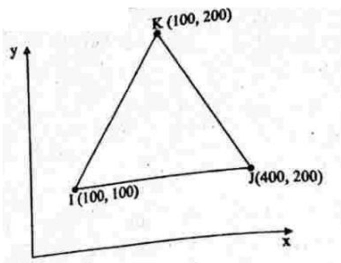
$x_1 = 100$ $\quad\quad$ $y_1 = 100$

$x_2 = 400$ $\quad\quad$ $y_2 = 100$

$x_3 = 200$ $\quad\quad$ $y_3 = 400$

Youngs Modulus E = 2× 10^5 N/mm^2.

t = 20 mm



Solution:-

Strain – displacement Matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \quad ----(1)$$

Where

$$q_1 = y_2 - y_3 = 100 - 400 = -300$$
$$q_2 = y_3 - y_1 = 400 - 100 = 300$$
$$q_3 = y_1 - y_2 = 100 - 100 = 0$$
$$r_1 = x_3 - x_2 = 200 - 400 = -200$$
$$r_2 = x_1 - x_3 = 100 - 200 = -100$$
$$r_3 = x_2 - x_1 = 400 - 100 = 300$$

Substitute the above values

$$B = \frac{1}{2A}\begin{bmatrix} -300 & 0 & 300 & 0 & 0 & 0 \\ 0 & -200 & 0 & -100 & 0 & 300 \\ -200 & -300 & -100 & 300 & 300 & 0 \end{bmatrix}$$

A = area of element

$$= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 100 & 100 \\ 1 & 400 & 100 \\ 1 & 200 & 400 \end{vmatrix}$$

$$= \frac{1}{2} \times \left[1(400 \times 400 - 200 \times 100) - 100(400 - 100) + 100(200 - 400)\right]$$

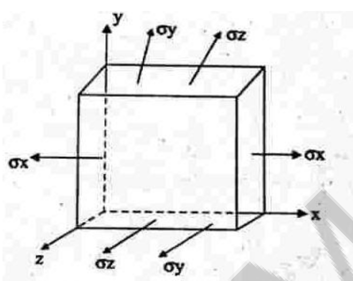
A = 45000m^2.

Sub A value in above eqn.

$$[B] = \frac{1}{2 \times 45000}\begin{bmatrix} -300 & 0 & 300 & 0 & 0 & 0 \\ 0 & -200 & 0 & -100 & 0 & 300 \\ -200 & -300 & -100 & 300 & 300 & 0 \end{bmatrix}$$

$$[B] = \frac{1}{900}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 3 \\ -2 & -3 & -1 & 3 & 3 & 0 \end{bmatrix}$$

15. Stress – Strain Relationship Matrix. Hooks law states that when a material is loaded with in its elastic limit, the stress is directly proportional to the strain. (Nov /Dec 2008)



Hooks law states that when a material is loaded with in its elastic limit, the stress is directly proportional to the strain.

Stress ∝ Strain

$$\sigma \propto \rho$$
$$\sigma = E_{\rho}$$
$$\rho = \frac{\sigma}{E}$$

Where ρ = strain

σ = stress N/mm^2.

E = youngs modulus

$$\rho_x^1 = \frac{\sigma}{E}$$

Where

$$-\rho_x^{11} = \frac{v\sigma_y}{E}$$
$$-\rho_x^{11} = \frac{-v\sigma_y}{E}$$

v = poisons ratio

Similarly

$$-\rho_x^{111} = \frac{v\sigma_z}{E}$$
$$\rho_x^{111} = \frac{-v\sigma_z}{E}$$

By applying super position principle

$$\rho_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

Strain in y direction $\rho_x = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$

Strain in z direction $\rho_x = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E} + \frac{\sigma_z}{E}$

Therefore $\sigma_x = \frac{E}{(1+v)(1-2v)}\left[\rho_x(1-v) + v\rho_y + v\rho_z\right]$

$$\sigma_x = \frac{E}{(1+v)(1-2v)}\left[v\rho_x + (1-v)\rho_y + v\rho_z\right]$$
$$\sigma_x = \frac{E}{(1+v)(1-2v)}\left[v\rho_x + v\rho_y + (1-v)\rho_z\right]$$

The shear strain relationship

$$\tau = G\gamma$$

T → Shear stress

W → Shear strain

G → Modulus of rigidity

$$\tau_{xy} = G\gamma_{xy}$$
$$\tau_{yz} = G\gamma_{yz}$$
$$\tau_{zx} = G\gamma_{zx}$$
$$G = \frac{E}{2(1+v)}$$
$$\tau_{xy} = \frac{E}{2(1+v)}\gamma_{xy}$$
$$\tau_{yz} = \frac{E}{(1+v)(1-2v)} \times \left(\frac{1-2v}{2}\right) \times \gamma_{yz}$$
$$\tau_{zx} = \frac{E}{(1+v)(1-2v)} \times \left(\frac{1-2v}{2}\right) \times \gamma_{zx}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} \rho_x \\ \rho_y \\ \rho_z \\ \gamma_x \\ \gamma_y \\ \gamma_z \end{Bmatrix}$$

The above equation is of form

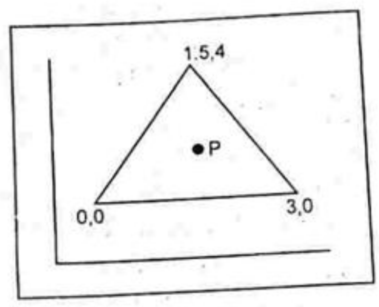
$$\{\sigma\} = [D]\{e\}$$

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

Where E = modulus of elastic

v = poisons ratio

16. (i) The x, y co – ordinates of nodes i, j, and k of a triangular element are given by (0, 0) and (1.5, 4) mm respectively Evaluate the shape functions N1, N2 and N3 at an interior point P(2, 2.5) mm for the element. (Nov/Dec 2009)



$$x = N_1x_1 + N_2x_2 + N_3x_3$$
$$Y = N_1y_1 + N_2y_2 + N_3y_3$$
$$2 = N_1 \times 0 + N_2 \times 3 + N_3 \times 1.5$$
$$2.5 = N_1 \times 0 + N_2 \times 0 + N_3 \times 4$$
$$3N_2 + 1.5N_3 = 2 \qquad ----(1)$$
$$4N_3 = 2.5 \qquad ----(2)$$
$$N_3 = 2.5/4$$
$$N_3 = 0.625$$

Sub N3 in equn (1)

$$3N_2 + 1.5 \times 0.625 = 2$$
$$N_2 = 0.354$$

We know that $N_1 + N_2 + N_3 = 1$

$N_1 = 1 - N_2 - N_3$

$= 1 - 0.625 - 0.354$

16. (ii) For the same triangular element, obtain the stain – displacement relation matrix B. (Nov/Dec 2009)

$$x_1 = 0 \quad y_1 = 0$$
$$x_2 = 3 \quad y_2 = 0$$
$$x_3 = 1.5 \quad y_3 = 4$$

Assume $E = 2 \times 10^5 \ N/mm^2$

t = 20 mm

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & `0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \qquad ---(1)$$

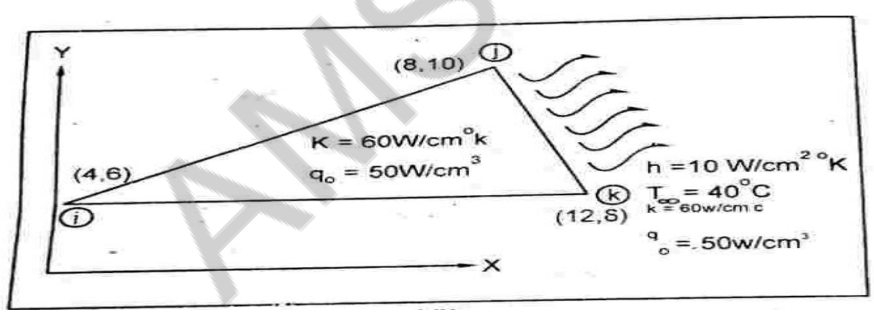
$$q_1 = y_2 - y_3 \Rightarrow 0 - 4 = -4$$
$$q_2 = y_3 - y_1 \Rightarrow 4 - 0 = 4$$
$$q_3 = y_1 - y_2 \Rightarrow 0 - 0 = 0$$
$$r_1 = x_3 - x_2 \Rightarrow 1.5 - 3 = -1.5$$
$$r_2 = x_1 - x_3 \Rightarrow 0 - 1.5 = -1.5$$
$$r_3 = x_2 - x_2 \Rightarrow 3 - 0 = 3$$

Sub the values in (1)

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1.5 & 4 \end{vmatrix}$$
$$= \frac{1}{2}\left[1(12-0) - 0(4-0) + 0(1.5-3)\right]$$
$$= 6 \text{ mm}^2$$

$$B = \frac{1}{2 \times 6}\begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & -1.5 & 0 & 3 \\ 1.5 & -4 & -1.5 & 4 & 3 & 0 \end{bmatrix}$$
$$= \frac{1}{12}\begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & -1.5 & 0 & 3 \\ 1.5 & -4 & -1.5 & 4 & 3 & 0 \end{bmatrix}$$

17. (b) Compute element materials and vectors for the element shown in Fig. 3, when the edge kj experience convection heat loss. (Nov/Dec 2009)



$$h = 10 \ \text{W/cm}^2\text{K}$$
$$T_\infty = 40°\text{C}$$

Stiffness matrix for conduction is given as

$$[k_1]_e = \frac{k_x}{4Ae}\begin{bmatrix} b_i^2 & b_ib_j & b_ib_k \\ b_ib_j & b_j^2 & b_jb_k \\ b_ib_k & b_jb_k & b_k^2 \end{bmatrix} + \frac{k_y}{4Ae}\begin{bmatrix} c_i^2 & c_1c_j & c_ic_k \\ c_jc_j & c_j^2 & c_j^k \\ c_ic_k & c_jc_k & c_k^2 \end{bmatrix}$$

$$= \frac{k}{4Ae}\begin{bmatrix} b_i^2 - c_i^2 & b_1c_j - c_ic_j & b_ib_k - c_ic_k \\ b_ib_j - c_ic_j & b_j^2 - c_j^2 & b_jb_k + c_jc_k \\ b_ib_k + c_ic_k & b_jb_k - c_jc_k & b_k^2 - c_k^2 \end{bmatrix}$$

$$b_i = y_2 - y_3 = 10 - 8 = 2$$

$$b_j = y_3 - y_1 = 8 - 6 = 2$$

$$b_k = y_1 - y_2 = 6 - 10 = 4$$

$$c_i = x_3 - x_2 = 12 - 8 = 4$$

$$c_j = x_1 - x_3 = 4 - 12 = -8$$

$$c_k = x_2 - x_1 = 8 - 4 = 4$$

$$A_e = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 4 & 6 \\ 1 & 8 & 10 \\ 1 & 12 & 8 \end{vmatrix}$$

$$= \frac{1}{2}\left[(64 - 120) - 4(8 - 10) + 6(12 - 8)\right]$$

$$= 12 \text{ cm}^2$$

$$[k_1]_e = \frac{60}{4\times 12}\begin{bmatrix} 2^2 + 4^2 & 4 - 32 & -8 + 16 \\ 4 - 32 & 2^2 - 8^2 & -8 - 32 \\ -8 + 16 & -8 - 32 & -4^2 + 4^2 \end{bmatrix}$$

$$= \frac{60}{48}\begin{bmatrix} 20 & -28 & 8 \\ -28 & 68 & -40 \\ 8 & -40 & 32 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 35 & 10 \\ -35 & 85 & -50 \\ 10 & -50 & 40 \end{bmatrix}$$

$$[k_2] = \frac{h_{kj}\ell_{kj}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

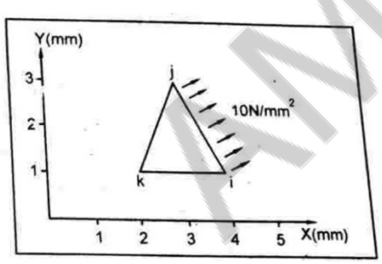
Length of edge

$$\ell_{jk} = \sqrt{(x_k - x_i)^2 - (y_k - y_i)^2}$$
$$= \sqrt{(12-8)^2 + (8-10)}$$
$$\ell_{jk} = 4.47 \text{ cm}$$
$$k_2 = \frac{10 \times 4.47}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 14.9 & 7.45 \\ 0 & 7.45 & 14.9 \end{bmatrix}$$

For triangular element

$$p_1 = \frac{q_0 A_e}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{50 \times 12}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$
$$p_2 = 0$$
$$p_3 = \frac{H_{kj} T_\alpha \ell_{kj}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = \frac{10 \times 40 \times 4.47}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$
$$p_3 = \begin{Bmatrix} 0 \\ 994 \\ 994 \end{Bmatrix}$$

18. The triangular element shown in Fig. 4 is subjected to a constant pressure 10 N/ mm^2 along the edge ij. Assume E = 200 GPa, Poisson's ratio μ = 0.3 and thickness of the element = 2 mm. The co – efficient of thermal expansion of the material is $\propto = 2 \times 10^{-6}$ ℓ°C and Δ T = 50°C. Determine the constitutive matrix (stress – strain relationship matrix D) and the nodal force vector for the element. (Nov/Dec 2009)



E= 200 GPa

$$\mu = 0.3$$
$$\alpha = 2 \times 10^{-6} \ell°C$$
$$\Delta T = 50°C$$

Given data

$$t = 2\,mm$$
$$E = 200\,GPa$$
$$\mu = 0.3$$
$$\alpha = 2\times10^{-6}\,\ell°c$$
$$p = 10/\,Nmm^2 = 2\times10^5\,N/mm^2 \qquad \Delta T = 50°C$$
$$x_1y_1 = (4,1)$$
$$x_2y_2 = (3,3)$$
$$x_3y_3 = 2,1$$

Area $= \frac{1}{2}bh = \frac{1}{2}\times2\times2$

A = 2 mm^2.

