IC8451 CONTROL SYSTEMS

SYLLABUS

COURSE OBJECTIVES

The student should be made to:

- To understand the use of transfer function models for analysis physical systems and introduce the control system components.
- To provide adequate knowledge in the time response of systems and steady state error analysis.
- To accord basic knowledge in obtaining the open loop and closed–loop frequency responses of systems.
- To introduce stability analysis and design of compensators
- To introduce state variable representation of physical systems

UNIT I SYSTEMS AND REPRESENTATION 9

Basic elements in control systems: – Open and closed loop systems – Electrical analogy of mechanical and thermal systems – Transfer function – AC and DC servomotors – Block diagram reduction techniques – Signal flow graphs.

UNIT II TIME RESPONSE 9

Time response: – Time domain specifications – Types of test input – I and II order system response – Error coefficients – Generalized error series – Steady state error – Root locus construction- Effects of P, PI, PID modes of feedback control –Time response analysis.

UNIT III FREQUENCY RESPONSE 9

Frequency response: – Bode plot – Polar plot – Determination of closed loop response from open loop response - Correlation between frequency domain and time domain specifications

UNIT IV STABILITY AND COMPENSATOR DESIGN 9

Characteristics equation – Routh Hurwitz criterion – Nyquist stability criterion- Performance criteria –Effect of Lag, lead and lag-lead compensation on frequency response-Design of Lag, lead and laglead compensator using bode plots.

UNIT V STATE VARIABLE ANALYSIS 9

Concept of state variables – State models for linear and time invariant Systems – Solution of state and output equation in controllable canonical form – Concepts of controllability and observability.

L	T	P	C
3	**2**	**0**	**4**

TOTAL (L: 45+T:30): 75 PERIODS

COURSE OUTCOMES

At the end of the course, the student should have the:

- Ability to develop various representations of system based on the knowledge of Mathematics, Science and Engineering fundamentals.
- Ability to do time domain and frequency domain analysis of various models of linear system.
- Ability to interpret characteristics of the system to develop mathematical model.
- Ability to design appropriate compensator for the given specifications.
- Ability to come out with solution for complex control problem.

- Ability to understand use of PID controller in closed loop system.

TEXT BOOKS

T1. Nagarath, I.J. and Gopal, M., “Control Systems Engineering”, New Age International Publishers, 2017.
T2. Benjamin C. Kuo, “Automatic Control Systems”, Wiley, 2014.

REFERENCES

R1. Katsuhiko Ogata, “Modern Control Engineering”, Pearson, 2015.
R2. Richard C.Dorf and Bishop, R.H., “Modern Control Systems”, Pearson Education,2009.
R3. John J.D., Azzo Constantine, H. and Houpis Sttuart, N Sheldon, “Linear Control System Analysis and Design with MATLAB”, CRC Taylor& Francis Reprint 2009.
R4. Rames C.Panda and T. Thyagarajan, “An Introduction to Process Modelling Identification and Control of Engineers”, Narosa Publishing House, 2017.
R5. M.Gopal, “Control System: Principle and design”, McGraw Hill Education, 2012.
R6. NPTEL Video Lecture Notes on “Control Engineering “by Prof. S. D. Agashe, IIT Bombay.

Unit – I
SYSTEMS COMPONENTS AND THEIR REPRESENTATION
Part – A

1. What is control system? Nov/Dec 2016
A system consists of a number of components connected together to perform a specific function. In a system when the output quantity is controlled by varying the input quantity then the system is called control system

2. Define open loop control systems Nov/Dec 2017
The control system in which the output quantity has no effect upon the input quantity is called open loop control system. This means that the output is not feedback to the point for correction

3. Define closed loop control systems Nov/Dec 2017
The control system in which the output has an effect upon the input quantity so as to maintain the desired output values are called closed loop control systems

4. What are the components of feedback control system? Nov/Dec 2016
The component of feedback control system are plant, feedback path elements, error detector actuator and controller

5. Distinguish between open loop and closed loop system May/June 2013, 2016, Nov/Dec 2019

S.No.	OPEN LOOP	CLOSED LOOP
1.	Inaccurate	Accurate
2.	Simple and economical	Complex and costlier
3.	The changes in output due to external disturbance are not corrected	The changes in output due to external disturbance are corrected automatically
4.	Always stable	Generally great efforts are needed to design a stable system

6. Define transfer function Nov/Dec 2011& April/May 2017
The transfer function of a system is defined as the ratio of the Laplace transform of output to Laplace transform of input with zero initial conditions.

7. What are the basic elements used for modeling mechanical translational system May/June13/Nov/Dec16&17/April/May 19
   - Mass M,.Kg
   - Stiffness of spring K, N/m
   - Viscous friction coefficient dashpot B,N-sec/m

8. What are the basic elements used for modeling mechanical rotational system? April/May 2019
   - Moment of inertia J,Kg- m^2 /rad
   - Dashpot with rotational frictional coefficient B,N-m/(rad/sec)
   - Torsional spring with stiffness K,N-m/rad

9. Name two types of electrical analogous for mechanical system
The two types of analogies for the mechanical system are
   - Force voltage analogy
   - Force current analogy

10. What is block diagram? Nov/Dec 2015, April/May 2017
A Block diagram of a system is a pictorial representation of the functions performed by each component of the system and shows the flow of signals.

11. What are the basic component s of block diagram? April/May 2017
The basic elements of block diagram are blocks, branch point and summing point

12. What is the basis for framing the rules of block diagram reduction technique?
The rules for block diagram reduction technique are framed such that any modification made on the diagram does not alter the input output relation

13. What is a signal flow graph?
A signal flow graph is a diagram that represents of set of simultaneous algebraic equations. By taking Laplace transform the time domain differential equations governing a control system can be transferred to a set of algebraic equations in a s-domain.

14. What is transmittance?
The transmittance is the gain acquired by the signal when it travels from one node to another node is signal flow graph.

15. What is sink and source?
Source is the input node in the signal flow graph and it has only outgoing branches. Sink is a output node in the signal flow graph and it has only incoming branches.

16. Write Masons Gain formula April/May 2015/2016, April/May 2018, April/May 2019
Masons gain formula states that the overall gain of the system as follows overall gain,

$$T(s) = \frac{\Sigma_K \Delta_K P_K}{\Delta}$$

T(s) = Transfer Function of the system
K = Number of forward path in the signal flow
P_K = Forward path gain of the Kth forward pain
Δ = 1- (Sum of individual loop gains) + (Sum of gain products of all possible combinations of two both touching loops) – (Sum of gain products of all possible combinations of three non touching loops) +….
Δ_K = (Δ for the part of the graph which is not touching Kth forward path)

17. Write the analogues electrical elements in force voltage analogy for the elements of mechanical translational system
Force – f – Voltage, e
Velocity, V – current , i
Displacement, x – charge, q
Fricitional coefficient , B – Resistance, R
Mass, M – inductance, L
Stiffness, K – Inverse of capacitance 1/C
Newton's second law – Kirchhoff's voltage law.

18. Write the analogous electrical elements in force current analogy for the elements of mechanical translational system
Force, f – current, i
Velocity, V – Voltage, e
Displacement, x – flnx Φ
Fricitional coefficient, B – Conductance, G = 1/R
Mass, M – capacitance C
Stiffness, K – Inverse of inductance, 1/L
Newton's second law – Kirchhoff's current law

19. Write the analogous electrical elements in torque voltage analogy for the elements of mechanical rotational system
Torque, T – Voltage, e
Angular Velocity, ω - current , i
Angular Displacement, θ - charge, q

Frictional coefficient, B – Resistance, R
Moment of Inertia, J– inductance, L
Stiffness of the spring, K-– Inverse of capacitance 1/C
Newton's second law – Kirchhoff's current law

20. Write the analogous electrical elements in torque current analogy for the elements of mechanical rotational system
Torque, t – current , i
Angular Velocity, ω - voltage , e
Angular Displacement, θ - flux, Φ
Frictional coefficient , B- Conductance , G = 1/R
Moment of Inertia, I- capactitance, C
Stiffness of the spring , K – Inverse of inducatance, 1/L
Newton's second law – Kirchhoff's current law

21. Write the force balance equation of an ideal mass, dashpot and spring element
Let a force f be applied to an ideal mass M. The mass will offer an opposing for f_m which is proportional to acceleration.

$$f = f_m = Md^2X / dt^2$$

Let a force f be applied to an ideal dashpot, with viscous frictional coefficient B. the dashpot will offer an opposing force f_b which is proportional to velocity.

$$f = f_b = B\frac{dX}{dt}$$

Let a force f be applied to an ideal spring, with spring constant K. The spring will offer an opposing force f_k which is proportional to displacement.

$$f = f_k = K\ X$$

22. Why negative feedback is invariably preferred in closed loop system?
The negative feedback results in better stability in steady state and rejects any disturbance signals.

23. State the principles of homogeneity (or) superposition
The principles of superposition and homogeneity states that if the system has responses
$y_1(t)$ and $y_2(t)$ for the inputs $x_1(t)$ and $x_2(t)$ respectively then the system response to the linear combination of the individual outputs $a_1x_1(t) + a_2x_2(t)$ its given by linear combination of the individual outputs $a_1y_1(t) + a_2y_2(t)$ where a_1 ,a_2 are constant.

$y_1(t)$ and $y_2(t)$ for the inputs $x_1(t)$ and $x_2(t)$ respectively then the system respsonse to the linear combination of the individual outputs $a_1x_1(t) + a_2x_2(t)$is given by linear combination of the individual outputs $a_1y_1(t) + a_2y_2(t)$ where $a_{1,}$ a_2 are constant

24. What are the basic properties of signal flow graph?
The basic properties of signal flow graph are
- Signal flow graph is applicable to linear systems
- It consists of nodes and branches
- A node adds the signal of all incoming branches and transmits this sum to all outgoing branches.
- Signals travel along branches only in the marked direction and is multiplied by the gain of the branch.
- The algebraic equations must be in form of cause and effect relationship

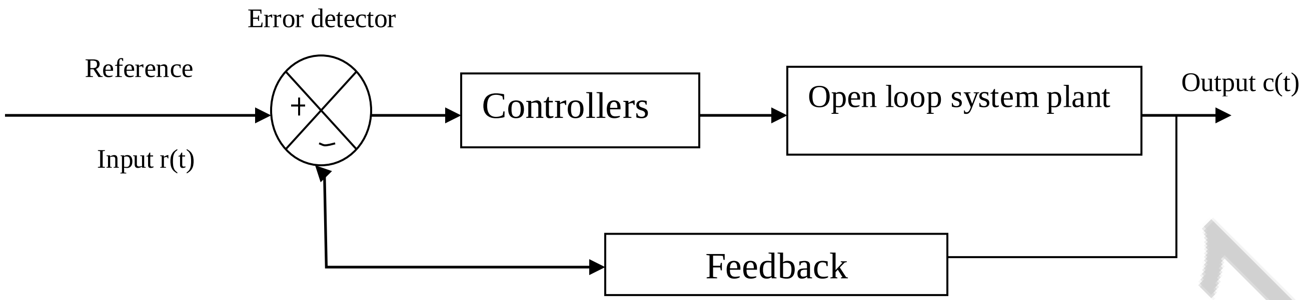
25. Define non touching loop
The loops are said to be non touching if they do not have common nodes

26. List the advantages of closed loop system? Nov/Dec 2015 & April/May 2017
It is accurate.

The change in output due to external disturbances are corrected automatically.