Stress – strain relationship is given as

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} = \frac{2\times10^5}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 1 & \frac{1-0.3}{2} \end{bmatrix}$$
$$= 21.9\times10^4\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

2. Find angle for node i + j

$$\sin\theta = \frac{x_1 - x_2}{\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}}$$
$$= \frac{4-2}{\sqrt{4-3^2} + (1-3)^2}$$
$$\sin\theta = 0.894$$
$$\theta = \sin^{-1}(0.894)$$
$$\theta = 63.43°$$

Pressure on node i and j

$$P_{ij} = P\times\sin\theta$$
$$= 10\times\sin 63.43$$
$$= 8.95\,N/mm^2$$
$$P_{yi} = 10\times\cos 63.43$$
$$= 4.47\,N/mm^2$$

$$P_{xi} = P_{xj}$$
$$P_{yi} = P_{yj}$$

We know that strain displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$$q_1 = y_2 - y_3 = 3 - 1 = 2$$

$$q_2 = y_3 - y_1 = 1 - 1 = 0$$

$$q_3 = y_1 - y_2 = 2 - 3 = 1$$

$$r_1 = x_3 - x_2 = 2 - 3 = -1$$

$$r_2 = x_1 - x_3 = 4 - 2 = 2$$

$$r_3 = x_2 - x_1 = 2 - 3 = -1$$

$$[B] = \frac{1}{2 \times 2}\begin{bmatrix} 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ -1 & 2 & 2 & 0 & -1 & 2 \end{bmatrix}$$

$$[B]^T = 0.25\begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$[B]^T[D] = 0.25\begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix} \times 21.97 \times 10^4 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$= 54925\begin{bmatrix} 2 & 0.6 & -0.35 \\ -0. & -1 & 0.7 \\ 0 & 0 & 0.7 \\ 0.6 & 2 & 0 \\ 2 & 0.6 & -0.35 \\ -0.3 & -1 & 0.7 \end{bmatrix}$$

Nodal force vector $= \frac{tA}{3}\begin{Bmatrix} P_{xi} \\ P_{yi} \\ P_{xj} \\ P_{yj} \\ P_{xk} \\ P_{yk} \end{Bmatrix} = \frac{2 \times 2}{3}\begin{Bmatrix} 8.94 \\ 4.47 \\ 8.94 \\ 4.47 \\ 0 \\ 0 \end{Bmatrix}$

$$[F] = \begin{Bmatrix} 11.92 \\ 5.96 \\ 11.92 \\ 5.96 \\ 0 \\ 0 \end{Bmatrix}$$

Temperature force vector

$$[F] = [B]^T [D] \{e_0\} At$$

$$\{e_0\} = \begin{bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{bmatrix} = \begin{Bmatrix} 2\times10^{-6}\times50 \\ 2\times10^{-6}\times50 \\ 0 \end{Bmatrix}$$

$$\{e_0\} = 1\times10^{-6}\begin{Bmatrix} 100 \\ 100 \\ 0 \end{Bmatrix}$$

$$[F] = 54925\begin{bmatrix} 2 & 0.6 & -0.35 \\ -0.3 & -1 & 0.7 \\ 0 & 0 & 0.7 \\ 0.6 & 2 & 0 \\ 2 & 0.6 & -0.35 \\ -0.3 & -1 & 0.7 \end{bmatrix} \times 1\times10^{-6}\begin{Bmatrix} 100 \\ 100 \\ 0 \end{Bmatrix} \times 2\times 2$$

$$= 0.2197\begin{bmatrix} 260 \\ -130 \\ 0 \\ 260 \\ 260 \\ -130 \end{bmatrix}$$

$$[F] = \begin{bmatrix} 57.12 \\ 28.56 \\ 0 \\ 57.12 \\ 57.12 \\ -28.56 \end{bmatrix}$$

19. A thin plate a subjected to surface traction as shown in Fig. Calculate the global stiffness matrix (NOV / DEC 2011)

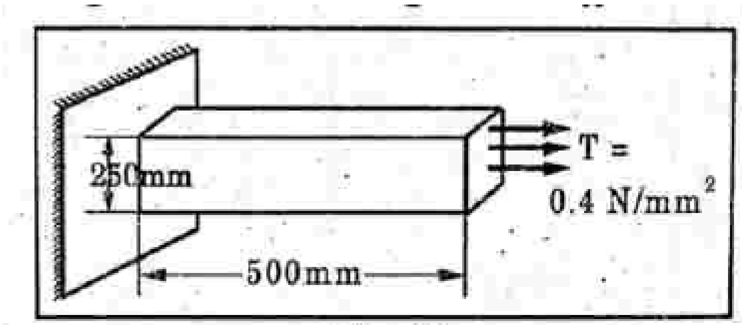


Table t = 25 mm, E = 2 × 10^5 N/ mm^2 ; V = 0.30. Assume plane stress condition.

T = 25 mm; E = 2 = 10^5 N/mm^2; V = 0.30

B = 250 mm; ℓ = 500 mm; T 0.4 N/mm^2

The tensile surface inaction is commented into nodal force.

$$F = \frac{1}{2}TA = \frac{1}{2}T \times (b \times t) = 1250\,N$$

Given

Thickness, t = 25 mm

Young's modulus, E = 2× 10^5 N/ mm^2

Poisson's ratio, v = 0.30

Breadth, b = 250 mm

Length, ℓ= 500 mm

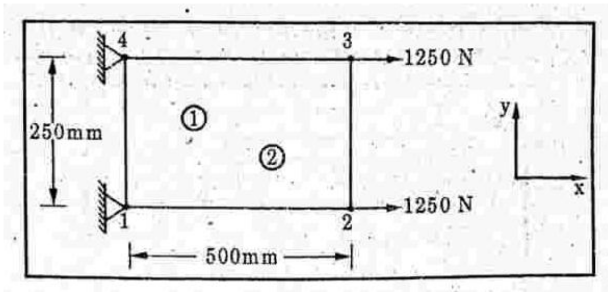
Tensile surface traction, T = 0.4 N/mm^2.

The tensile surface traction is converted into nodal force.

$$\Rightarrow F = \frac{1}{2}TA = \frac{1}{2}T \times (b \times t)$$

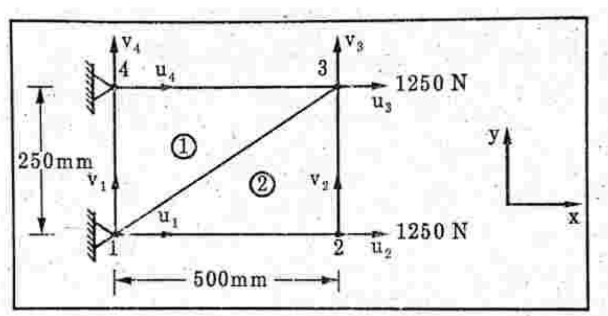
$$\Rightarrow \frac{1}{2} \times 0.4 \times 250 \times 25$$

Nodal force, F= 1250 N

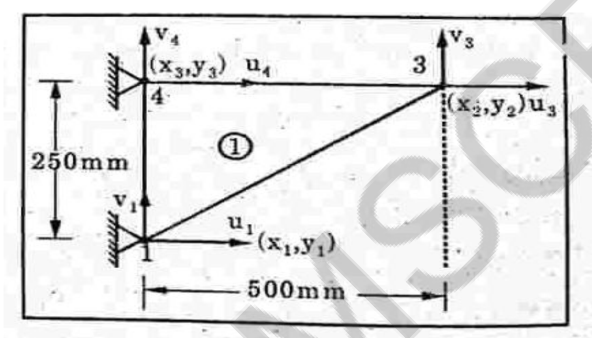


Find 1: Global stiffness matrix [K]

Solution



For element (1): [Nodal displacements are u1 v_2, u3 v_3 and u_4 v_4]



Take node 1as origin

For node 1: $\begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix}$

For node 2: $\begin{pmatrix} x_2 & y_2 \\ 500 & 250 \end{pmatrix}$

For node 3: $\begin{pmatrix} x_3 & y_3 \\ 0 & 250 \end{pmatrix}$

We know that,

Stiffness matrix, $[K]_1 = [B]^T [D] [B] At$ --(1)

Where A = Area of the triangular element

$$= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 500 & 250 \\ 1 & 0 & 250 \end{vmatrix}$$

$$= \frac{1}{2} \times 1(500 \times 250 - 0)$$

$$A = 62500\,\text{mm}^2 = 62.5 \times 10^3\,\text{mm}^2$$

$$A = 62.5 \times 10^3\ \text{mm}^2 \qquad ---(2)$$

Strain – Displacement matrix,

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \qquad ----(3)$$

$q_1 = y_2 - y_3 = 250 - 250 = 0$
$q_2 = y_3 - y_1 = 250 - 0 = 250$
$q_3 = y_1 - y_2 = 0 - 250 = -250$
$r_1 = x_3 - x_2 = 0 - 500 = -500$
$r_2 = x_1 - x_3 = 0 - 0 = 0$
$r_3 = x_2 - x_1 = 500 - 0 = 500$

Substitute the above values in equation (3),

$$\Rightarrow B = \frac{1}{2A}\begin{bmatrix} 0 & 0 & 250 & 0 & -250 & 0 \\ 0 & -500 & 0 & 0 & 0 & 500 \\ -500 & 0 & 0 & 250 & 500 & -250 \end{bmatrix}$$

Substitute Area value

$$\Rightarrow B = \frac{1}{2 \times 62.5 \times 10^3}\begin{bmatrix} 0 & 0 & 250 & 0 & -250 & 0 \\ 0 & -500 & 0 & 0 & 0 & 500 \\ -500 & 0 & 0 & 250 & 500 & -250 \end{bmatrix}$$

$$= \frac{250}{2 \times 62.5 \times 10^3}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix} \qquad ---(4)$$

Stress – strain relationship matrix [D] for plane stress problems is,

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

$$= \frac{2\times10^5}{1-(0.3)^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$[D] = \frac{2\times10^5}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \qquad ---(5)$$

$$\Rightarrow [D][B] = \frac{2\times10^5}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$\times\frac{250}{2\times62.5\times10^3}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.6 & 1 & 0 & -1 & 0.6 \\ 0 & -2 & 0.3 & 0 & -0.3 & 2 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \end{bmatrix} \qquad ---(6)$$

We know that,

$$[B] = \frac{250}{2\times 62.5\times 10^3}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

$$[B]^T = 2\times 10^{-3}\begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$[B]^T = 2\times 10^{-3}$$

$$\times 439.56\begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & -0.6 & 1 & 0 & -1 & 0.6 \\ 0 & -2 & 0.3 & 0 & -0.3 & 2 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \end{bmatrix}$$

$$= \begin{bmatrix} 1.4 & 0 & 0 & -0.7 & -1.4 & 0.7 \\ 0 & 4 & -0.6 & 0 & 0.6 & -4 \\ 0 & -0.6 & 1 & 0 & -1 & 0.6 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \\ -1.4 & 0.6 & -1 & 0.7 & 2.4 & -1.3 \\ 0.7 & -4 & 0.6 & -0.35 & -1.3 & 4.35 \end{bmatrix}$$

Substitute $[B]^T$ [D] [B] and A, t values in equation (1),

Stiffness matrix

$$[K]_1 = 0.8791\begin{bmatrix} 1.4 & 0 & 0 & -0.7 & -1.4 & 0.7 \\ 0 & 4 & -0.6 & 0 & 0.6 & -4 \\ 0 & -0.6 & 1 & 0 & -1 & 0.6 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \\ -1.4 & 0.6 & -1 & 0.7 & 2.4 & -1.3 \\ 0.7 & -4 & 0.6 & -0.35 & -1.3 & 4.35 \end{bmatrix}\times 6.25\times 10^3\times 25$$

$$[K]_1 = 1373.59\times 10^3\begin{bmatrix} 1.4 & 0 & 0 & -0.7 & -1.4 & 0.7 \\ 0 & 4 & -0.6 & 0 & 0.6 & -4 \\ 0 & -0.6 & 1 & 0 & -1 & 0.6 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \\ -1.4 & 0.6 & -1 & 0.7 & 2.4 & -1.3 \\ 0.7 & -4 & 0.6 & -0.35 & -1.3 & 4.35 \end{bmatrix}$$

$$[K]_1 = 1\times 10^3\begin{bmatrix} 1923.026 & 0 & 0 & -961.513 & -1923.026 & 961.513 \\ 0 & 5494.36 & 824.154 & 0 & 824.154 & -5494.36 \\ 0 & -824.154 & 1373.59 & 0 & -1373.59 & 824.154 \\ -961.513 & 0 & 0 & 480.7665 & 961.513 & -480.7565 \\ -1923.026 & 824.154 & -1373.59 & 961.513 & 3296.616 & -1785.667 \\ 961.513 & -5494.36 & 824.154 & -480.7565 & -1785.667 & 5975.1165 \end{bmatrix}$$

For element (1), nodal displacements are u_1, v_1, u_3, u_4, v_4 [Refer Fig. (iv)]

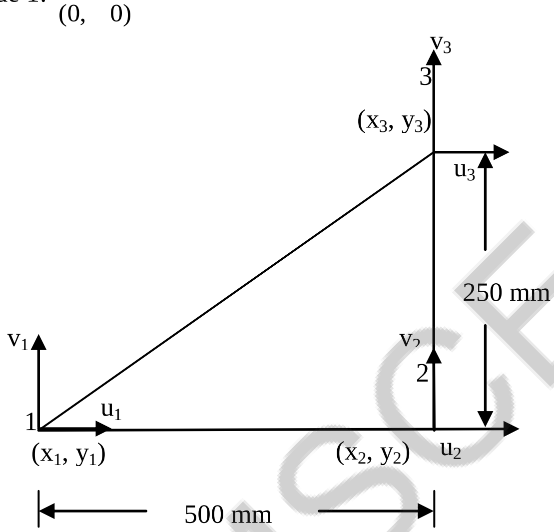
Stiffness matrix $[K]_1 = u_1 \quad v_1 \quad u_3 \quad v_3 \quad u_4 \quad v_4$

$$1\times10^3 \times \begin{bmatrix} 1923.026 & 0 & 0 & -961.513 & -1923.026 & 961.513 \\ 0 & 5494.36 & 824.154 & 0 & 824.154 & -5494.36 \\ 0 & -824.154 & 1373.59 & 0 & -1373.59 & 824.154 \\ -961.513 & 0 & 0 & 480.7665 & 961.513 & -480.7565 \\ -1923.026 & 824.154 & -1373.59 & 961.513 & 3296.616 & -1785.667 \\ 961.513 & -5494.36 & 824.154 & -480.7565 & -1785.667 & 5975.1165 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad --(7)$$

Form element (2):

Take, node 1 as origin:

For node 1: $\begin{matrix} x_1 & y_1 \\ (0, & 0) \end{matrix}$



For node 2: $\begin{matrix} x_2 & y_2 \\ (500, & 0) \end{matrix}$

We know that,

Stiffness matrix, $[K]_2 = [B]^T\,[D]\,[B]\,A\,t$ ---(8)

Where, A = Area of the triangular element

$$= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 500 & 0 \\ 1 & 500 & 250 \end{vmatrix}$$

$$= \frac{1}{2}\times1\times(500\times250-0)$$

$$A = 62.5\times10^3 \text{mm}^2$$

Strain – Displacement matrix,

$$[B]=\frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \qquad ---(9)$$

Where,

$$q_1 = y_2 - y_3 = 0 - 250 = -250$$
$$q_2 = y_3 - y_1 = 250 - 0 = 250$$
$$q_3 = y_1 - y_2 = 0 - 0 = 0$$
$$r_1 = x_3 - x_2 = 500 - 500 = 0$$
$$r_2 = x_1 - x_3 = 0 - 500 = 500$$
$$r_3 = x_2 - x_1 = 500 - 0 = 500$$

Substitute the above values in equation (9),

$$\Rightarrow [B]=\frac{1}{2A}\begin{bmatrix} -250 & 0 & 250 & 0 & 0 & 0 \\ 0 & 0 & 0 & -500 & 0 & 500 \\ 0 & -250 & -500 & 250 & 500 & 0 \end{bmatrix}$$

Substitute Area value

$$\Rightarrow [B]=\frac{250}{2\times 62.5\times 10^3}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & -2 & 1 & 2 & 0 \end{bmatrix} \qquad ---(10)$$

Stress – Strain relationship matrix [D] for plane stress problems is,

$$[D]=\frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} = \frac{2\times 10^5}{1-(0.3)^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$[D]=\frac{2\times 10^5}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \qquad ---(11)$$

$$\Rightarrow [D][B]=\frac{2\times 10^5}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\times\frac{250}{2\times 62.5\times 10^3}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & -2 & 1 & 2 & 0 \end{bmatrix}$$

$$= 439.56\begin{bmatrix} -1 & 0 & 1 & -0.6 & 0 & 0.6 \\ -0.3 & 0 & 0.3 & -2 & 0 & 2 \\ 0 & -0.35 & -0.7 & 0.35 & 0.7 & 0 \end{bmatrix} \qquad ---(12)$$

We know that,

$$[B] = \frac{250}{2\times 62.5\times 10^3}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & -2 & 1 & 2 & 0 \end{bmatrix} \text{ [eqn no.10]}$$

$$[B]^T = 2\times 10^{-3}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$[B]^T[D][B] = 2\times 10^{-3}\times 439.56\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & -0.6 & 0 & 0.6 \\ -0.3 & 0 & 0.3 & -2 & 0 & 2 \\ 0 & -0.35 & -0.7 & 0.35 & 0.7 & 0 \end{bmatrix}$$

$$[B]^T[D][B] = 0.8791\begin{bmatrix} 1 & 0 & -1 & 0.6 & 0 & -0.6 \\ 0 & 0.35 & 0.7 & -0.35 & -0.7 & 0 \\ -1 & 0.7 & 2.4 & -1.3 & -1.4 & 0.6 \\ 0.6 & -0.35 & -1.3 & 4.35 & 0.7 & -4 \\ 0 & -0.7 & -1.4 & 0.7 & 1.4 & 0 \\ -0.6 & 0 & 0.6 & -4 & 0 & 4 \end{bmatrix}$$

Substitute $[B]^T$ [D] [B] and A, t value in equation (8), Stiffness matrix

$$[K]_2 = 0.8791\begin{bmatrix} 1 & 0 & -1 & 0.6 & 0 & -0.6 \\ 0 & 0.35 & 0.7 & -0.35 & -0.7 & 0 \\ -1 & 0.7 & 2.4 & -1.3 & -1.4 & 0.6 \\ 0.6 & -0.35 & -1.3 & 4.35 & 0.7 & -4 \\ 0 & -0.7 & -1.4 & 0.7 & 1.4 & 0 \\ -0.6 & 0 & 0.6 & -4 & 0 & 4 \end{bmatrix}\times 62.5\times 10^3\times 25$$

$$[K]_2 = 1373.59\times 10^3\begin{bmatrix} 1 & 0 & -1 & 0.6 & 0 & -0.6 \\ 0 & 0.35 & 0.7 & -0.35 & -0.7 & 0 \\ -1 & 0.7 & 2.4 & -1.3 & -1.4 & 0.6 \\ 0.6 & -0.35 & -1.3 & 4.35 & 0.7 & -4 \\ 0 & -0.7 & -1.4 & 0.7 & 1.4 & 0 \\ -0.6 & 0 & 0.6 & -4 & 0 & 4 \end{bmatrix}$$

$$[K]_2 = 1\times 10^3\begin{bmatrix} 1373.59 & 0 & -1373.59 & 824.154 & 0 & -824.154 \\ 0 & 480.7565 & 961.513 & -480.7565 & -961.513 & 0 \\ -1373.59 & 961.513 & 3296.616 & -1785.667 & -1923.026 & 824.154 \\ 824.154 & -480.7565 & -1785.667 & 5975.1165 & 961.513 & -5494.36 \\ 0 & -961.513 & -1923.0.26 & 961.513 & 1923.026 & 0 \\ -824.154 & 0 & 824.154 & -5494.36 & 0 & 5494.36 \end{bmatrix}$$

For element (2) nodal displacements are u1, v_1, u_2, v_2, u_3, v_3 [Refer Fig. (v)]

$$[K]_2 = 1\times10^3 \begin{bmatrix} 1373.59 & 0 & -1373.59 & 824.154 & 0 & -824.154 \\ 0 & 480.7565 & 961.513 & -480.7565 & -961.513 & 0 \\ -1373.59 & 961.513 & 3296.616 & -1785.667 & -1923.026 & 824.154 \\ 824.154 & -480.7565 & -1785.667 & 5975.1165 & 961.513 & -5494.36 \\ 0 & -961.513 & -1923.0.26 & 961.513 & 1923.026 & 0 \\ -824.154 & 0 & 824.154 & -5494.36 & 0 & 5494.36 \end{bmatrix} \quad ----(13)$$

Global stiffness matrix, [K]:

Assemble the stiffness matrix equation (7) and (13),

Global stiffness matrix [K] =

	u1	v1	u2	v2	u3	v3	u4	v4	
	1923.026 + 1373.59	0 + 0	-1373.59	824.154	0 + 0	-961.513 + -824.154	-1923.026	961.513	u1
	0 + 0	5494.36 + 480.7565	961.513	-480.7565	-824.154 + -961.513	0 + 0	824.154	-5494.36	v1
1×10^3	-1373.59	961.513	3296.616	-1785.667	-1923.0.26	824.154	0	0	u2
	824.154	-480.7565	-1785.667	5975.1165	961.513	-5494.36	0	0	v2
	0 + 0	-824.154 + -961.513	0 + 1923.0.26	0 + 961.513	1373.59 + 1923.026	0 + 0	-1373.59 + 0	824.154 + 0	u3
	-961.513 + -824.154	0 + 0	0 + 824.154	0 + -5494.36	0 + 0	480.7565 + 5494.36	961.513 + 0	-480.7565 + 0	v3
	-1923.026	824.154	0	0	-1373.59	961.513	3296.616	-1785.667	u4
	961.513	-5494.36	0	0	824.154	-480.7565	-1785.667	5975.1165	v4

Global stiffness matrix [K] =

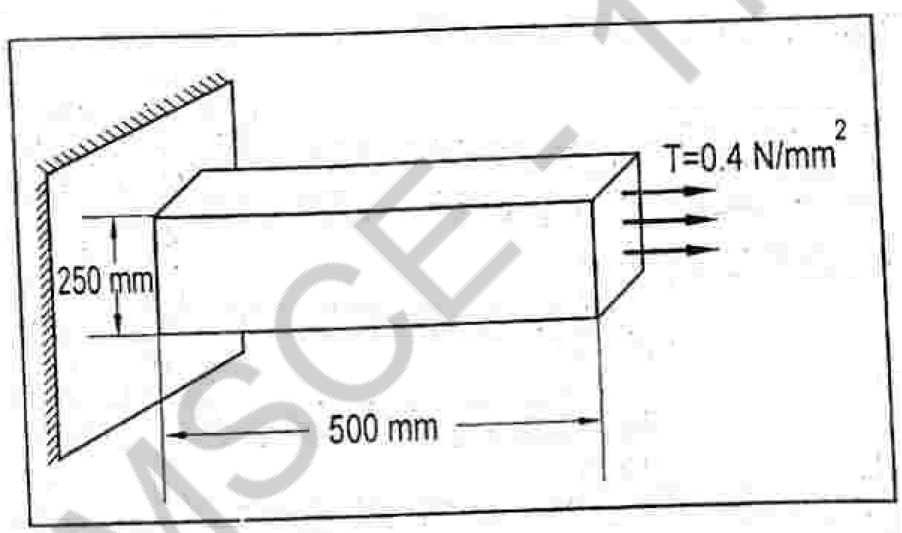
	u1	v1	u2	v2	u3	v3	u4	v4	
	3296.616	0	-1373.59	824.154	0	-1785.667	-1923.026	961.513	u1
	0	5975.1165	961.513	-480.7565	-1785.665	0	824.154	-5494.36	v1
	-1373.59	961.513	3296.616	-1785.667	-1923.0.26	824.154	0	0	u2
	824.154	-480.7565	-1785.667	5975.1165	961.513	-5494.36	0	0	v2
	0	-824.154 + -961.513	1923.0.26	961.513	1373.59 + 1923.026	0	-1373.59	824.154	U3
1×10^3	-1785.667	0	824.154	-5494.36	0	5975.1165	961.513	-480.7565	V3
	-1923.026	824.154	0	0	-1373.59	961.513	3296.616	-1785.667	u4
	961.513	-5494.36	0	0	824.154	-480.7565	-1785.667	5975.1165	v4

Result:

Global stiffness matrix [K] = $10^3 \times$

$$\begin{bmatrix} 3296.616 & 0 & -1373.59 & 824.154 & 0 & -1785.667 & -1923.026 & 961.513 \\ 0 & 5975.1165 & 961.513 & -480.7565 & -1785.667 & 0 & 824.154 & -5494.36 \\ -1373.59 & 961.513 & 3296.616 & -1785.667 & 1923.206 & 824.154 & 0 & 0 \\ 824.154 & -480.7565 & -1785.667 & 5975.1165 & 961.513 & -549.36 & 0 & 0 \\ 0 & -1785.665 & -1923.026 & 961.513 & 3296.616 & 0 & -1373.59 & 824.154 \\ -1785.667 & 0 & 824.154 & -5494.36 & 0 & 5975.1165 & 961.513 & -480.7565 \\ -1923.026 & 824.154 & 0 & 0 & -1373.59 & 961.513 & 3296.616 & -1785.667 \\ 961.513 & -5494.36 & 0 & 0 & 824.154 & -480.7565 & -1785.667 & 5975.1165 \end{bmatrix}$$

20. A thin plate is subjected to surface traction as shown in fig. Calculate the global striggness matrix. (MAY / JUNE 2010)



Take, $v = 0.30$ $t = 25\text{mm}$, $E = 2\times10^5\ \text{N/mm}^2$ and $v = 0.30$

Assume plane stress condition.