Part B and C question & Answers

1. Explain the features of closed loop feedback Control system May/ June 2015
   Control system in which the output has an effect upon the input quantity in order to maintain the desired output values are called closed loop systems.



The open loop system can be modified as closed loop system by providing a feedback. The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed loop system is also called automatic control systems

The general block diagram of an automatic control system is shown in fig. In consists of an error detector, a controller, plant and feedback path elements.

The reference signal (or input signal) corresponds to desired output. The feedback path elements samples the output and converts it to a signal of same type as that of reference signal. The feedback signal is proportional to output signal and it is fed to the error detector.

The error signal generated by the error detector is difference between reference signal and feedback signal. The controller modifier and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output.

Advantages of closed loop system:

1. The closed loop systems are accurate.
2. The closed loop systems are accurate even in the presence of non linearties.
3. The sensitivity of the systems may be made small to make the system more stable
4. The closed loop systems are less affected by noise.

Disadvantages of closed loop systems.

1. The closed loop systems are complex and costly
2. The feedback in closed loop system may lead to oscillatory response.
3. The feedback reduces the overall gain of the system
4. Stability is a major problem in closed loop system and more care is needed to design a stable closed loop system

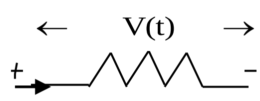
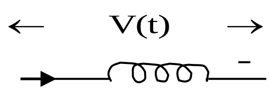
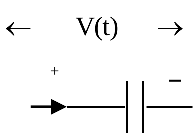
2.Compare open loop and closed loop control system **Nov/Dec 2016, Nov/Dec 2018**

S.No.	Open Loop	Closed loop
1.	Any change in output has no effect on the input (i.e.) feedback does not exists	Changes in output, affects the input which is possible by use of feedback
2.	Output measurement is not required for operation of system	Output measurement is necessary

3.	Feedback element is absent	Feedback element is present
4.	Error detector is absent	Error detector is necessary
5.	It is inaccurate and unreliable	Highly accurate and reliable
6.	Highly sensitive to the disturbances	Less sensitive to the disturbances
7.	Highly sensitive to the environmental changes	Less sensitive to the environmental changes
8.	Bandwidth is small	Bandwidth is large
9.	Simple to construct and cheap	Completed to design and hence costly
10.	Generally are stable in nature	Stability is the major consideration while designing
11.	Highly affected by non linearities	Reduced effect of non linearities

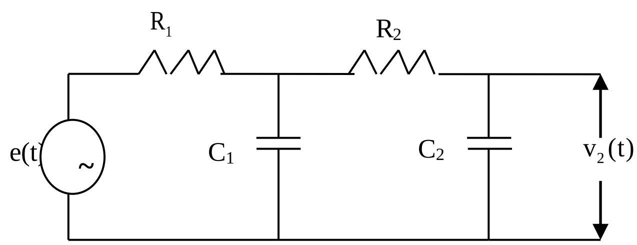
Mathematical Models of Electrical Systems

- → The basic elements of electrical system are resistor, inductor, capacitor
- → The differential equations of the electrical systems can be formed by applying Kirchoff's laws

COMPONENTS	VOLTAGE ACROSS THE ELEMENT	CURRENT THRO' THE ELEMENT
Resistors 	$V(t) = Ri(t)$	$i(t) = \frac{V(t)}{R}$
Inductor 	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int v(t)\,dt$
Capacitor 	$v(t) = \frac{1}{C}\int i(t)dt$	$i(t) = C\frac{dv(t)}{dt}$

Problems

1. Obtain the transfer function of the electrical network shown in fig.

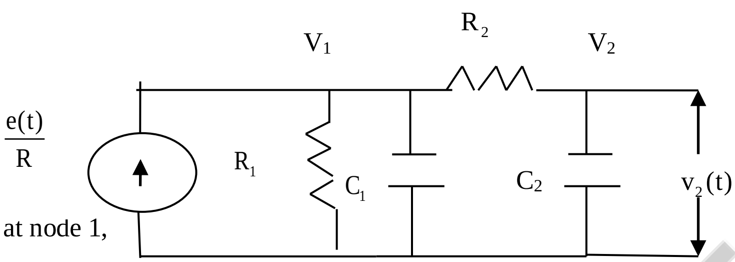


Input – e(t)

Output – $V_2(t)$

Transfer function $= \dfrac{V_2(s)}{E(s)}$

Using source transformation technique, voltage source is converted into current source.



Apply KCL at node 1,

$$\frac{V_1}{R_1} + C_1\frac{dV_1}{dt} + \frac{V_1 - V_2}{R_2} = \frac{e(t)}{R} \qquad (1)$$

Apply KCL at node 2

$$\frac{V_2 - V_1}{R_2} + C_2\frac{dV_2}{dt} = 0 \qquad (2)$$

Equations (1) and (2) forms differential equations/mathematical form of the electrical network shown in fig,

To find transfer function $V_2(s)/E(s)$

Apply Laplace transform to eqn (1)

$$\frac{V_1(s)}{R_1} + C_1 sV(s) + \frac{V_1(s)}{R_2} - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$V_1(s)\left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1} \qquad (3)$$

Apply Laplace transform to eqn (2)

$$\frac{V_2(s)}{R_2} - \frac{V_1(s)}{R_2} + C_2 s V_2(s) = 0$$

$$\therefore V_2(s)\left[\frac{1}{R_2} + C_2 s\right] - V_1(s)\left[\frac{1}{R_2}\right] = 0$$

$$V_1(s) = \left[1 + sC_2R_2\right]V_2(s) \qquad \rightarrow (4)$$

$$L\left[x(t)\right] = X(s)$$

$$L\left[\frac{dx(t)}{dt}\right] = sX(s)$$

$$L\left[\frac{d^2x(t)}{dt^2}\right] = s^2X(s)$$

$$L\left[\int x(t)dt\right] = \frac{X(s)}{s}$$

Substituting for $V_1(s)$ from (4) in (3), we get

$$V_2(s)[1 + sC_2R_2]\left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

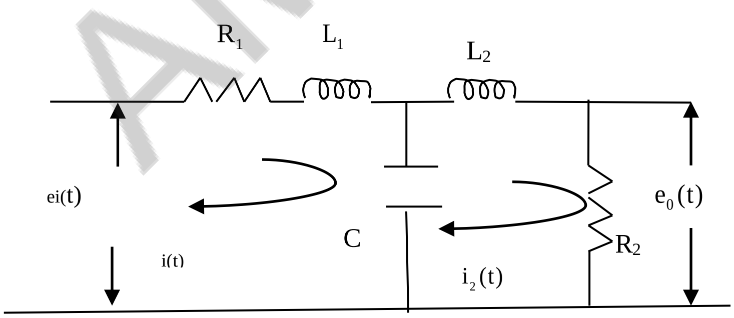
$$V_2(s)\left[(1 + sC_2R_2)\left(\frac{R_2 + sC_1R_1R_2 + R_1}{R_1R_2}\right) - \frac{R_1}{R_1R_2}\right] = \frac{E(s)}{R_1}$$

$$V_2(s)\left[\frac{(1 + sC_2R_2)(R_2 + R_1 + sC_1R_1R_2) - R_1}{R_1R_2}\right] = \frac{E(s)}{R_1}$$

$$\frac{V_2(s)}{E(s)} = \frac{R_2}{(1 + sC_2R_2)(R_1 + R_2 + sC_1R_1R_2) - R_1} \qquad (5)$$

Eqn (5) Is the required transfer function

2. Obtain the transfer function of the following n/w



Input → $e_i(t)$

Output $\rightarrow e_o(t)$

Transfer function = $\dfrac{E_0(s)}{E_i(s)}$

Applying KVL to mesh 1,

$$e_i(t) = R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C}\int (i_i - i_2)\,dt \qquad \rightarrow (1)$$

Applying KVL to mesh 2,

$$0 = L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C}\int (i_2 - i_1)\,dt \qquad \rightarrow (2)$$

$$e_0(t) = i_2(t)\ R_2 \qquad \rightarrow (3)$$

Applying Laplace transform to eqns. (1), (2) and (3)

$$E_i(s) = R_1 I_1(s) + L_1 s I_1(s) + \frac{1}{Cs}\left[I_1(s) - I_2(s)\right]$$

$$E_i(s) = \left[R_1 + L_1 s + \frac{1}{Cs}\right] I_1(s) - \frac{1}{Cs} I_2(s) \qquad (4)$$

$$0 = L_2 s I_2(s) + R_2 I_2(s) + \frac{1}{Cs}\left[I_2(s) - I_1(s)\right]$$

$$0 = -\frac{1}{Cs} I_1(s) + \left[R_2 L_2 s \frac{1}{Cs}\right] I_2(s) \qquad \rightarrow (5)$$

$$E_0(s) = R_2 I_2(s) \qquad \rightarrow (6)$$

Expressing eqn (4) & (5) in matrix form

$$\begin{bmatrix} E_i(s) \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + L_1 s + \frac{1}{Cs} & -\frac{1}{Cs} \\ -\frac{1}{Cs} & R_2 + L_2 s + \frac{1}{Cs} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}$$

$$\Delta = \begin{vmatrix} R_1 + L_1 s + \frac{1}{Cs} & -\frac{1}{Cs} \\ -\frac{1}{Cs} & R_2 + L_2 s + \frac{1}{Cs} \end{vmatrix}$$

$$= \left[\left(R_1 + L_1 s + \frac{1}{Cs}\right)\left(R_2 + L_2 s + \frac{1}{Cs}\right) - \left(\frac{1}{Cs}\right)^2\right]$$