Given Thickness, $t = 25\text{mm}$

Young's modulus, $E = 2\times10^5\ \text{N/mm}^2$

Poisson's ratio, $v = 0.30$

Breadth, $b = 250\text{mm}$

Length, $l = 500\text{mm}$

Tensile surface traction, $T = 0.4\,N/mm^2$

The tensile surface traction is converted into nodal force.

$$\Rightarrow F = \frac{1}{2}TA = \frac{1}{2} \times T \times (b \times 1) = \frac{1}{2} \times 0.4 \times 250 \times 25$$

Nodal force, $F = 1250N$

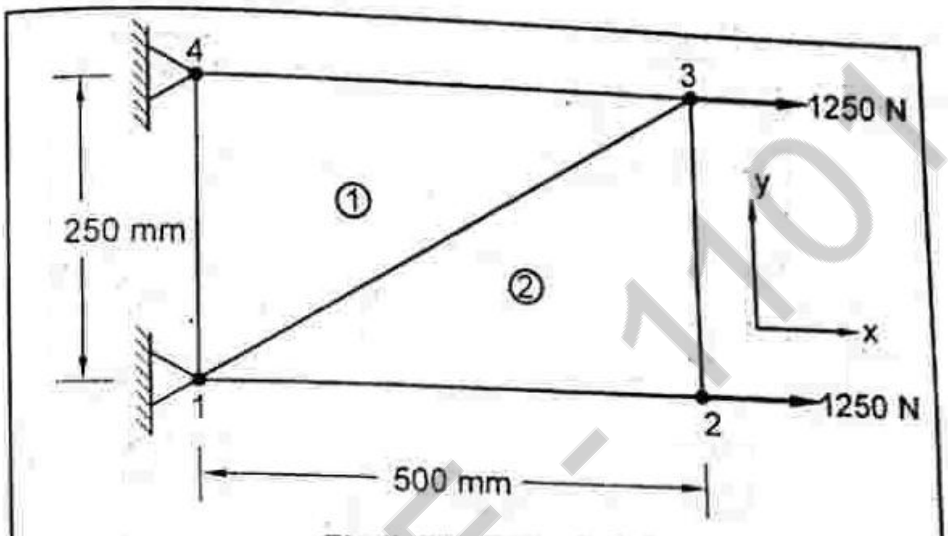


Fig ii Discretized plate

To find: 1. Global stiffness matrix [K]

Solution

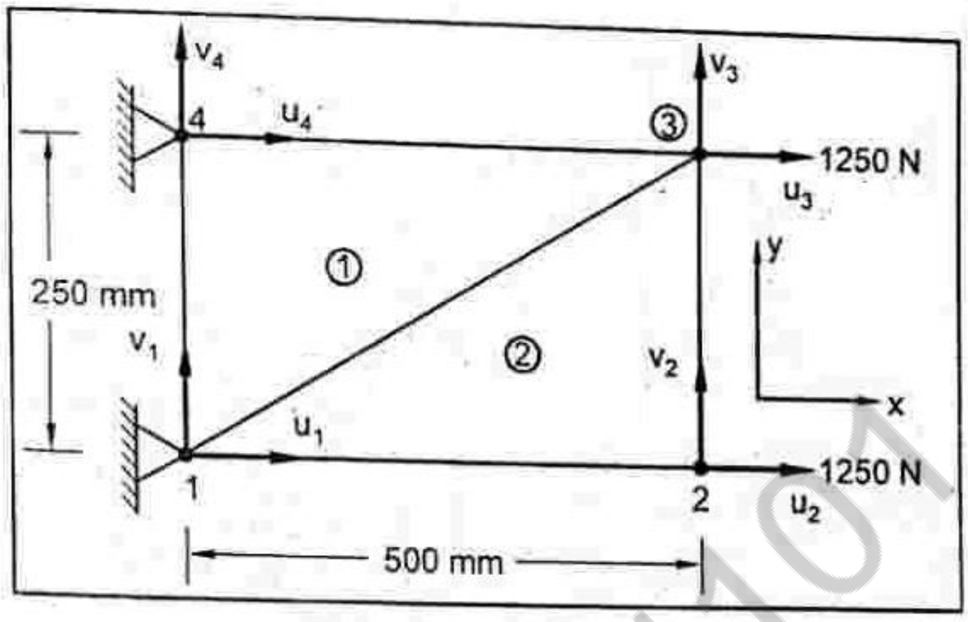


Fig. (iii)

For element (1): Nodal displacement are $u_1 v_1$, $u_3 v_3$ and $u_4 v_4$

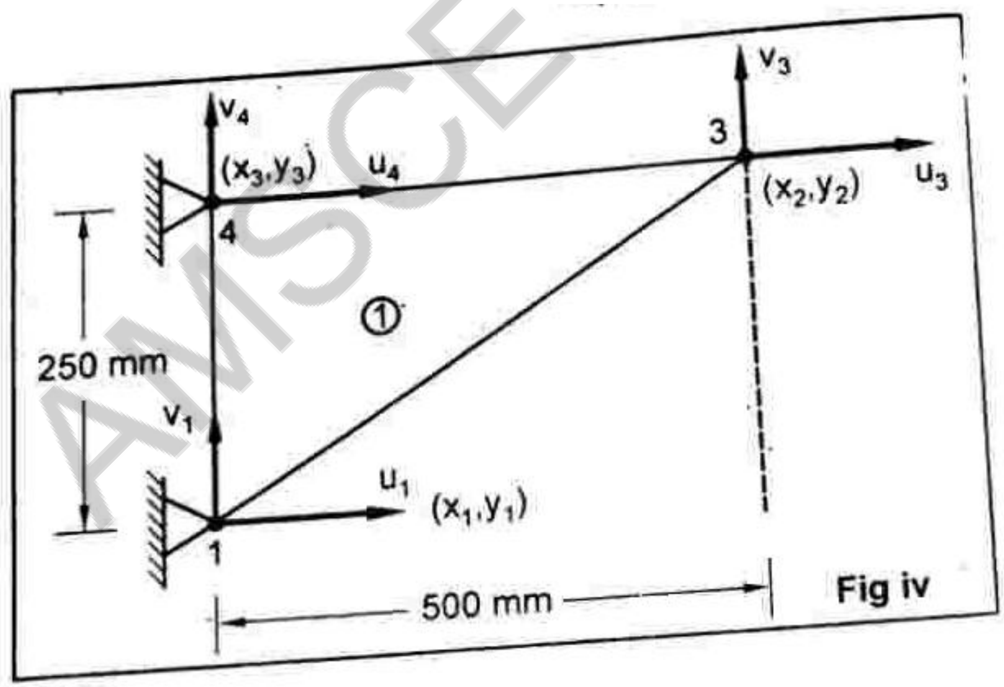


Fig iv

Take node 1 as origin

$x_1 \quad y_1$

For node 1: $(0, \quad 0)$

$x_2 \quad y_2$

For node 3: $(500, \quad 250)$

$x_3 \quad y_3$

For node 4: $(0, \quad 250)$

We know that,

Stiffness matrix, $[K]_1 = [B]^T [D][B] A_t$

Where, A = Area of the triangular element

$$= \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 500 & 250 \\ 1 & 0 & 250 \end{bmatrix}$$

$$= \frac{1}{2}\times(500\times 250 - 0)$$

$$A = 62500 mm^2 = 62.5\times 10^3 mm^2$$

$$A = 62.5\times 10^3 mm^2$$

Stain Displacement matrix,

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

Where, $q_1 = y_2 - y_3 = 250 - 250 = 0$

$q_2 = y_3 - y_1 = 250 - 0 = 250$

$q_3 = y_1 - y_2 = 0 - 250 = -250$

$r_1 = x_3 - x_2 = 0 - 500 = -500$

$r_2 = x_1 - x_3 = 0 - 0 = 0$

$r_3 = x_2 - x_1 = 500 - 0 = 500$

Substitute the above values in equation (3),

$$\Rightarrow B = \frac{1}{2A}\begin{bmatrix} 0 & 0 & 250 & 0 & -250 & 0 \\ 0 & -500 & 0 & 0 & 0 & 500 \\ -500 & 0 & 0 & 250 & 500 & -250 \end{bmatrix}$$

Substitute Area value,

$$\Rightarrow B = \frac{1}{2\times 62.5\times 10^3}\begin{bmatrix} 0 & 0 & 250 & 0 & -250 & 0 \\ 0 & -500 & 0 & 0 & 0 & 500 \\ -500 & 0 & 0 & 250 & 500 & -250 \end{bmatrix}$$

$$= \frac{250}{2\times 62.5\times 10^3}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

Stress-strain relationship matrix $[D]$ for plane stress problem is,

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

$$= \frac{2\times 10^5}{\left(1-(0.3)^2\right)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$[D] = \frac{2\times 10^5}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \quad \ldots(5)$$

$$\Rightarrow [D][B] = \frac{2\times 10^5}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$\frac{250}{2\times 62.5\times 10^3}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

$$= 439.56\begin{bmatrix} 0 & -0.6 & 1 & 0 & -1 & 0.6 \\ 0 & -2 & 0.3 & 0 & -0.3 & 2 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \end{bmatrix} \quad \ldots(6)$$

We know that,

$$B = \frac{250}{2 \times 62.5 \times 10^3} \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix} \text{ from equation no. (4)}$$

$$[B]^T = 2 \times 10^{-3} \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$[B]^T [D][B] = 2 \times 10^{-3} \times 439.56 \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -0.6 & 1 & 0 & -1 & 0.6 \\ 0 & -2 & 0.3 & 0 & -0.3 & 2 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \end{bmatrix}$$

$$= 0.8791 \begin{bmatrix} 14 & 0 & 0 & -0.7 & -1.4 & 0.7 \\ 0 & 4 & -0.6 & 0 & 0.6 & -4 \\ 0 & -0.6 & 1 & 0 & -1 & 0.6 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \\ -1.4 & 0.6 & -1 & 0.7 & 2.4 & -1.35 \\ 0.7 & -4 & 0.6 & -0.35 & -1.3 & 4.35 \end{bmatrix}$$

Substitute $[B]^T [D][B]$ and A, t values in equation (1),

Stiffness matrix

$$[K_1] = 0.8791 \begin{bmatrix} 14 & 0 & 0 & -0.7 & -1.4 & 0.7 \\ 0 & 4 & -0.6 & 0 & 0.6 & -4 \\ 0 & -0.6 & 1 & 0 & -1 & 0.6 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \\ -1.4 & 0.6 & -1 & 0.7 & 2.4 & -1.35 \\ 0.7 & -4 & 0.6 & -0.35 & -1.3 & 4.35 \end{bmatrix} \times 6.25 \times 10^3$$

$$[K_1] = 1\times10^3 \begin{bmatrix} 1923.026 & 0 & 0 & -961.513 & -1923.026 & 961.513 \\ 0 & 5494.36 & -824.154 & 0 & 524.154 & -5494.36 \\ 0 & -824.154 & 1373.59 & 0 & -1373.59 & 824.154 \\ -961.513 & 0 & 0 & 480.7565 & 961.513 & -480.7565 \\ -1923.026 & 824.154 & -1373.59 & 961.513 & 3296.616 & -1785.667 \\ 961.513 & -5494.36 & 824.154 & -480.7565 & -1785.667 & 5975.1165 \end{bmatrix}$$

For element (1), nodal displacements are u_1, v_1, u_3, u_4, v_4 [Refer Fig. (iv)]

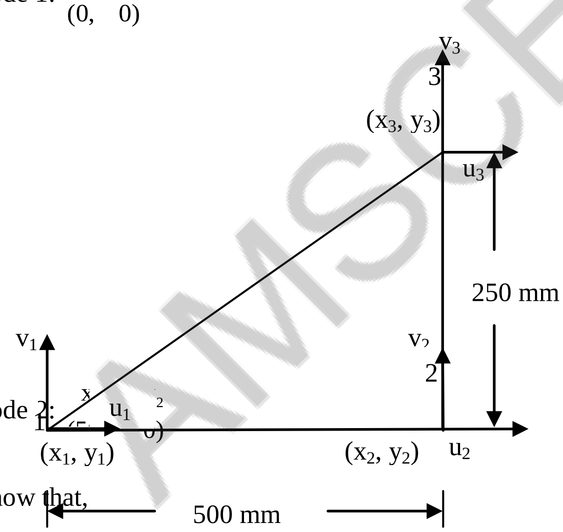
Stiffness matrix $[K]_1 = u_1 \quad v_1 \quad u_3 \quad v_3 \quad u_4 \quad v_4$

$$1\times10^3 \times \begin{bmatrix} 1923.026 & 0 & 0 & -961.513 & -1923.026 & 961.513 \\ 0 & 5494.36 & 824.154 & 0 & 824.154 & -5494.36 \\ 0 & -824.154 & 1373.59 & 0 & -1373.59 & 824.154 \\ -961.513 & 0 & 0 & 480.7665 & 961.513 & -480.7565 \\ -1923.026 & 824.154 & -1373.59 & 961.513 & 3296.616 & -1785.667 \\ 961.513 & -5494.36 & 824.154 & -480.7565 & -1785.667 & 5975.1165 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad --(7)$$

Form element (2):

Take, node 1 as origin:

For node 1: $\begin{matrix} x_1 & y_1 \\ (0, & 0) \end{matrix}$



For node 2:

We know that,

Stiffness matrix, $[K]_2 = [B]^T [D] [B] A t$ ---(8)

Where, A = Area of the triangular element

$$= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 500 & 0 \\ 1 & 500 & 250 \end{vmatrix}$$

$$= \frac{1}{2} \times 1 \times (500 \times 250 - 0)$$

$$A = 62.5 \times 10^3 \text{mm}^2$$

Strain – Displacement matrix,

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \qquad ---(9)$$

Where,

$$q_1 = y_2 - y_3 = 0 - 250 = -250$$
$$q_2 = y_3 - y_1 = 250 - 0 = 250$$
$$q_3 = y_1 - y_2 = 0 - 0 = 0$$
$$r_1 = x_3 - x_2 = 500 - 500 = 0$$
$$r_2 = x_1 - x_3 = 0 - 500 = 500$$
$$r_3 = x_2 - x_1 = 500 - 0 = 500$$

Substitute the above values in equation (9),

$$\Rightarrow [B] = \frac{1}{2A}\begin{bmatrix} -250 & 0 & 250 & 0 & 0 & 0 \\ 0 & 0 & 0 & -500 & 0 & 500 \\ 0 & -250 & -500 & 250 & 500 & 0 \end{bmatrix}$$

Substitute Area value

$$\Rightarrow [B] = \frac{250}{2 \times 62.5 \times 10^3}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & -2 & 1 & 2 & 0 \end{bmatrix} \qquad ---(10)$$

Stress – Strain relationship matrix [D] for plane stress problems is,

$$[D]=\frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{1-v}{2}\end{bmatrix}=\frac{2\times10^5}{1-(0.3)^2}\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & \frac{1-0.3}{2}\end{bmatrix}$$

$$[D]=\frac{2\times10^5}{0.91}\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.35\end{bmatrix} \qquad ---(11)$$

$$\Rightarrow [D][B]=\frac{2\times10^5}{0.91}\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.35\end{bmatrix}\times\frac{250}{2\times62.5\times10^3}\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 0 & 2\\ 0 & -1 & -2 & 1 & 2 & 0\end{bmatrix}$$

$$=439.56\begin{bmatrix}-1 & 0 & 1 & -0.6 & 0 & 0.6\\ -0.3 & 0 & 0.3 & -2 & 0 & 2\\ 0 & -0.35 & -0.7 & 0.35 & 0.7 & 0\end{bmatrix} \qquad ---(12)$$

We know that,

$$[B]=\frac{250}{2\times62.5\times10^3}\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 0 & 2\\ 0 & -1 & -2 & 1 & 2 & 0\end{bmatrix} \quad \text{[eqn no.10]}$$

$$[B]^T=2\times10^{-3}\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 1 & 0 & -2\\ 0 & -2 & 1\\ 0 & 0 & 2\\ 0 & 2 & 0\end{bmatrix}$$

$$[B]^T[D][B]=2\times10^{-3}\times439.56\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 1 & 0 & -2\\ 0 & -2 & 1\\ 0 & 0 & 2\\ 0 & 2 & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 1 & -0.6 & 0 & 0.6\\ -0.3 & 0 & 0.3 & -2 & 0 & 2\\ 0 & -0.35 & -0.7 & 0.35 & 0.7 & 0\end{bmatrix}$$

$$[B]^T[D][B]=0.8791\begin{bmatrix}1 & 0 & -1 & 0.6 & 0 & -0.6\\ 0 & 0.35 & 0.7 & -0.35 & -0.7 & 0\\ -1 & 0.7 & 2.4 & -1.3 & -1.4 & 0.6\\ 0.6 & -0.35 & -1.3 & 4.35 & 0.7 & -4\\ 0 & -0.7 & -1.4 & 0.7 & 1.4 & 0\\ -0.6 & 0 & 0.6 & -4 & 0 & 4\end{bmatrix}$$

Substitute $[B]^T$ [D] [B] and A, t value in equation (8), Stiffness matrix

$$[K]_2 = 0.8791\begin{bmatrix} 1 & 0 & -1 & 0.6 & 0 & -0.6 \\ 0 & 0.35 & 0.7 & -0.35 & -0.7 & 0 \\ -1 & 0.7 & 2.4 & -1.3 & -1.4 & 0.6 \\ 0.6 & -0.35 & -1.3 & 4.35 & 0.7 & -4 \\ 0 & -0.7 & -1.4 & 0.7 & 1.4 & 0 \\ -0.6 & 0 & 0.6 & -4 & 0 & 4 \end{bmatrix} \times 62.5\times10^3 \times 25$$

$$[K]_2 = 1373.59\times10^3\begin{bmatrix} 1 & 0 & -1 & 0.6 & 0 & -0.6 \\ 0 & 0.35 & 0.7 & -0.35 & -0.7 & 0 \\ -1 & 0.7 & 2.4 & -1.3 & -1.4 & 0.6 \\ 0.6 & -0.35 & -1.3 & 4.35 & 0.7 & -4 \\ 0 & -0.7 & -1.4 & 0.7 & 1.4 & 0 \\ -0.6 & 0 & 0.6 & -4 & 0 & 4 \end{bmatrix}$$

$$[K]_2 = 1\times10^3\begin{bmatrix} 1373.59 & 0 & -1373.59 & 824.154 & 0 & -824.154 \\ 0 & 480.7565 & 961.513 & -480.7565 & -961.513 & 0 \\ -1373.59 & 961.513 & 3296.616 & -1785.667 & -1923.026 & 824.154 \\ 824.154 & -480.7565 & -1785.667 & 5975.1165 & 961.513 & -5494.36 \\ 0 & -961.513 & -1923.0.26 & 961.513 & 1923.026 & 0 \\ -824.154 & 0 & 824.154 & -5494.36 & 0 & 5494.36 \end{bmatrix}$$

For element (2) nodal displacements are u1, v_1, u_2, v_2, u_3, v_3 [Refer Fig. (v)]

$$[K]_2 = 1\times10^3\begin{bmatrix} 1373.59 & 0 & -1373.59 & 824.154 & 0 & -824.154 \\ 0 & 480.7565 & 961.513 & -480.7565 & -961.513 & 0 \\ -1373.59 & 961.513 & 3296.616 & -1785.667 & -1923.026 & 824.154 \\ 824.154 & -480.7565 & -1785.667 & 5975.1165 & 961.513 & -5494.36 \\ 0 & -961.513 & -1923.0.26 & 961.513 & 1923.026 & 0 \\ -824.154 & 0 & 824.154 & -5494.36 & 0 & 5494.36 \end{bmatrix} \quad ----(13)$$

Global stiffness matrix, [K]:

Assemble the stiffness matrix equation (7) and (13),

Global stiffness matrix [K] =

	u1	v1	u2	v2	u3	v3	u4	v4	
	1923.026 + 1373.59	0 + 0	-1373.59	824.154	0 + 0	-961.513 + -824.154	-1923.026	961.513	u1
	0 + 0	5494.36 + 480.7565	961.513	-480.7565	-824.154 + -961.513	0 + 0	824.154	-5494.36	v1
	-1373.59	961.513	3296.616	-1785.667	-1923.0.26	824.154	0	0	u2
1×10^3	824.154	-480.7565	-1785.667	5975.1165	961.513	-5494.36	0	0	v2
	0 + 0	-824.154 + -961.513	0 + 1923.0.26	0 + 961.513	1373.59 + 1923.026	0 + 0	-1373.59 + 0	824.154 + 0	u3
	-961.513	0	0	0	0	480.7565	961.513	-480.7565	v3

	+ -824.154	+ 0	+ 824.154	+ -5494.36	+ 0	+ 5494.36	+ 0	+ 0	
	-1923.026	824.154	0	0	-1373.59	961.513	3296.616	-1785.667	u4
	961.513	-5494.36	0	0	824.154	-480.7565	-1785.667	5975.1165	v4

Global stiffness matrix [K] =

	u1	v1	u2	v2	u3	v3	u4	v4	
	3296.616	0	-1373.59	824.154	0	-1785.667	-1923.026	961.513	u1
	0	5975.1165	961.513	-480.7565	-1785.665	0	824.154	-5494.36	v1
	-1373.59	961.513	3296.616	-1785.667	-1923.0.26	824.154	0	0	u2
1×10^3	824.154	-480.7565	-1785.667	5975.1165	961.513	-5494.36	0	0	v2
	0	-1785.667	0	824.154	-5494.36	0	5975.1165	961.513	- 480.7565
	-1923.026	824.154	0	0	-1373.59	961.513	3296.616	-1785.667	u4
	961.513	-5494.36	0	0	824.154	-480.7565	-1785.667	5975.1165	v4

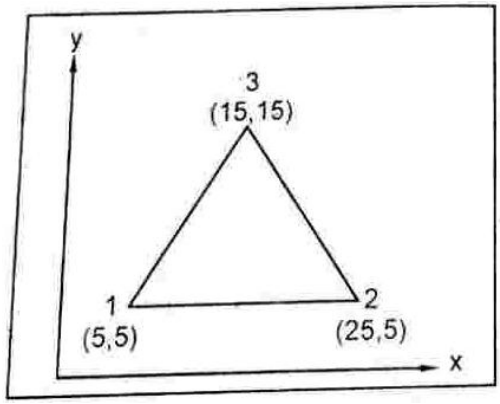
Result:

Global stiffness matrix [K] = $10^3 \times$

$$\begin{bmatrix} 3296.616 & 0 & -1373.59 & 824.154 & 0 & -1785.667 & -1923.026 & 961.513 \\ 0 & 5975.1165 & 961.513 & -480.7565 & -1785.667 & 0 & 824.154 & -5494.36 \\ -1373.59 & 961.513 & 3296.616 & -1785.667 & 1923.206 & 824.154 & 0 & 0 \\ 824.154 & -480.7565 & -1785.667 & 5975.1165 & 961.513 & -549.36 & 0 & 0 \\ 0 & -1785.665 & -1923.026 & 961.513 & 3296.616 & 0 & -1373.59 & 824.154 \\ -1785.667 & 0 & 824.154 & -5494.36 & 0 & 5975.1165 & 961.513 & -480.7565 \\ -1923.026 & 824.154 & 0 & 0 & -1373.59 & 961.513 & 3296.616 & -1785.667 \\ 961.513 & -5494.36 & 0 & 0 & 824.154 & -480.7565 & -1785.667 & 5975.1165 \end{bmatrix}$$

21. For the plane strain element shown in the Figure 6, the nodal, displacements are given as

$u_1 = 0.005mm, u_2 = 0.002mm, u_3 = 0.0mm, u_4 = 0.0mm, u_5 = 0.004mm, u_6 = 0.0mm$. Determine the element stresses. Take E = 200 GPa and $\gamma = 0.3$. Use unit thickness for plane strain. (MAY / JUNE 2010)