$$\Delta I_2(s) = \begin{vmatrix} R_1 + L_1 s + \frac{1}{Cs} & E_i(s) \\ -\frac{1}{Cs} & 0 \end{vmatrix} = E_i(s)\frac{1}{Cs}$$

$$I_2(s) = \frac{\Delta I_2(s)}{\Delta} = \frac{E_i(s)\frac{1}{Cs}}{\left[\left(R_1 + L_1 s + \frac{1}{Cs}\right)\left(R_2 + L_2 s + \frac{1}{Cs}\right) - \left(\frac{1}{Cs}\right)^2\right]}$$

$$E_0(s) = I_2(s) R_2$$

$$= \frac{E_i(s)\frac{1}{Cs}R_2}{\left[\left(R_1 + L_1 s + \frac{1}{Cs}\right)\left(R_2 + L_2 s + \frac{1}{Cs}\right) - \left(\frac{1}{Cs}\right)^2\right]} = \frac{E_i(s)\frac{1}{\cancel{C}s}R_2}{\frac{\left[\left(R_1Cs + L_1Cs^2 + 1\right)\left(R_2Cs + L_2\left(s^2+1\right) - 1\right)\right]}{C^{\cancel{2}}s^{\cancel{2}}}}$$

$$= \frac{E_i(s) R_2 Cs}{\left[\left(R_1Cs + L_1Cs^2 + 1\right)\left(R_2Cs + L_2Cs^2 + 1\right) - 1\right]}$$

$$\boxed{\frac{E_0(s)}{E_i(s)} = \frac{R_2 C_s}{\left[L_1Cs^2 + R_1Cs + 1\right]\left[L_2Cs^2 + R_2Cs + 1\right] - 1}}$$

3. An electrical circuit is shown in fig. obtain the transfer function relating the output voltage $e_o(t)$ to the input voltage $e_i(t)$ in the form
$$\frac{E_0(s)}{E_i(s)} = K_g \frac{(1+sT_1)}{(1+sT_2)}$$

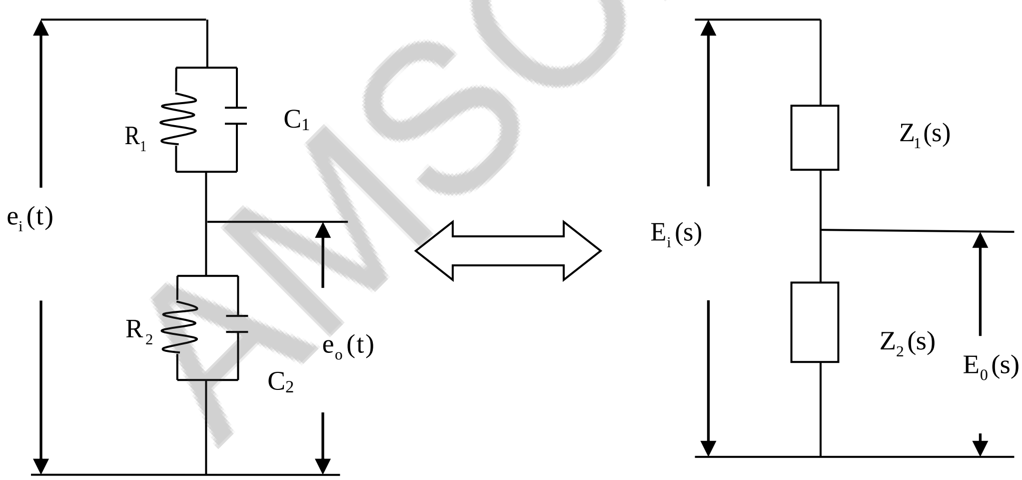


Fig (a) fig(b)

Sol: The components R_1 & C_1 form one parallel combination R_2 & C_2 form are one parallel combination and representation in fig (b)

$$\therefore Z_1(s) = \frac{R_1 \frac{1}{Cs}}{R_1 + \frac{1}{Cs}} = \frac{R_1}{(1 + R_1 C_1 s)}$$

similarly,

$$Z_2(s) = \frac{R_2}{(1 + R_2 C_2 s)}$$

By voltage division d rule,

$$E_0(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}, E_i(s)$$

$$\frac{E_0(s)}{E_i(s)} - \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{\frac{R_2}{1 + R_2 C_2 s}}{\frac{R_1}{1 + R_1 C_1 s} + \frac{R_2}{1 + R_2 C_2 s}}$$

$$= \; = \frac{\frac{R_2}{\cancel{(1 + R_2 C_2 s)}}}{\frac{R_1 (1 + R_2 C_2 s) + (1 + R_1 C_1 s) R_2}{(1 + R_1 C_1 s)\cancel{(1 + R_2 C_2 s)}}}$$

$$= \frac{R_2 (1 + R_1 C_1 s)}{R_1 (1 + R_2 C_2 s) + R_2 (1 + R_1 C_1 s)} = \frac{R_2 (1 + R_1 C_1 s)}{R_1 + R_2 + R_1 R_2 C_1 s + R_1 R_2 C_2 s}$$

$$= \frac{R_2 (1 + R_1 C_1 s)}{(R_1 + R_2)\left[1 + \frac{R_1 R_2 C_1 s + R_1 R_2 C_2 s}{R_1 + R_2}\right]}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{R_2 (1 + R_1 C_1 s)}{(R_1 + R_2)\left[1 + \frac{R_1 R_2 C_1 + R_1 R_2 C_2}{R_1 + R_2}\right]^3}$$

$$\frac{E_0(s)}{E_i(s)} = K_g \frac{(1 + sT_1)}{(1 + sT_2)} \quad \text{where} \quad K_g = \frac{R_2}{R_1 + R_2};$$

$$T_1 = R_1 C_1; \quad T_2 = \left[1 + \frac{R_1 R_2 C_1 + R_1 R_2 C_2}{R_1 + R_2}\right]$$

Mathematical Modeling of Mechanical Systems

1. What are the basic elements of Mechanical rotational systems? Write its torque balance equations Nov/Dec 2015, May/ June 2015

   The basic elements of Mechanical rotational system are

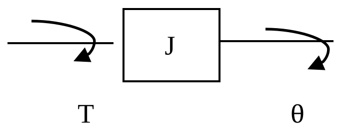
   (i) Moment of Inertia (J)

   (ii) Viscous fiction (B)

   (iii) Torsional stiffness (K)

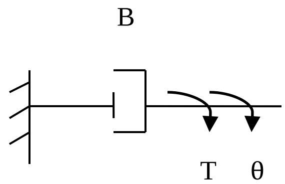
Torque Balance Equation

1. Moment of Inertia



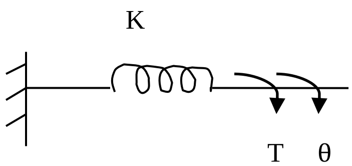
$$T_J(t) = J.\frac{d^2\theta}{dt^2}$$

2. Dashpot [one end is fixed]



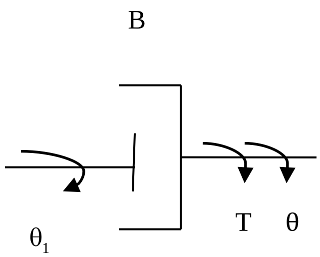
$$T_B(t) = B\frac{d\theta}{dt}$$

3. Torsional Spring [one end is fixed]



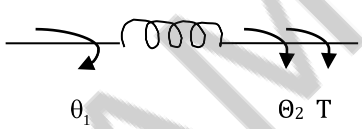
$$T_K(t) = K\theta$$

4. Dashpot [both ends are free]



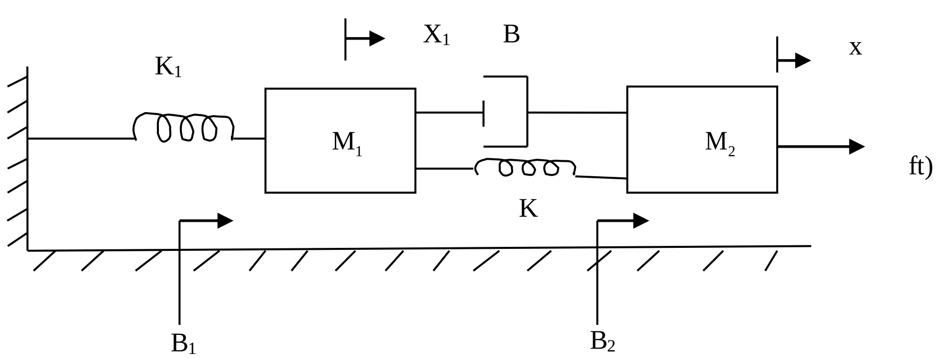
$$T_B(t) = B\frac{d}{dt}(\theta_1 - \theta_2)$$

5. Spring [both ends are free]



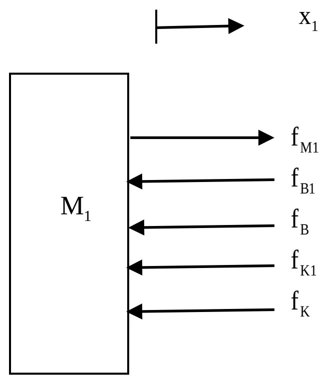
$$T_K(t) = K(\theta_1 - \theta_2)$$

2. Write the differential equations governing the mechanical system and determine the transfer function **(May/June 2016) (April/May 2019)**



Sol

Free body diagram for Mass 1



$$f_{M1} = M_1 \frac{d^2x_1}{dt^2} \qquad f_{K1} = K_1X_1$$

$$f_{B1} = B_1.\frac{dx_1}{dt} \qquad f_K = K(x_1 - x)$$

$$f_B = B.\frac{d}{dt}(x_1 - x)$$

By Newton's second law,

Σ applied force = Σ opposing force

$$f(t) = f_{M2} + f_{B2} + f_B + f_K$$

$$f(t) = M_2\frac{d^2x}{dt^2} + B_2\frac{dx}{dt} + B.\frac{d}{dt}(x - x_1) + K(x - x_1) \rightarrow (3)$$

On taking Laplace transform,

$$H_2s^2X(s) + B_2sX(s) + Bs[X(s) - X_1(s)]$$
$$+K[X(s) - X_1(s)] \;=\; F(s)$$
$$X(s) = [M_2s^2 + (B_2 + B)s + k] - X_1(s)[Bs + K] = F(s) \quad (4)$$

Substituting eqn (2) in eqn (4)

$$X(s) = \left[M_2s^2 + (B_2 + B)s + K\right] - X(s)\frac{(Bs + K)^2}{M_1s^2 + (B_1 + B)s + (K + K_1)}$$

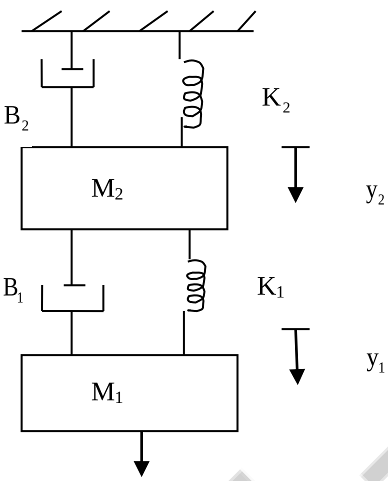
$$X(s)\left[\frac{\left[M_1s^2 + (B_1 + B)s + (K_1 + K)\right]\left[M_2s^2 + (B_2 + B)s + K\right] - (Bs + K)^2}{M_1s^2 + (B_1 + B)s + (K_1 + K)}\right] = F(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{M_1 s^2 + (B_1 + B)s + (K_1 + K)}{[M_1 s^2 + (B_1 + B)s + (K_1 + K)][M_2 s^2 + (B_2 + B)s + K] - (Bs + K)^2} \rightarrow (5)$$

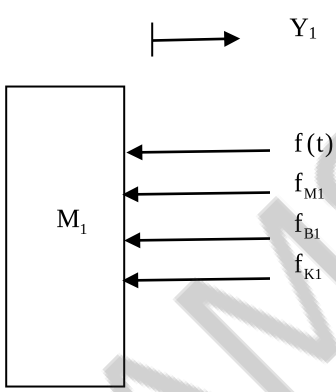
Eqn (1) and (3) forms the Mathematical model/differential equation of the given mechanical system equation (V) is the required transfer function

3.For the mechanical translational system shown in fig. determine the differential equation and obtain the transfer function $\frac{Y_2(s)}{F(s)}$

Nov/Dec 2019



Sol: Free body diagram for M_1



$$f_{M1} = M_1 \frac{d^2 y_1}{dt^2}$$

$$f_{K1} = K_1 (y_1 - y_2)$$

$$f_{B1} = B_1 \frac{d}{dt}(y_1 - y_2)$$

By Newton's second law,

$$f(t) = f_{M1} + f_{B1} + f_{K1}$$

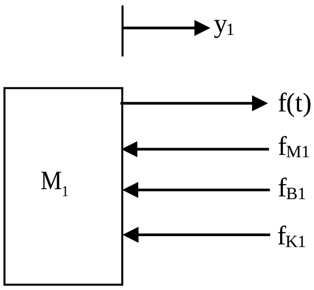
$$f(t) = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{d}{dt}(y_1 - y_2) + K_1 (y_1 - y_2) \rightarrow (1)$$

Taking Laplace transform

$$F(s) = M_1 s^2 Y_1(s) + B_1 s\left[Y_1(s) - Y_2(s)\right] + K_1\left[Y_1(s) - Y_2(s)\right]$$

$$F(s) = \left[M_1 s^2 + B_1 s + K_1\right] Y_1(s) - \left[B_1 s + K_1\right] Y_2(s) \qquad \rightarrow (2)$$

Free body diagram for M_2



$$f_{M2} = M_2 . \frac{d^2 y_2}{dt^2} \qquad f_{K1} = K_1 (y_2 - y_1)$$

$$f_{B1} = B_1 . \frac{d}{dt}(y_2 - y_1) \qquad f_K = K_2 (y_2)$$

$$f_B = B_2 . \frac{dy_2}{dt}$$

By Newton's second Law,

$$f_{M2} + f_{B1} + f_B + f_{K1} + f_K = 0$$

$$M_2 \frac{d^2 y_2}{dt^2} + B_1 \frac{d}{dt}(y_2 - y_1) + B\frac{dy_2}{dt} + K_1 (y_2 - y_1) + K_2 y_2 = 0 \qquad \rightarrow (3)$$

Taking Laplace transform

$$M_2 s^2 Y_2(s) + B_1 s\left[Y_2(s) - Y_1(s)\right] + BsY_2(s) + K_1\left[Y_2(s) - Y_1(s)\right] + K_2 Y_2(s) = 0$$

$$\left[M_2 s^2 + B_1 s + Bs + K_1 + K_2 Y_2(s)\right] - \left[B_1 s + K_1\right] Y_1(s) = 0$$

$$\left[M_2 s^2 + (B_1 + B)s + (K_1 + K_2)\right] Y_2(s) = \left[B_1 s + K_1\right] Y_1(s)$$

$$Y_1(s) = \frac{M_2 s^2 + (B_1 + B)s + (K_1 + K_2)}{(B_1 s + K_1)} Y_2(s) \qquad \rightarrow (4)$$

Substituting equation (4) in equation (2)

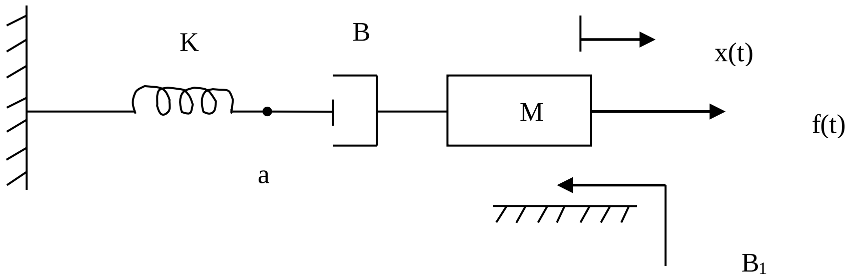
$$F(s) = \frac{(M_1 s^2 + B_1 s + K_1)\left[M_2 s^2 + (B_1 + B)s + (K_1 + K_2)\right] Y_2(s)}{(B_1 s + K_1)} - \left[B_1 s + K_1\right] Y_2(s)$$

$$F(s) = \frac{(M_1 s^2 B_1 s + K_1)\left[M_2 s^2 + (B_1 + B)s + (K_1 + K_2)\right] - \left[B_1 s + K_1\right]^2}{(B_1 s + K_1)} Y_2(s)$$

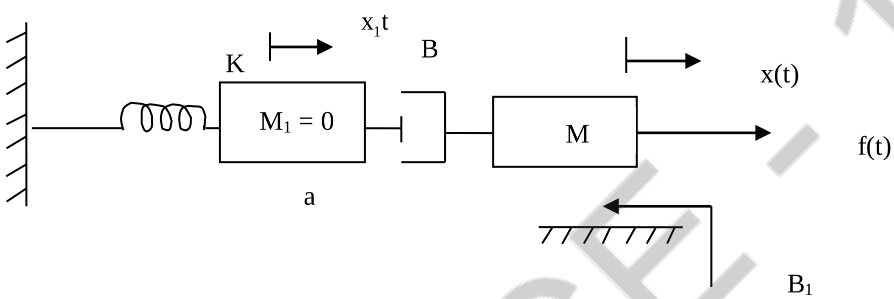
$$\frac{Y_2(s)}{F(s)} = \frac{B_1 s + K_1}{\left[M_1 s^2 + B_1 s + K_1\right]\left[M_2 s^2 + (B_1 + B)s + (K_1 + K_2)\right] - \left[B_1 s + K_1\right]^2} \rightarrow (5)$$

Eqn (1) and (3) forms the mathematical model/ differential equations of the given system eqn (5) gives the required transfer function of the given mechanical system

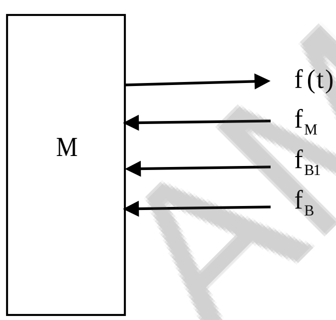
4. Determine the transfer function of the system shown in fiq.



Sol: Given system can be redrawn as follows



Free body diagram for Mass M



$$f_M = M\frac{d^2x}{dt^2}$$

$$f_{B1} = B_1\frac{dx}{dt}$$

$$f_B = B.\frac{d}{dt}(x - x_1)$$

By Newton's second law

$$f(t) = f_M + f_{B1} + f_B$$

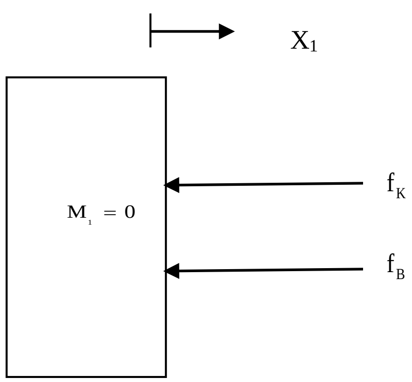
$$M\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + B.\frac{d}{dt}(x - x_1) = f(t) \qquad \rightarrow (1)$$

On taking Laplace transform

$$Ms^2X(s) + B_1sX(s) + Bs[X(s) - X_1(s)] = F(s)$$

$$[Ms^2 + (B_1 + B)s]X(s) - BsX_1(s) = F(s) \qquad \rightarrow (2)$$

Free body diagram for $M_1 = 0$



$$f_K = Kx_1$$

$$f_B = B.\frac{d}{dt}(x_1 - x)$$

By Newton's second law,

$$f_B + f_K = 0$$

$$\therefore B\frac{d}{dt}(x_1 - x) + Kx_1 = 0 \qquad \rightarrow (3)$$

On taking Laplace transform

$$Bs[X_1(s) - X(s)] + KX_1(s) = 0$$

$$[Bs + K]X_1(s) + BsX(s) = 0$$

$$X_1(s)[Bs + K] = Bs.X(s)$$

$$X_1(s) = \frac{Bs}{(Bs + K)}.X(s) \qquad \rightarrow (4)$$

On substituting eqn. (4) in eqn (2)

$$[Ms^2 + (B_1 + B)s]X(s) - \frac{(Bs)^2}{Bs + K}X(s) = F(s)$$

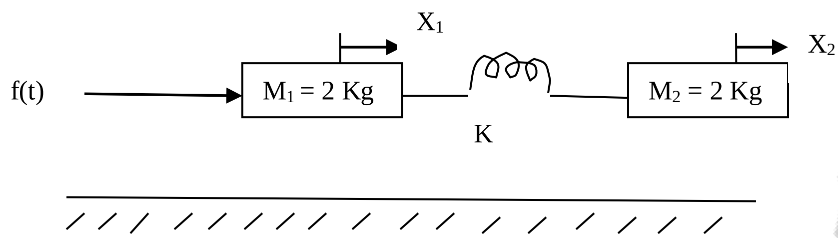
$$\left[\frac{\left[Ms^2+(B_1+B)s\right]\left[Bs+K\right]-\left(Bs\right)^2}{Bs+K}\right]X(s)=F(s)$$

$$\frac{X(s)}{F(s)}=\frac{Bs+K}{\left[Ms^2+\left(B_1+B\right)s\right]\left[Bs+K\right]-\left(Bs\right)^2} \qquad \rightarrow (5)$$