$x_1 = 5$

$x_2 = 25$

$x_3 = 15$

$y_1 = 5$

$y_2 = 5$

$y_3 = 15$

$u_1 = 0.005\text{mm}$

$u_5 = 0.004\text{mm}$

$u_2 = 200\text{GPa}$

$= 200 \times 109$

$= 2 \times 10^5 \ \text{N/mm}^2$

$u_3 = u_4 = u_6 = 0$

$\gamma = 0.3t = 1$

WKT Strain displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & v_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 5 & 5 \\ 1 & 25 & 5 \\ 1 & 15 & 15 \end{bmatrix}$$

$$=\frac{1}{2}\left[1(25\times15-15\times5)-5(15\times1-5\times1)+5(15\times1-25\times1)\right]$$

$$=\frac{1}{2}\left[(375-75)-5(15-5)+5(15-25)\right]$$

$$=\frac{1}{2}\left[300-50-50\right]$$

$$=100\text{mm}^2$$

$$q_1=y_2-y_3=15-15=-10$$

$$q_2=y_3-y_1=15-5=10$$

$$q_3=y_1-y_2=5-25=-20$$

$$r_1=x_3-x_2=15-25=-10$$

$$r_2=x_1-x_3-15-15=-10$$

$$r_3=x_2-x_1=25-5=20$$

$$[B]=\frac{1}{100}\begin{bmatrix}-10 & 0 & 10 & 0 & -20 & 0\\ 0 & -10 & 0 & -10 & 0 & 20\\ -10 & -10 & -10 & 10 & 20 & -20\end{bmatrix}\qquad\ldots(1)$$

We known that

Stress strain relationship matrix $[D]$ for plain strain is

$$[D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix}1-v & v & 0\\ v & 1-v & 0\\ 0 & 0 & \frac{1-2v}{2}\end{bmatrix}$$

$$=\frac{200\times10^3}{(1+0.3)(1-2\times0.3)}\begin{bmatrix}1-0.3 & 0.3 & 0\\ 0.3 & 1-0.3 & 0\\ 0 & 0 & \frac{1-2\times0.3}{2}\end{bmatrix}$$

$$=384.62\times10^3\begin{bmatrix}0.7 & 0.3 & 0\\ 0.3 & 0.7 & 0\\ 0 & 0 & 0.2\end{bmatrix}\qquad\ldots(2)$$

$$[D][B] = 384.62\times10^3 \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$\times\frac{1}{100}\begin{bmatrix} -10 & 0 & 10 & 0 & -20 & 0 \\ 0 & -10 & 0 & -10 & 0 & 20 \\ -10 & -10 & -10 & 10 & 20 & -20 \end{bmatrix}$$

$$= 3846.2 \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$\begin{bmatrix} -10 & 0 & 10 & 0 & -20 & 0 \\ 0 & -10 & 0 & -10 & 0 & 20 \\ -10 & -10 & -10 & 10 & 20 & -20 \end{bmatrix}$$

$$= 3846.2 \begin{bmatrix} -7 & -3 & 7 & 3 & -14 & 6 \\ -3 & -7 & 3 & -7 & -6 & 14 \\ -2 & 2 & -2 & 2 & 4 & -4 \end{bmatrix}$$

WKT

$$\{\sigma\} = [D][B]\{u\}$$

$$3846.2 \begin{bmatrix} -7 & -3 & 7 & 3 & -14 & 6 \\ -3 & -7 & 3 & -7 & -6 & 14 \\ -2 & 2 & -2 & 2 & 4 & -4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$= 3846.2 \begin{bmatrix} -7 & -3 & 7 & 3 & -14 & 6 \\ -3 & -7 & 3 & -7 & -6 & 14 \\ -2 & 2 & -2 & 2 & 4 & -4 \end{bmatrix} \begin{Bmatrix} 0.005 \\ 0.002 \\ 0 \\ 0 \\ 0.004 \\ 0 \end{Bmatrix}$$

$$= 3846.2 \begin{bmatrix} -0.035 & -0.006 & +0 & -0 & -0.056 & -0 \\ 0.005 & -0.14 & -0 & -0 & -0.024 & +0 \\ -0.010 & -0.004 & +0 & +0 & -0.008 & +0 \end{bmatrix}$$

$$= 3846.2\begin{bmatrix} -0.097 \\ -0.179 \\ -0.006 \end{bmatrix}$$

$$\{0\} = \begin{bmatrix} -373.08 \\ -688.46 \\ -23.07 \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}\begin{bmatrix} -373.08 \\ -688.46 \\ -23.07 \end{bmatrix}$$

Maximum normal stress

$$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left[\frac{\sigma_x + \sigma_y}{2}\right]^2 \cdot \tau_{xy}^2}$$

$$= 373.08$$

$$\frac{373.08 + 688.46}{2}\sqrt{\left(\frac{373.08 + 688.46}{2}\right)^2 + (23.07)^2}$$

$$530.77 + \sqrt{(530.77)^2 + (23.07)^2}$$

$$23.07\ \text{N/mm}^2$$

$$\sigma_{min} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left[\frac{\sigma_x + \sigma_y}{2}\right]^2 - \tau x^2 y}$$

$$= -\frac{373.08 + 688.46}{2} - \sqrt{\left(\frac{-373.08 + 688.46}{2}\right)^2 + (-23.07)^2}$$

$$= -530.77 - 530.77 - 23.07$$

$$= -1037.8\ \text{N/mm}^2$$

Result

$$\sigma_x = -373.08\ \text{N/mm}^2$$

$$\sigma_y = -688.46\ \text{N/mm}^2$$

$$Z_{xy} = -23.07\ \text{N/mm}^2$$

Maximum stress $\sigma_1 = 23.07\ \text{N/mm}^2$

Minimum stress $\sigma_2 = -1037.8\ \text{N/mm}^2$

UNIT IV

31. What is constitute law and give constitute law axi – symmetric problems? (Nov/Dec 2008)

For a finite element, the stress – strain relations are expressed as follows.

{σ} = [D] {e}

Where {σ} = stress

{e} = strain

[D] = Stress – strain relationship matrix (or) constitute matrix.

This equation is known as constitutive law constitutive matrix [D] for axisymmetric triangular element is given by

$$[D] = \frac{E}{(1+V)(1-2V)} \begin{bmatrix} 1-V & V & V & 0 \\ V & 1-V & V & 0 \\ V & V & 1-V & 0 \\ 0 & 0 & 0 & \frac{1-2V}{V} \end{bmatrix}$$

Where E = young's modulus

V = Poisson's ratio.

32. Give one example each for plane stress and plan strain problems. (Nov/Dec 2008)

Generally, members that are thin (those with a small dimension compared to the in plane x and y dimension) and whose loads act only in the n- y plane can be considered to be under plane stress.

Plates with holes and plates with fillets are coming, under plane stress analysis problems. Dams and pipes subjected to loads that remains constant over their length are coming underplane stain analysis problem.

33. Write down the governing differential equation for a two dimensional steady – state heat transfer problems. (Nov/Dec 2009)

$$\frac{\partial}{\partial x}\left(kx\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(ky\frac{\partial T}{\partial y}\right) + C = 0$$

34. What is meant by axi – symmetric field problem? (Nov/Dec 2009)

Given an example.

In some three dimensional solids like flywheel, turbine discs etc, the material is symmetric with respect to their axes. Hence the stress developed is also symmetric. Such solids are known as axisymmetric solids. Due to this condition, there 3 dimensional elements.

35. What are the difference between 2 dimensional scalar variable and vector variable elements? (Nov/Dec 2009)

$$\left.\begin{matrix}\text{2D Scalar}\\ \text{Variable}\end{matrix}\right\} = \text{Elements have only one direction and has implement variable node}$$

"Stiffness matrix is 3×3

$$\left\{\begin{matrix}\text{2D vector}\\ \text{Variable}\end{matrix}\right\} = \text{Elements have direction dependent variable at each node.}$$

"Stiffness matrix is 6×6

36. List the required conditions for a problem assumed to be axisymmetrix. (MAY / JUNE 2010)

The condition to be axisymmetrix is as follows

a.

1. Problem domain must be symmetric about the axis of revolution

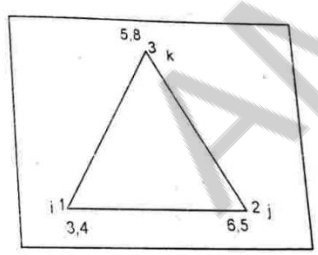
2. all boundary conditions must be sommetric alout the axis of revolution

3. all loading conditions must be synmetic about the axis of revolution

16 MARKS

11. The x, y co – ordinates of nodes i, j, and k an axisymmetric triangular element are given by (3, 4,) 6, 5) and (5, 8) cm respectively. The element displacement in cm vector is given as

q = [0.002, 0.001, 0.001, 0.004, -0.003, 0.007]^T. Determine the element strains. (Nov/Dec 2009)



We know that Strain displacement matrix

$$\text{Area } A = \frac{1}{2}\begin{bmatrix}1 & r_1 & z_1\\ 1 & r_2 & z_2\\ 1 & r_3` & z_3\end{bmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 6 & 5 \\ 1 & 6 & 8 \end{vmatrix}$$

$$= \frac{1}{2}\left[(48-30) - 3(8-5) + 4(6-6)\right]$$

$$= 5\text{cm}^2$$

We know

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{3+6+5}{3}$$

$$r = 4.66$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{4+5+8}{3}$$

$$z = 5.66$$

$$\alpha_1 = r_2 z_2 - r_3 z_3 = 6\times 8 - 5\times 5$$

$$\alpha_1 = 23$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = 5\times 4 - 3\times 8$$

$$\alpha_2 = -4$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = 3\times 5 - 6\times 4$$

$$\alpha_3 = -9$$

$$\beta_1 = z_2 - z_3 = 5 - 8 \quad \beta_1 = -3$$

$$\beta_2 = z_3 - z_1 = 8 - 4 \quad \beta_2 = 4$$

$$\beta_3 = z_1 - z_2 = 4 - 5 \quad \beta_2 = -1$$

$$\gamma_1 = r_3 - r_2 = 5 - 6 \quad \gamma_1 = -1$$

$$\gamma_2 = r_1 - r_3 = 3 - 5 \quad \gamma_2 = -2$$

$$\gamma_3 = r_2 - r_1 = 6 - 3 \quad \gamma_3 = 3$$

We know that

$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{23}{4.66} - 3 + \frac{-1\times 5.66}{4.56} = 0.72$$

$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_3 z}{r} = \frac{-4}{4.66} + 4 + \frac{-2\times 5.66}{4.66} = 0.712$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-9}{4.66} - 1 + \frac{3\times 5.66}{4.66} = 0.712$$

$$[B] = \frac{1}{2\times 3}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0.72 & 0 & 0.712 & 0 & 0.712 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix}$$

Strain {e} = [B] {u}

$$\begin{Bmatrix} e_r \\ e_0 \\ e_z \\ \tau_{rz} \end{Bmatrix} = [B] \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0.72 & 0 & 0.712 & 0 & 0.712 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix} \begin{Bmatrix} 0.001 \\ 0.001 \\ 0.001 \\ 0.004 \\ -0.003 \\ 0.007 \end{Bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} -0.006 + 0 + 0.004 + 0 + 0.003 + 0 \\ 0.00144 + 0 + 0.712 + 0 - 0.002136 + 0 \\ 0 - 0.001 + 0 + 0.008 + 0 + 0.021 \end{bmatrix}$$

$$= \begin{Bmatrix} 1 \times 10^{-4} \\ -1.6 \times 10^{-6} \\ 1.2 \times 10^{-3} \\ -0.7 \times 10^{-3} \end{Bmatrix}$$

12. Determine the element stiffness matrix and the thermal load vector for the plane stress element shown in Figure 7. The element experiences 20°C increase in temperature. Take $E = 15e6\,N/cm^2$, $\gamma = 0.25$, $t = 0.5cm$. (MAY / JUNE 2010)

$t = 0.5cm$

$E = 15 \times 10^6\ N/cm^2$

$a = 0.25cm^2$

$\alpha = 6 \times 10^{-6}/C$

$v = 0.25$

$r_1 = 0$

$r_2 = 2$

$r_3 = 1$

$z_1 = 0$

$z_2 = 0$

$z_3 = 3$

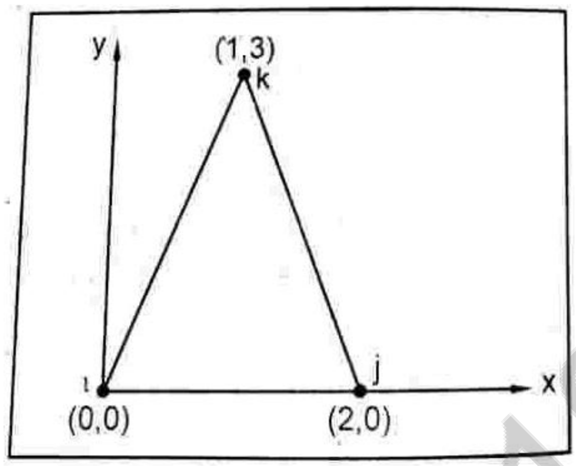


Fig. 7 Triangular elastic elements

$\alpha = 6\times10^{-6}/C$

$v = 0.25$

$E = 15\times10^{6}\ N/cm^{2}$

$\Delta T = 20°C$

We known that

$k = 2\pi rA[B]^T[B][D] \qquad (1)$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1-\frac{\gamma_1^2}{r} & 0 & \frac{\alpha_2}{r}+\beta_2-\frac{\gamma_2^2}{r} & 0 & \frac{\alpha_3}{r}+\beta_3-\frac{\gamma_3^2}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\alpha_1 = r_2z_3 - r_3z_2 = 2\times3 - 1\times0 = 6$