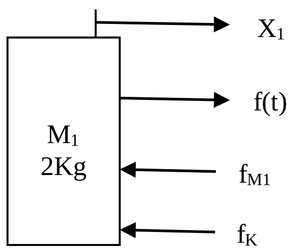
Equation (1) and (3) are differential equations governing the given system

Equation (5) is the required transfer function of the given mechanical translational system

5.Derive the transfer function of system shown in fig May/June 2015



Freebody diagram for M_1



$$f_{M1}=M_1\frac{d^2x_1}{dt^2}$$
$$f_K=K\left(x_1-x_2\right)$$

By Newton's second law,

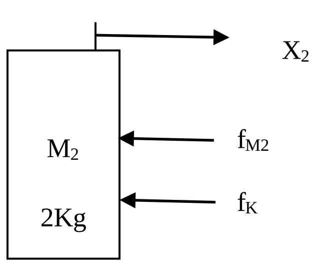
$$f(t)=f_{M1}+f_K$$
$$f(t)=M_1\frac{d^2x_1}{dt^2}+K\left(x_1-x_2\right) \qquad \rightarrow (1)$$

On taking Laplace transform,

$$F(s)=M_1s^2X_1(s)+K\left(X_1(s)-X_2(s)\right)$$

$$F(s)=\left(M_1s^2+K\right)X_1(s)-KX_2(s) \qquad \rightarrow (2)$$

Free body diagram for M_2



$$f_{M2} = M_2 . \frac{d^2x_2}{dt^2}$$

$$f_K = K\left(x_2 - x_1\right)$$

By Newton' second law

$$f_{M2} + f_K = 0$$

$$M_2 \frac{d^2x_2}{dt^2} + K\left(x_2 - x_1\right) = 0 \qquad \rightarrow (3)$$

On taking Laplace transform

$$M_2 s^2 X_2(s) + K\left[X_2(s) - X_1(s)\right] = 0$$

$$\left(M_2 s^2 + K\right) X_2(s) - KX_1(s) = 0$$

$$\left(M_2 s^2 + K\right) X_2(s) = KX_1(s)$$

$$X_2(s) = \frac{K}{M_2 s^2 + K} X_1(s) \qquad \rightarrow (4)$$

$$X_1(s) = \frac{M_2 s^2 + K}{K} X_2(s) \qquad \rightarrow (5)$$

On substituting eqn (4) in (2)

$$F(s) = \left(M_1 s^2 + K\right) X_1(s) - \frac{K^2}{M_2 s^2 + K} X_1(s)$$

$$F(s) = \frac{\left(M_1 s^2 K\right)\left(M_2 s^2 + K\right) - K^2}{M_2 s^2 + K} X_1(s)$$

$$\frac{X_1(s)}{F(s)} = \frac{M_2 s^2 + K}{\left(M_1 s^2 + K\right)\left(M_2 s^2 + K\right) - K^2} \qquad \rightarrow (6)$$

Put $M_1 = 2$; $M_2 = 2$ in eqn (6)

$$\frac{X_1(s)}{F(s)} = \frac{2s^2}{\left(2s^2 + K\right)\left(2s^2 + k\right) - K^2}$$

$$\frac{X_1(s)}{F(s)} = \frac{2s^2}{\left(2s^2 + K\right)^2 - K^2}$$

$$= \frac{2s^2}{4s^4 + K^2 + 4Ks^2 - K^2}$$

$$= \frac{2s^2}{4s^4 + 4Ks^2} = \frac{s^2}{2\left(s^4 + Ks^2\right)}$$

$$= \frac{s^2}{2s^2\left(s^2 + K\right)} = \frac{1}{2\left(s^2 + K\right)}$$

$$\frac{X_1(s)}{F(s)} = \frac{1}{2\left(s^2 + K\right)} \qquad \rightarrow (7)$$

On substituting eqn (5) in eqnn (2)

$$F(s) = \frac{(M_1 s^2 + K)(M_2 s^2 + K)}{K} X_2(s) - K X_2(s)$$

$$F(s) = \left[\frac{(M_1 s^2 + K)(M_2 s^2 + K) - K^2}{K}\right] X_2(s)$$

$$\frac{X_2(s)}{F(s)} = \frac{K}{\left[\left(M_1 s^2 + K\right)\left(M_2 s^2 + K\right) - K^2\right]} \qquad \rightarrow (8)$$

put $M_1 = 2,\ M_2 = 2$

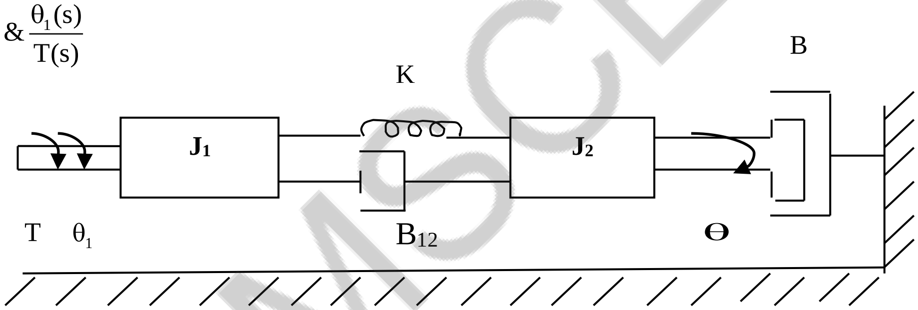
$$\therefore \frac{X_2(s)}{F(s)} = \frac{K}{\left[\left(2s^2 + K\right)^2 - K^2\right]} = \frac{K}{2\left(s^4 + Ks^2\right)}$$

$$\therefore \frac{X_2(s)}{F(s)} = \frac{K}{2s^2\left(s^2 + K\right)} \qquad \rightarrow (9)$$

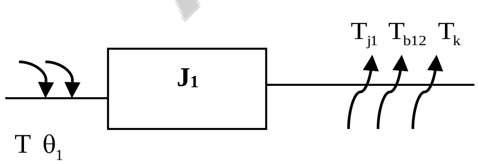
Eqn (7) & (9) are the required transfer function.

6.For the mechanical rotational system shown in fig determine the transfer function.

$$\frac{\theta(s)}{T(s)} \ \& \ \frac{\theta_1(s)}{T(s)}$$



Sol : Free body diagram for mass with moment of intertia J_1



$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2};\ T_{b12} = B_{12}\frac{d}{dt}(\theta_1 - \theta);\ T_K = K(\theta_1 - \theta)$$

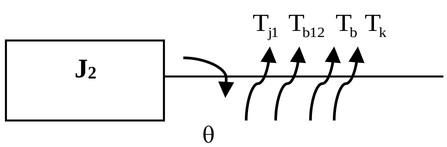
By Newton's second law, $T_{j1} + T_{b12} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_{12}.\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T \qquad (1)$$

On taking Laplace transform,

$$J_1 s^2 \theta_1(s) + sB_{12}\left[\theta_1(s) - \theta(s)\right] + K\left[\theta_1(s) - \theta(s)\right] = T(s)$$
$$\theta_1(s)\left[J_1 s^2 + sB_{12} + K\right] - \theta(s)\left[sB_{12} + K\right] = T(s) \qquad (2)$$

Free body diagram of mass with moment if intertia J_2



$$T_{J2} = J_2 \frac{d^2\theta}{dt^2};\ T_{b12} = B_{12}\frac{d}{dt}(\theta - \theta_1)$$
$$T_b = B\frac{d\theta}{dt};\ T_K = K(\theta - \theta_1)$$

By Newton's second law

$$T_{j2} + T_{b12} + T_b + T_k = 0$$

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} + B_{12}\frac{d}{dt}(\theta - \theta_1) + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0 \qquad \rightarrow (3)$$

On taking Laplace transform,

$$J_2 s^2 \theta(s) + B_{12} s\left[\theta(s) - \theta_1(s)\right] + Bs\left[\theta(s)\right] + K\left[\theta(s) - \theta_1(s)\right] = 0$$
$$\theta(s)\left[J_2 s^2 + s(B_{12} + B) + K\right] - \theta_1(s)\left[B_{12}s + K\right] = 0$$
$$\theta_1(s) = \left[\frac{J_2 s^2 + s(B_{12} + B) + K}{B_{12}s + K}\right]\theta(s) \qquad (4)$$
$$\theta(s) = \left[\frac{B_{12}s + K}{J_2 s^2 + s(B_{12} + B) + K}\right]\theta_1(s) \qquad (5)$$

Substituting equation (4) in eqn (2)

$$\frac{\left[J_1s^2+B_{12}s+K\right]\left[J_2s^2+s(B_{12}+B)+K\right]}{(B_{12}s+K)}\theta(s)-\left[B_{12}s+K\right]\theta(s)=T(s)$$

$$\frac{\left[J_1s^2+B_{12}s+K\right]\left[J_2s^2+s(B_{12}+B)+K\right]-(B_{12}s+K)^2}{B_{12}s+K}\theta(s)=T(s)$$

$$\boxed{\frac{\theta(s)}{T(s)}=\frac{B_{12}s+K}{\left[J_1s^2+B_{12}s+K\right]\left[J_2s^2+s(B_{12}+B)+K\right]-\left(B_{12}s+K\right)^2}} \qquad (6)$$

Substituting eqn (5) in eqn (2)

$$\theta_1(s)\left[J_1s^2+sB_{12}+K\right]-\frac{(B_{12}s+K)^2}{J_2s^2+s(B_{12}+B)+K}\theta_1(s)=T(s)$$

$$\theta_1(s)\frac{\left[(J_1s^2+B_{12}s+K)(J_2s^2+s(B_{12}+B)+K)-(B_{12}s+K)^2\right]}{J_2s^2+s(B_{12}+B)+K}=T(s)$$

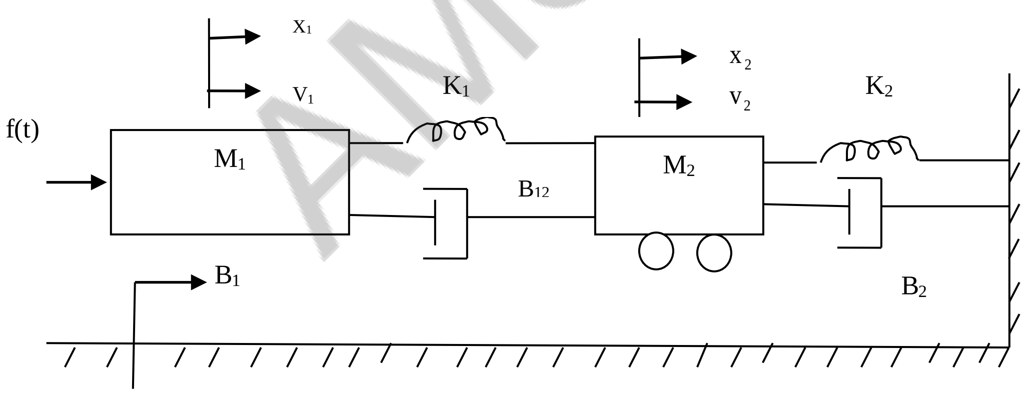
$$\boxed{\frac{\theta_1(s)}{T(s)}=\frac{J_2s^2+s(B_{12}+B)+K}{(J_1s^2+B_{12}s+K)(J_2s^2+s(B_{12}+B)+K)-(B_{12}s+K)^2}} \qquad (7)$$

The equation (1) and (3) are called differential equations of the given mechanical rotational systems.