$\alpha_2 = r_3z_1 - r_1z_3 = 1\times0 - 0\times3 = 0$

$\alpha_3 = r_1z_2 - r_2z_1 = 0\times0 - 2\times0 = 0$

$\beta_1 = z_2 - z_3 = 0 - 3 = -3$

$\beta_2 = z_3 - z_1 = 3 - 0 = 3$

$\beta_3 = z_1 - z_2 = 0 - 0 = 0$

$\gamma_1 = r_3 - r_2 = 1 - 2 = -1$

$\gamma_2 = r_1 - r_3 = 0 - 1 = -1$

$\gamma_3 = r_2 - r_1 = 2 - 0 = 2$

$$r = \frac{r_1 + r_2 + r_3}{3}$$

$$= \frac{0 + 2 + 1}{3} = \frac{3}{3}$$

$r = 1$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{0 + 0 + 3}{3}$$

$z = 1$

$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1^2}{r} = \frac{6}{1} - 3 + \frac{-1 \times 1}{1} = 2$$

$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2^2}{r} = \frac{0}{1} + 3 + \frac{-1 \times 1}{1} = 2$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3^2}{r} = \frac{0}{1} + 0 + \frac{2 \times 1}{1} = 2$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(6 - 0) + 0 + 0\right]$$

$A = 3\text{cm}^2$

$$[B] = \frac{1}{2 \times 3}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 6 & -3 & 0 & 3 & 0 & 0 \end{bmatrix} \ldots(1)$$

$$[B]^T = 0.1667\begin{bmatrix} -3 & 2 & 0 & 6 \\ 0 & 0 & 6 & -3 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We know that stress strain displacement matrix

$$[D] = \frac{E}{(1+\gamma)(1-2\gamma)}\begin{bmatrix} 1-\gamma & \gamma & \gamma & 0 \\ \gamma & 1-\gamma & \gamma & 0 \\ \gamma & \gamma & 1-\gamma & 0 \\ 0 & 0 & 0 & \frac{1-2\gamma}{2} \end{bmatrix}$$

$$= \frac{15\times10^6}{(1+0.25)(1-2\times0.25)}\begin{bmatrix} 1-0.25 & 0.25 & 0.25 & 0 \\ 0.25 & 1-0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 1-0.25 & 0 \\ 0 & 0 & 0 & \frac{1-2\times0.25}{2} \end{bmatrix}$$

$$= 8.5\times8571.4\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$= 0.25\times8571.4\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[D] = 2142.85\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B]^T[D] = 0.1667\begin{bmatrix} -3 & 2 & 0 & 6 \\ 0 & 0 & 6 & -3 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 357.2\begin{bmatrix} -7 & 3 & -1 & 6 \\ 6 & 6 & 18 & -3 \\ 11 & 9 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 2 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[B]^T D[B] = 357.2\begin{bmatrix} -7 & 3 & -1 & 6 \\ 6 & 6 & 18 & -3 \\ 11 & 9 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 2 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\times\frac{1}{6}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 6 & -3 & 0 & 3 & 0 & 0 \end{bmatrix}$$

$$= 58.7\begin{bmatrix} 63 & -24 & -15 & 18 & 6 & 0 \\ -24 & 117 & 30 & -9 & 12 & 0 \\ -15 & 0 & 44 & 0 & 18 & 0 \\ 18 & -9 & 0 & 9 & 0 & 0 \\ 6 & 12 & 18 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Substitute $[B]^T D[B]$ in (1)

$$[k] = 2\pi\times 1\times 3\times 58.7\begin{bmatrix} 63 & -24 & -15 & 18 & 6 & 0 \\ -24 & 117 & 30 & -9 & 12 & 0 \\ -15 & 0 & 44 & 0 & 18 & 0 \\ 18 & -9 & 0 & 9 & 0 & 0 \\ 6 & 12 & 18 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k] = 1106.46\begin{bmatrix} 63 & -24 & -15 & 18 & 6 & 0 \\ -24 & 117 & 30 & -9 & 12 & 0 \\ -15 & 0 & 44 & 0 & 18 & 0 \\ 18 & -9 & 0 & 9 & 0 & 0 \\ 6 & 12 & 18 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thermal Force Vector

$$[F]_t = [B]^T [D]\{r\}_2\, 2\pi r A$$

$$[e]_t = \begin{Bmatrix} \alpha\Delta T & 5\times10^5 & 15 & 50 \\ \alpha\Delta T & 5\times10^5 & 10 & 50 \\ 0 & 5\times10^5 & 0 & 0 \\ \alpha\Delta T & 6\times10^6 & 15 & 50 \end{Bmatrix}$$

$$[F]_t = 357.2 \begin{bmatrix} -7 & 3 & -1 & 6 \\ 6 & 6 & 18 & -3 \\ 11 & 9 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 2 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^6 \begin{Bmatrix} 90 \\ 90 \\ 0 \\ 90 \end{Bmatrix} \times 2\pi \times 1.3$$

$$= 6.73\times10^{-3} \begin{Bmatrix} 180 \\ 810 \\ 1800 \\ 270 \\ 558 \\ 0 \end{Bmatrix}$$

$$[F]_t = \begin{Bmatrix} 1.21 \\ 5.45 \\ 12.11 \\ 1.82 \\ 3.70 \\ 0 \end{Bmatrix}$$

UNIT V

2 MARKS

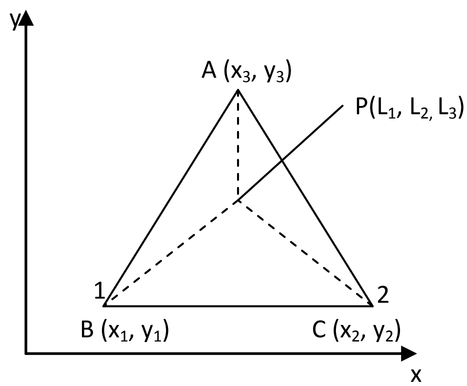
32. What are the advantages of natural coordinates over global co- ordinates? (Nov/Dec 2008)

A natural co – ordinates system is used to define any point inside the element by a set of dimension less number whose magnitude never exceeds unity. This system is very useful in assembling of stiffness matrices. But in the global co – ordinate system, the points in the entire structure are defined using co – ordinates.

33. Define Iso parametric elements. (Nov/Dec 2008)

It is difficult to represent the curved boundaries by straight edges finite elements. A large number of finite elements may be used to obtain reasonable between original body and the assemblage. In order to overcome this drawback, isopacemetric element are used for problems involving curved boundaries a family of element known as " Isotroparametric elements" are used.

34. Write the natural co – ordinates for the point 'P' of the triangular element. The point 'P' is the C.G of the triangle (Nov/Dec 2008)



Solution:-

Natural co – ordinates

35. Distinguish between essential boundary conditions and natural boundary conditions and naturals boundary conditions. (Nov/Dec 2009)

Primary boundary condition

The boundary condition which in terms of field variables is known as primary boundary condition

Natural boundary conditions

The boundary conditions which are in the differential form of field variables is known as natural boundary conditions.

36. Write down the interpolation function of a field variable for three-node triangular element. (MAY / JUNE 2010)

$$J^{-1} = \frac{1}{|J|}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

16 MARKS

9. Why higher order elements are needed? Determine the shape function of an eight noded rectangular element. (Nov /Dec 2008)

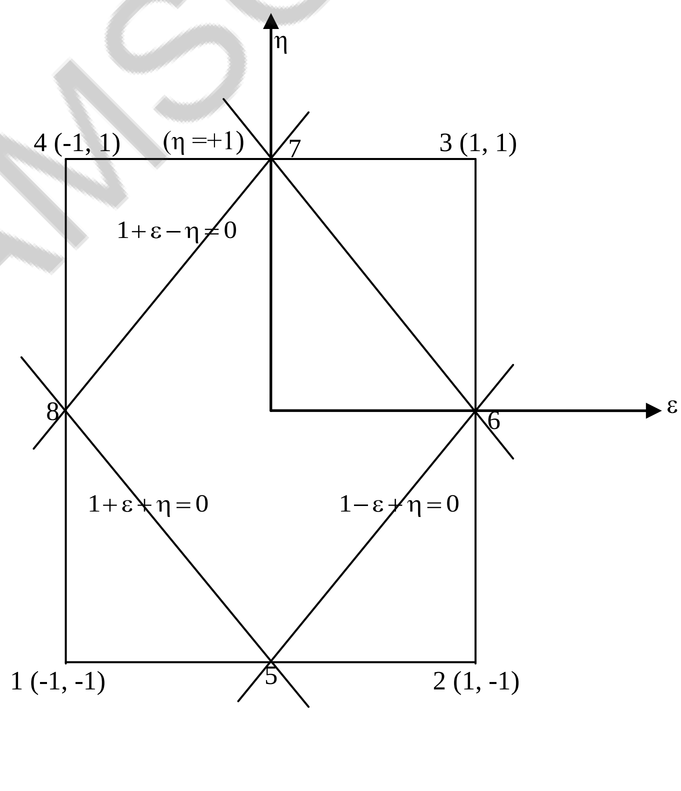
Solution:-

The higher order element can be capture variation in stress in element, such as the stress occurring near filters, holes etc.

Eight noded rectangular element is shown below.

We know that shape function N1 = 1 at node 1 and 0 at all other nodes. Thus N1 has to vanish along the lines

$\varepsilon = +1,\ \eta = +1$ and $\varepsilon + \eta = -1$



At node – 1

(co – ordinate ε = -1, η = -1)

Shape function N_1 = 1 at node 1

N_2 = 0 at another node

$$N_1 = C(1-\varepsilon)(1-\eta)(1+\varepsilon+\eta) \qquad ----(1)$$

Where C = constant

$$\varepsilon = -1,\ \eta = -1 \text{is}\ (1)$$
$$N_1 = C(1+1)(1+1)(1-1)$$
$$N_1 = -4C_1$$
$$C = -1/4$$

$$N_1 = -1/4(1-\varepsilon)(1-\eta)(1+\varepsilon+\eta) \qquad ---(2)$$

At node (Co – ordinate ε =1, η = -1)

N_2 = 1 at node 2

N_2 = 0

$$N_2 = C(1+\varepsilon)(1-\eta)(1-\varepsilon+\eta) \qquad -----(3)$$

ε =1, η = -1 in (3)

$$N_2 = C(1+1)(1+1)(-1)$$
$$N_2 = -4C$$
$$C = -1/4$$

$$N_2 = -1/4(1+\varepsilon)(1-\eta)(1-\varepsilon+\eta) \qquad ---(4)$$

At node – 3 (Co – ordinate ε = 1, η = 1)

N_3 = 1 at node 3

N3 = C(1+ε)(1+η)(1-ε-η) -----------(5)

Sub ε = 1, η = 1 in (5)

$$N_3 = C(1+1)(1+\eta)(-1)$$
$$1 = -4C$$
$$C = -1/4$$
$$N_3 = -1/4(1+\varepsilon)(1+\eta)(1-\varepsilon-\eta) \qquad -----(6)$$

Similarly all notes are same as substation

$$N_4 = -\frac{1}{4}(1-\varepsilon)(1+\eta)(1+\varepsilon-\eta)$$

$$N_5 = \frac{1}{2}(1-\varepsilon)^2(1-\eta)$$

$$N_6 = \frac{1}{2}(1+\varepsilon)(1-\eta^2)$$

$$N_7 = \frac{1}{2}(1-\varepsilon^2)(1+\eta)$$

$$N_8 = \frac{1}{2}(1-\varepsilon)(1-\eta^2)$$

Where at node 8 is differ

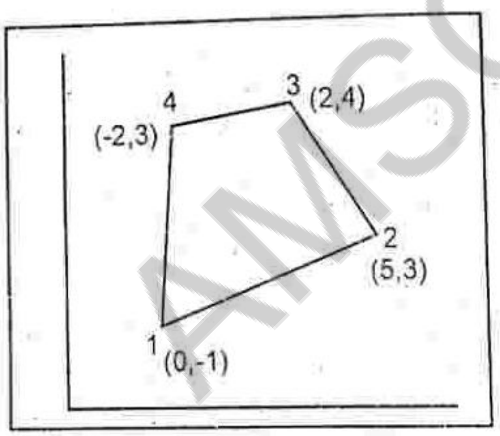
$$C = \frac{1}{2}$$

Where shape function

$$N_8 = C(1+1)(1-0)$$

$$N_8 = \frac{1}{2}(1-\varepsilon)(1-\eta^2)$$

10. The Cartesian (global) coordinates of the corner nodes of a quadrilateral element are given by (0, -1), (-2,3), (2, 4) and (5, 3). Find the coordinate transformation between the global and local (natural) co ordinates. Using this, determine the Cartesian co ordinates of the defined by (r, s) = 0.5, 0.5) in the global co ordinate system. (Nov/Dec 2009)