The equation (6) and (7) are the required transfer functions of the given mechanical rotational systems.

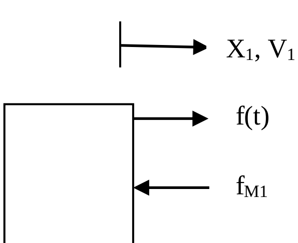
Analogous Systems

7.Write the differential equations governing the mechanical system shown in fig. Draw the force voltage and force current electrical analogous circuits and verify by writing mech and node equations.



Solution:

Freebody diagram for mass M_1



$$f_{M1} = M_1 \frac{d^2x_1}{dt^2}; \quad f_{K1} = K_1(x_1 - x_2)$$

$$f_{B1} = B_1 \frac{dx_1}{dt}$$

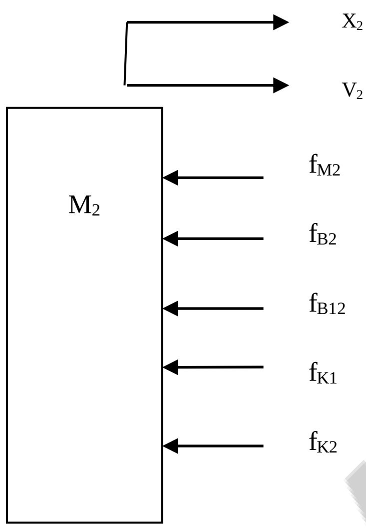
$$f_{B12} = B_{12} \frac{d}{dt}(x_1 - x_2)$$

By Newtons second law,

$$f_{M1} + f_{B1} + f_{B12} + f_{K1} = f(t)$$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = f(t) \qquad \rightarrow (1)$$

Free body diagram for Mass M_2



$$f_{M2} = M_2 \frac{d^2x_2}{dt^2}$$

$$f_{B2} = B_2 \frac{dx_2}{dt}$$

$$f_{B12} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

$$f_{K1} = K_1(x_2 - x_1)$$

$$f_{K2} = K_2 x_2$$

By Newtons second law,

$$f_{M2} + f_{B2} + f_{K2} + f_{B12} + f_{K1} = 0$$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + B_{12} \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) = 0 \qquad \rightarrow (2)$$

On replacing the displacements by velocity in differential equations (1) and (2) of the mechanical system

$$\left[(\text{i.e.,})v, \frac{d^2x}{dt^2} = \frac{dv}{dt}; \frac{dx}{dt} = v; x = \int v\,dt\right]$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B_{12}(v_1 - v_2) + K_1 \int (v_1 - v_2)\,dt = f(t) \qquad \rightarrow (3)$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + K_2 \int v_2\,dt + B_{12}(v_2 - v_1) + K_1 \int (v_2 - v_1)\,dt = 0 \qquad \rightarrow (4)$$

FORCE VOLTAGE ANALOGOUS CIRCUIT

The electrical analogous elements for mechanical system are given below

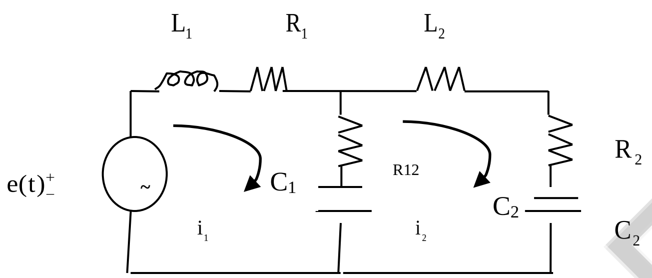
$f(t) \rightarrow e(t);\ v_1 = i;\ v_2 = i_2;$

$M_1 \rightarrow L_1 \quad B_1 \rightarrow R_1 \quad K_1 = 1/C_1$

$M_2 \rightarrow L_2 \quad B_2 \rightarrow R_2 \quad K_2 = 1/C_2$

$B_{12} \rightarrow R_{12}$

Force voltage electrical analogous circuits is shown below



Applying KVL to mesh 1

$$L_1 \frac{di_1}{dt} + R_1 i_1 + R_{12}(i_1 - i_2) + \frac{1}{c_1}\int (i_1 - i_2)\,dt = e(t) \qquad (5)$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{c_2}\int i_2\,dt + R_{12}(i_2 - i_1) + \frac{1}{c_1}\int (i_2 - i_1)\,dt = 0 \quad (6)$$

It is observed that the mesh basis equations

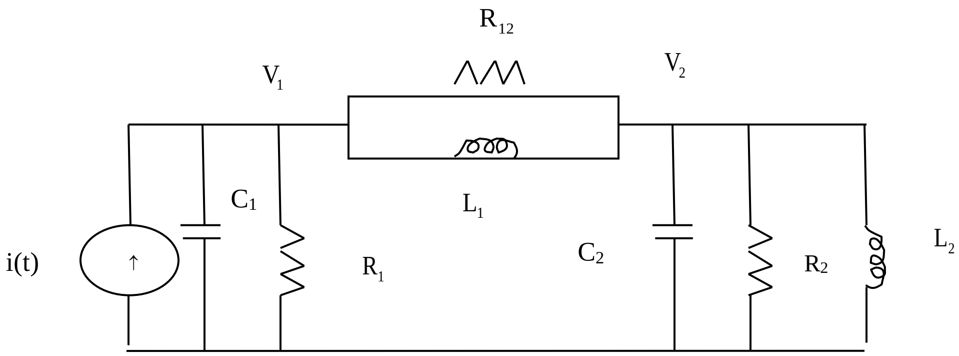
Eqn (5) and (6) ar similar to the differential equations

Eqn (3) and (4) governing the mechanical system

FORCE CURRENT ANALOGOUS CIRCUIT

The electrical analogous elements for the elements of mechanical system

$$f(t) = i(t) \quad M_1 \to C_1 \quad B_1 \to 1/R_1 \quad K_1 \to 1/L_1$$
$$v_1 = V_1 \quad M_2 \to C_2 \quad B_2 \to 1/R_2 \quad K_2 \to 1/L_2$$
$$v_2 = V_2 \quad B_{12} \to 1/R_{12}$$



Force-current electrical analogous circuit

Applying KCL at node 1

$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1} V_1 + \frac{1}{R_{12}}(V_1 - V_2) + \frac{1}{L}\int (V_1 - V_2)\,dt = i(t) \qquad \to (7)$$

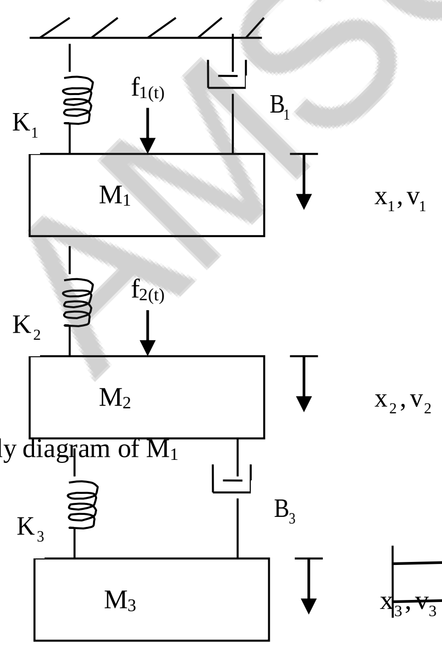
$$C_2 \frac{dV_2}{dt} + \frac{1}{R_2} V_2 + \frac{1}{L_2}\int V_2\,dt + \frac{1}{R_{12}}(V_2 - V_1) + \frac{1}{L}\int (V_2 - V_1)\,dt = 0 \qquad \to (8)$$

It is observed that the node basis equations

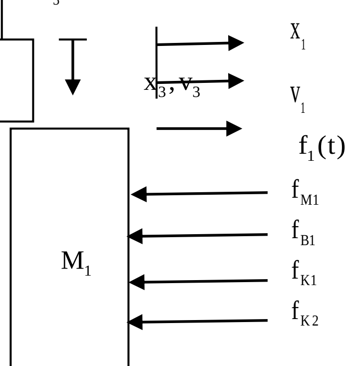
Eqn (7) and (8) are similar to the differential equations

Eqn (3) and (4) governing the mechanical system

8. Write the differential equations the mechanical system shown in fig.Draw the force – voltage and force current electrical analogous circuit and verify by writing mesh and node equations.   NOV/DEC   2015



Sol: Freebody diagram of M_1

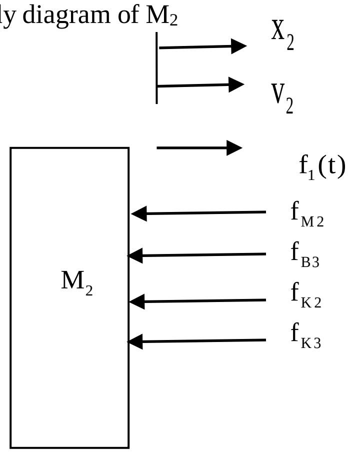


$$f_{M1} = M_1 . \frac{d^2 x_1}{dt^2} \qquad f_{B1} = B_1 . \frac{dx_1}{dt}$$
$$f_{K2} = K_2 \left(x_1 - x_2 \right); \; f_{K1} = K_1 x_1$$

By newton's second law,

$$f_{M1} + f_{B1} + f_{K1} + f_{K2} = f_1(t)$$
$$M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_2 \left(x_1 - x_2 \right) + K_1 x_1 = f_1(t) \qquad \rightarrow (1)$$

Free body diagram of M_2

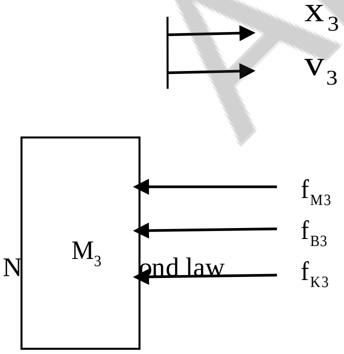


$$f_{M2} = M_2 . \frac{d^2 x_2}{dt^2} \qquad f_{B3} = B_3 . \frac{d}{dt} \left(x_2 - x_3 \right)$$
$$f_{K2} = K_2 \left(x_2 - x_1 \right); \; f_{K3} = K_3 \left(x_2 - x_3 \right)$$

By Newtons second law,

$$f_{M2} + f_{B3} + f_{K2} + f_{K3} = f_2(t)$$
$$M_2 \frac{d^2 x_2}{dt^2} + B_3 \frac{d}{dt} \left(x_2 - x_3 \right) + K_2 \left(x_2 - x_1 \right) + K_3 \left(x_2 - x_3 \right) = f_2(t) \qquad \rightarrow (2)$$

Free body diagram for M_3



$$f_{M3} = M_3 \frac{d^2 x_3}{dt^2}$$
$$f_{B3} = B_3 \frac{d}{dt} \left(x_3 - x_2 \right)$$
$$f_{K3} = K_3 \left(x_3 - x_2 \right)$$

By N ond law

$$f_{M3} + f_{B3} + f_{K3} = 0$$

$$M_3 \frac{d^2x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_2) + K_3(x_3 - x_2) = 0 \rightarrow (3)$$

On replacing the displacements by velocity in the differential equation (1) and (2) and (3) governing the mechanical system

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + K_1 \int v_1 dt + K_2 \int (v_1 - v_2) dt = f_1(t) \qquad \rightarrow (4)$$

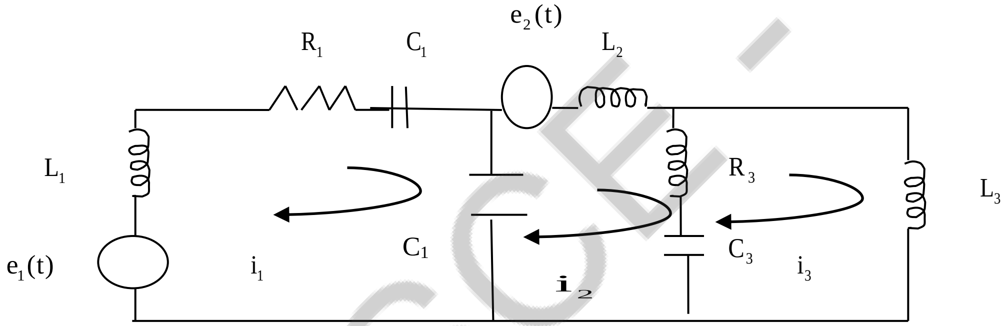
$$M_2 \frac{dv_2}{dt} + B_3(v_2 - v_3) + K_2 \int (v_2 - v_1) dt + K_3 \int (v_2 - v_3) dt = f_2(t) \qquad \rightarrow (5)$$

$$M_3 \frac{dv_3}{dt} + B_3(v_3 - v_2) + K_3 \int (v_3 - v_2) dt = 0 \qquad \rightarrow (6)$$

FORCE VOLTAGE ANALOGOUS CIRCUIT

The electrical analogous elements for the elements of mechanical system are given below.

$f_1(t) = e_1(t)$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$f_2(t) = e_2(t)$	$M_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$v_1 = i_1$	$M_3 \rightarrow L_3$		$K_3 \rightarrow 1/C_3$
$v_2 = i_2$			
$v_3 = i_3$			



Applying KVL to mesh (1), (2) and (3)

$$L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C_1}\int i_1dt+\frac{1}{C_2}\int(i_1-i_2)dt=e_1(t) \qquad (7)$$

$$L_2\frac{di_2}{dt}+R_3(i_2-i_3)+\frac{1}{C_3}\int(i_2-i_3)dt+\frac{1}{C_2}\int(i_2-i_1)dt=e_2(t) \quad (8)$$

$$L_3\frac{di_3}{dt}+R_3(i_3-i_2)+\frac{1}{C_3}\int(i_3-i_2)dt=0 \qquad (9)$$

It is observed that the mesh equations (7), (8) and (9) are similar to the differential equations (4), (5) and (6) governing the mechanical system

FORCE CURRENT ANALOGOUS CIRCUIT

The electrical analogous elements for the elements of mechanical system are given below

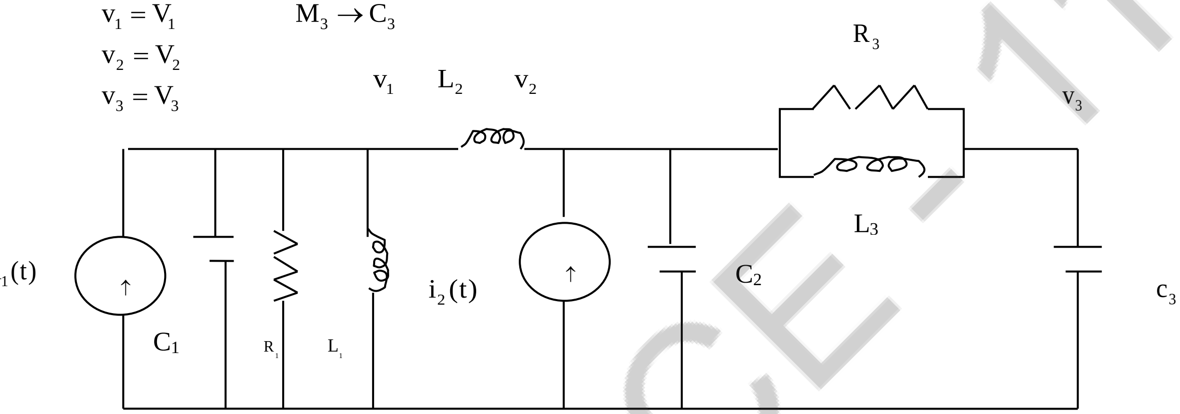
$f_1(t)=i_1(t)$ $\quad M_1\rightarrow C_1$ $\quad B_1\rightarrow 1/R_1$ $\quad K_1\rightarrow 1/L_1$

$f_2(t)=i_2(t)$ $\quad M_2\rightarrow C_2$ $\quad B_3\rightarrow 1/R_3$ $\quad K_2\rightarrow 1/L_2$

$v_1=V_1$ $\quad M_3\rightarrow C_3$

$v_2=V_2$

$v_3=V_3$



On applying KCL at node (1), (2) and (3)

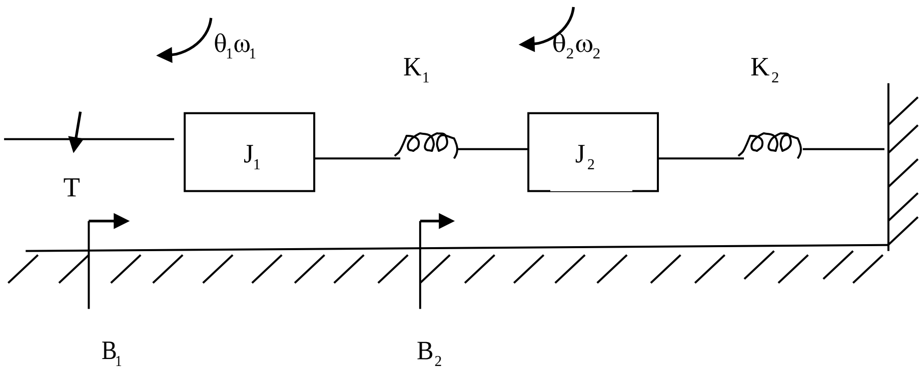
$$C_1\frac{dV_1}{dt}+\frac{1}{R_1}V_1+\frac{1}{L_1}\int V_1dt+\frac{1}{L_2}\int(V_1-V_2)dt=i_1(t) \qquad (10)$$

$$C_2\frac{dV_2}{dt}+\frac{1}{R_3}(V_2-V_3)+\frac{1}{L_3}\int(V_2-V_3)dt+\frac{1}{L_2}\int(V_2-V_1)dt=i_2(t) \quad (11)$$

$$C_3\frac{dV_3}{dt}+\frac{1}{R_3}(V_3-V_1)+\frac{1}{L_3}\int(V_3-V_1)dt=0 \qquad (12)$$

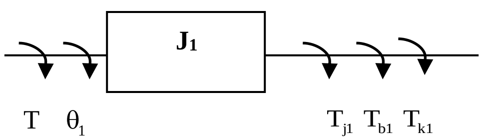
It is observed that node basis equations (10). (11) and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system

9. Write the differential equations governing the mechanical rotational system shown in fig. Draw the torque voltage and torque – current electrical analogous circuits and verifying by writing mesh and node equations.