We know that $\varepsilon = 0.5$, $\eta = 0.5$

Shape function for quad evened are

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta) = \frac{1}{4}(1-0.5)\times(1-0.5) = 0.0625$$

$$N_2 = \frac{1}{2}(1+\varepsilon)(1-\eta) = \frac{1}{4}(1+0.5)\times(1-0.5) = 0.1875$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta) = \frac{1}{4}(1+0.5)\times(1+0.5) = 0.5625$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta) = \frac{1}{4}(1-0.5)(1+0.5) = 0.1875$$

We know that

$$\begin{aligned}
X &= N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \\
&= 0.625\times 0 + 0.1875\times 5 + 0.5625\times 2 - 0.1875\times -2 \\
X &= 1.6875\,\text{mm} \\
Y &= N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 \\
&= 0.0625\times -1 + 0.1875\times 3 + 0.5625\times 4 + 0.1875\times 3 \\
Y &= 3.3125\,\text{mm}
\end{aligned}$$

Co ordinate of P = (1.6875, 3.3125)

11. Evaluate the integral $I = \int_{-1}^{1} (2 + x + x^2)dx$ and compare with exact results. (Nov/Dec 2009)

Solution:-

$$I = \int_{-1}^{1} (2 + x + x^2)dx$$

Since the integral is a polynomial of 2, we get

2n – 1 = 2

2n = 3

n = 1.5=2

Since it is a 2 sampling points we used from table

$$x_1 = -1\sqrt{1/3} = 0.577350.26$$
$$x_2 = \sqrt{1/3} = -0.57735026$$

And W1 = W2 = 1

We know that $f(x) = 2+x+x^2$

$f(x_1) = 2+x_1+x_1^2$

$= 2+ 0.57735026+ (+0.57735026)^2$.

= 2.9106836

$$W_1 f(x_1) = 1 \times 2.9106836$$
$$= 2.910636$$

$$f(x) = 2 + x + x_2^2$$
$$= 2 - 0.57735026 + (-0.57735026)^2$$
$$= 1.755983$$

$$W_2 f(x_2) = 1 \times 1.755983$$

Adding (1) and (2)

$$W_1 f_1(x_1) + W_2 f_1(x_2) = 2.9106836 + 1.755983$$
$$= 4.6666666$$

Exact solution

$$\int_{-1}^{1} (2 + x + x^2)dx = 2(x)_{-1}^{1} + \left(\frac{x^2}{2}\right)_{-1}^{1} + \left(\frac{x^3}{3}\right)_{-1}^{1}$$
$$= 2(1+1) + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{3}\right)$$
$$= 4 = 1 + \frac{2}{3}$$
$$= 5 + \frac{2}{3}$$
$$= 4.6666666$$

12. (i) The Cartesian (global) co ordinates of the corner nodes of an isoparametric quadrilateral element are given by (1, 0), (2, 0), (2.5, 1.5) and (1.5, 1). Find its Jacobina matrix. (Nov/Dec 2009)

$x_1 = 1 \quad x_2 = 2 \quad x_3 = 2.5 \quad x_4 = 1.5$

$y_1 = 0 \quad y_2 = 0 \quad y_3 = 1.5 \quad y_4 = 1$

We know that

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$
$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$
$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \qquad --(1)$$
$$y = N_1 y_1 + N_2 y_2 + N_3 x_3 + N_4 x_4 \qquad --(2)$$

Sub all N1, N2, N3 and N4 and all x and y values

$$x = \frac{1}{4}\left[(1-\varepsilon)(1-\eta)\times 1 + (1+\varepsilon)(1-\eta)2 + (1+\varepsilon)(1+\eta)\times 25 + (1-\varepsilon)(1+\eta)\times 1.5\right] \quad --(3)$$

$$y = \frac{1}{4}\left[(1-\varepsilon)(1-\eta)\times 0 + (1+\varepsilon)(1-\eta)0 + (1+\varepsilon)(1+\eta)\times 1.5 + (1-\varepsilon)(1+\eta)\times 1\right] \quad --(4)$$

Simplifying eq (3)

$$x = \frac{1}{4}\left[\begin{matrix}(1-\varepsilon-\eta+\varepsilon\eta+2+2\varepsilon-2\eta-2\varepsilon\eta+2.5 \\ +2.5\varepsilon+2.5\eta+2.5\varepsilon\eta+1.5-1.5\varepsilon+1.5\eta-1.5\varepsilon\eta\end{matrix}\right]$$

$$x = \frac{1}{4}\left[8+2\varepsilon+\eta+0\right]$$

$$\frac{\partial x}{\partial \varepsilon} = \frac{1}{4}\left[2+0+0\right]$$

$$= \frac{1}{2}$$

$$\frac{\partial x}{\partial \eta} = \frac{1}{4}\left[0+1++0\right]$$

$$= \frac{1}{4}$$

Simplify the eq (4)

$$Y = \frac{1}{4}\left[0+0+1.5+1.5\varepsilon+1.5\eta+1.5\varepsilon\eta-\varepsilon+\eta-\varepsilon\eta\right]$$

$$Y = \frac{1}{4}\left[2.5+0.5\varepsilon-2.5\eta-\varepsilon\eta\right]$$

$$\frac{\partial y}{\partial \varepsilon} = \frac{1}{4}\left[0+0.5+0+0\right]$$

$$= \frac{0.5}{4}$$

$$\frac{\partial y}{\partial \eta} = \frac{1}{4}\left[0+0+2.5+0.5\varepsilon\right]$$

$$= \frac{0.5}{4}\left[5+\varepsilon\right]$$

Jacobin matrix $\left[J\right] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$

$$J_{11} = \frac{\partial x}{\partial \varepsilon} J_{12} = \frac{\partial y}{\partial \varepsilon}$$

$$J_{21} = \frac{\partial x}{\partial \eta} J_{22} = \frac{\partial y}{\partial \eta}$$

$$\left[J\right] = \begin{bmatrix} \frac{1}{2} & 0.125(1+\eta) \\ \frac{1}{4} & 0.125(5+\varepsilon) \end{bmatrix}$$

12. (ii) Distinguish between sub parametric and super parametric elements. (Nov/Dec 2009)

Sub parametric element

No. of nodes used for defining the geometry is more than no. of needs for defining displacement.

Super parametric

If the no. of nodes used for defining the geometry is less than the no. of nodes used for defining the placement.

13. Evaluate the integral $I = \int_{-1}^{1} x^2 + \cos\left(\frac{x}{2}\right)dx$ using three point Gaussian and compare with exact solution: (NOV / DEC 2011)

$$I = \int_{-1}^{1} \left[x^2 + \cos\left(\frac{x}{2}\right)\right]dx$$
$$f(x) = x^2 + \cos\left(\frac{x}{2}\right)$$

For three point Gaussian quadration

$$x_1 = \sqrt{\frac{3}{5}} = 0.77 + 59669$$
$$x_2 = 0$$
$$x_3 = -\sqrt{\frac{3}{5}} = -0.774596669$$
$$\omega_1 = \frac{5}{9} \Rightarrow 0.555555$$
$$\omega_2 = \frac{8}{9} \Rightarrow 0.888888$$
$$\omega_{39} = \frac{5}{9} \Rightarrow 0.555555$$

$$f(x) = x^2 + \cos\left(\frac{x}{2}\right)$$
$$\Rightarrow f(x_1) = x_1^2 + \cos\left(\frac{x_1}{2}\right)$$
$$f(x_1) = 1.5259328$$
$$\Rightarrow \omega_1 f(x_1) = 0.8477396 \qquad ---(1)$$
$$\Rightarrow f(x_2) = 1$$
$$\Rightarrow \omega_2 f(x_2) \Rightarrow 0.888888 \qquad ---(2)$$
$$\Rightarrow f(x_3) = 1.5259326$$
$$\Rightarrow \omega_3 f(x_3) \Rightarrow 0.8477396 \qquad ---(3)$$

Adding Equation (1) , (2) and (3)

$$\int_{-1}^{1} \left(x^2 + \cos\frac{x}{2}\right)dx = 2.58436 \qquad ---(4)$$

Exact solution

$$\int_{-1}^{1}\left[x^2+\left(\cos\frac{x}{2}\right)\right]dx$$

$$\Rightarrow\left[\frac{x^3}{3}\right]_{-1}^{1}+\left[\frac{\sin\frac{x}{2}}{\left(\frac{1}{2}\right)}\right]_{-1}^{1}$$

$$\Rightarrow 2.58436 \qquad ---(5)$$

By comparing (4) and (5) Both answer one equal.

14. Define the following terms with suitable examples:

(i) Plane stress, Plane strain (ii) Node, Element and Shape functions

(iii) Iso-parametric element (iv) Axisymmetric analysis. (MAY / JUNE 2010)

(i) Plane stress and plane strain

Plane stress: A state of rest in which the normal stress and shear stress is directly proportions to the plane are assumed to be zero

Example: Plate with notes and plate with fillets

Plane strain: A state of strain in which the strain normal to the xy plane and shear stress are assumed to be zero.

Example: Dams and pipes subjected to loads remains constant over a length.

(ii) Node, Element and Shape Function

Node: Each kind of finite element has a specified structural shape and is interconnected with the adjacent elements by nodal points. The force will act only at nodes.

Element: Each nodes are connected by a line called element

Shape function: In FEM field variables within an element are generally expressed by

$$\phi(x,y)=N_1(x,y)\phi_1+N_2(x,y)\phi_2+N_3(x,y)\phi_3$$

(iii) Isoparametric element

If the number of nodes used for defining the geometry is same as no of nodes used for defining the displacements

(iv) Axisymmetric analysis

Many 3D three dimensional problems in engineering exhibit symmetry about an axis of rotation. Such types of problems as solved a special two dimensional element called axisymmetric element.

15. Use Gaussian quadrature to obtain an exact value of the integral (MAY / JUNE 2010)

$$I = \int_{-1}^{1}\int_{-1}^{1} \left(r^3 - 1\right)\left(s-1\right)^2 \, dr \;\; ds$$

or

$$I = \int_{-1}^{1}\int_{-1}^{1} \left(r^2 - 1\right)\left(S-1\right)^2 \, dr \;\; dS$$

Gaussian quadrature

We know the Gaussian quadrature function is

$$\int_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0)$$

Lets go by variable separate method

$$\int f(x)dx = \int_{-1}^{1} \left(r^3 - 1\right)dr \;\; \int_{-1}^{1} \left(S-1\right)^2 dS$$

$$\int_{-1}^{1} \left(r^3 - 1\right)dr = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0) \qquad ---(1)$$

$$f(r) = r^3 - 1$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = \left[-\sqrt{\frac{3}{5}}\right]^3 - 1$$

$$= -0.4648 - 1$$

$$= -1.4648 \qquad --(2)$$

$$f\left(\sqrt{\frac{3}{5}}\right) = \left(\sqrt{\frac{3}{5}}\right) - 1$$

$$= -0.5353 \qquad --(3)$$

$$f(0) = 0 - 1 = -1 \qquad --(4)$$

Sub 2, 3, 4 in (1)

$$\int_{-1}^{1} \left(r^3 - 1\right)dr = \frac{5}{9}\left[-1.4648 - 0.5353\right] + \frac{8}{9}(-1)$$

$$= \frac{5}{9}\left[-2.000\right] - \frac{8}{9}$$

Similar

$$f(S) = \left(S-1\right)^2$$

$$f\left(-\sqrt{\frac{3}{5}}\right)=\left(-\sqrt{\frac{3}{5}}-1\right)^2=3.149$$

$$f\left(\sqrt{\frac{3}{5}}\right)=\left(\sqrt{\frac{3}{5}}-1\right)^2=0.0508$$

$$f(0)=(0-1)^2=1$$

$$f(S)=(S-1)^2=\frac{5}{9}[3.1492+0.0508]-\frac{8}{9}(1)$$

$$=2.6667$$

$$\int_{-1}^{1}\int_{-1}^{1}(r^3-1)(s-1)^2\,dr\;dS=-2.000\times 2.6667$$

$$=-5.3336$$

Exact solution

$$I=\int_{-1}^{1}\int_{-1}^{1}(r^3-1)(S-1)^2\,dr\;dS$$

$$=\int_{-1}^{1}(r^3-1)(S-1)^2\,dr\,dS$$

$$=\left(\frac{r^4}{4}-r\right)_{-1}^{1}\left(\frac{(S-1)^3}{3}\right)_{-1}^{1}$$

$$=\left[\left(\frac{1}{4}-1\right)-\left(\frac{1}{4}+1\right)\times 0-\frac{(-1-1)^3}{3}\right]$$

$$=-2\times\frac{8}{3}$$

$$=-\frac{16}{3}$$

$$=-5.333$$