Sol

Free body diagram of J_1



$$T_{ji} = J_1 \frac{d^2\theta_1}{dt^2}$$

$$T_{b1} = B_1 \frac{d\theta_1}{dt}$$

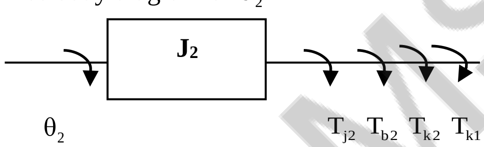
$$T_{k1} = K_1(\theta_1 - \theta_2)$$

By newtons second law

$$T_{j1} + T_{b1} + T_{k1} = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_2) = T \qquad (1)$$

Free body diagram of J_2



$$T_{j2} = J_2 \frac{d^2\theta_2}{dt^2}; \quad T_{b2} = B_2 \frac{d\theta_2}{dt}; \quad T_{k2} = K_2\theta_2; \; T_{k_1} = K_1(\theta_2 - \theta_1)$$

By Newtons second law,

$$T_{j2} + T_{b2} + T_{k2} + T_{k1} = 0$$

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_2\theta_2 + K_1(\theta_2 - \theta_1) \qquad (2)$$

On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system, we get

$$(\text{i.e } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt};\ \frac{d\theta}{dt} = \omega;\ \theta = \int \omega dt)$$

$$J_1 \frac{d\omega_1}{dt^2} + B_1\omega_1 + K_1\int (\omega_1 - \omega_2)\,dt = T \qquad \rightarrow (3)$$

$$J_2 \frac{d\omega_2}{dt^2} + B_2\omega_2 + K_2\int (\omega_2 dt) + K_1\int (\omega_2 - \omega_1)\,dt = 0 \quad \rightarrow (4)$$

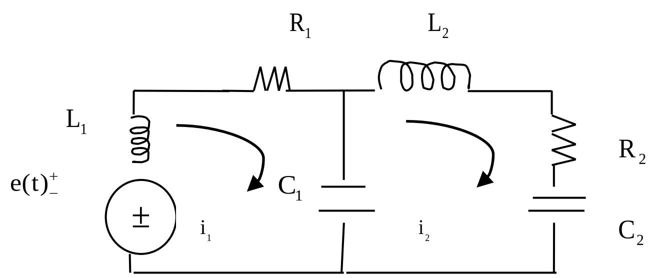
TORQUE – VOLTAGE ANALOGOUS CIRCUIT

The electrical analogous elements for the elements if mechanical rotational system are given below

$T \rightarrow e(t)$ $\quad J_1 \rightarrow L_1\ \ B_1 \rightarrow R_1 \quad K_1 \rightarrow 1/C_1$

$\omega_1 \rightarrow i_1$ $\quad J_2 \rightarrow L_2\ B_2 \rightarrow R_2 \quad K_2 \rightarrow 1/C_2$

$\omega_2 \rightarrow i_2$



Applying KVL to mesh (1) and (2)

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int (i_1 - i_2) = e(t) \qquad (5)$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2 dt + \frac{1}{C_1}\int (i_2 - i_1)dt = 0 \qquad (6)$$

It is observed that mesh basis equations (5) and (6) are similar to differential equations (3) and (4) governing the mechanical system.

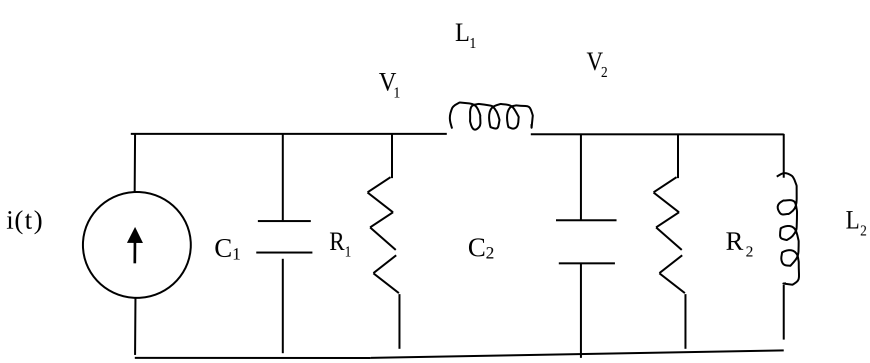
TORQUE CURRENT ANALOGOUS CIRCUIT

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)\ J_1 \rightarrow C_1\ \ B_1 \rightarrow 1/R_1 \quad K_1 \rightarrow 1/L_1$

$\omega_1 \rightarrow V_1\ J_2 \rightarrow C_2\ B_2 \rightarrow 1/R_2 \quad K_2 \rightarrow 1/L_2$

$\omega_2 \rightarrow V_2$



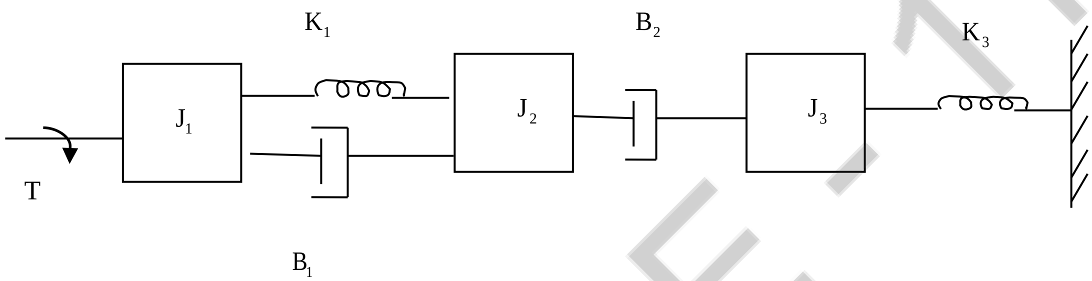
Applying KCL at node (1) and (2)

$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1} V_1 + \frac{1}{L_1} \int (V_1 - V_2) dt = i(t)$$

$$C_2 \frac{dV_2}{dt} + \frac{1}{R_2} V_2 + \frac{1}{L_2} \int V_2 dt + \frac{1}{L_1} \int (V_2 - V_1) dt = 0 \quad (8)$$

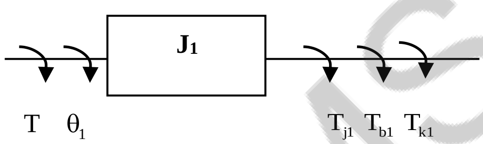
It is observed that the node basis equations (7) and (8) are similar to the differential equations (3) and (4) governing the mechanical system.

10.Write the differential equations governing the mechanical rotational system shown in fig. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing mesh and node equations.



Solution

Freebody diagram of J_1



$$T_{ji} = J_1 \frac{d^2\theta_1}{dt^2}$$

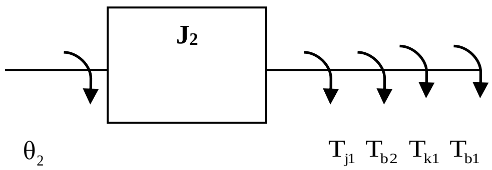
$$T_{b1} = B_1 \frac{d(\theta_1 - \theta_2)}{dt}$$

$$T_{k1} = K_1 (\theta_1 - \theta_2)$$

By newtons second law, $T_{j1} + T_{b1} + T_{k1} = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d(\theta_1 - \theta_2)}{dt} + K_1(\theta_1 - \theta_2) = T \quad (1)$$

Freebody diagram of J_2

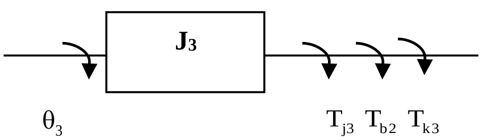


$$T_{j2} = J_2 \frac{d^2\theta}{dt^2}$$
$$T_{b2} = B_2 \frac{d(\theta_2 - \theta_3)}{dt}$$
$$T_{k1} = K_1(\theta_2 - \theta_1)$$
$$T_{b1} = B_1 \frac{d(\theta_2 - \theta_1)}{dt}$$

By newtons second law,

$$T_{j2} + T_{b2} + T_{b1} + T_{k1} = 0$$
$$J_2 \frac{d^2\theta}{dt^2} + B_2 \frac{d}{dt}(\theta_2 - \theta_3) + B_1 \frac{d(\theta_2 - \theta_1)}{dt} + K_1(\theta_2 - \theta_1) = 0 \qquad (2)$$

Free body diagram of J_3



$$T_{j3} = J_3 \frac{d^2\theta_3}{dt^2}$$
$$T_{b2} = B_2 \frac{d(\theta_3 - \theta_2)}{dt}$$
$$T_{k3} = K_3\theta_3$$

By Newton's second law,

$$T_{j3} + T_{b2} + T_{k3} = 0$$
$$J_3 \frac{d^2\theta_3}{dt^2} + B_2 \frac{d(\theta_3 - \theta_2)}{dt} + K_3\theta_3 = 0 \qquad (3)$$

On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system, we get

$$(\text{i.e} \frac{d^2\theta}{dt^2}; \frac{d\omega}{dt} = \frac{d\theta}{dt} = \omega;\ \theta = \int \omega dt)$$

$$J_1 \frac{d\omega_1}{dt} + B_1(\omega_1 - \omega_2) + K_1 \int (\omega_1 - \omega_2) dt = T \qquad \rightarrow (4)$$
$$J_2 \frac{d\omega_2}{dt} + B_1(\omega_2 - \omega_1) + B_2(\omega_2 - \omega_3) + K_1 \int (\omega_2 - \omega_1) dt = 0 \qquad \rightarrow (5)$$
$$J_3 \frac{d\omega_3}{dt} + B_2(\omega_3 - \omega_2) + K_3 \int \omega_3 dt = 0 \qquad \rightarrow (6)$$

TORQUE – VOLTAGE ANALOGOUS CIRCUIT

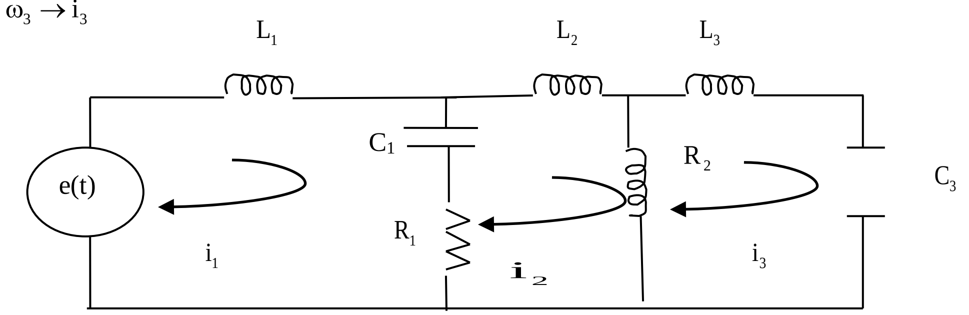
The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow e(t)$ $\quad J_1 \rightarrow L_1 \;\; B_1 \rightarrow R_1$ $\quad K_1 \rightarrow 1/C_1$

$\omega_1 \rightarrow i_1$ $\quad J_2 \rightarrow L_2 \; B_2 \rightarrow R_2$ $\quad K_3 \rightarrow 1/C_3$

$\omega_2 \rightarrow i_2$ $\quad J_3 \rightarrow L_3$

$\omega_3 \rightarrow i_3$



Applying KVL to mesh (1), (2) and (3)

$$L_1 \frac{di_1}{dt} + R_1(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)dt = e(t) \qquad (7)$$

$$L_2 \frac{di_2}{dt} + R_1(i_2 - i_1) + R_2\left(i_2 - i_3\right) + \frac{1}{C_2}\int (i_2 - i_1)dt = 0 \quad (8)$$

$$L_3 \frac{di_3}{dt} + R_2(i_3 - i_2) + \frac{1}{C_3}\int i_3 dt = 0 \qquad (9)$$

It is observed that mesh basis equation (7), (8) and (9) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

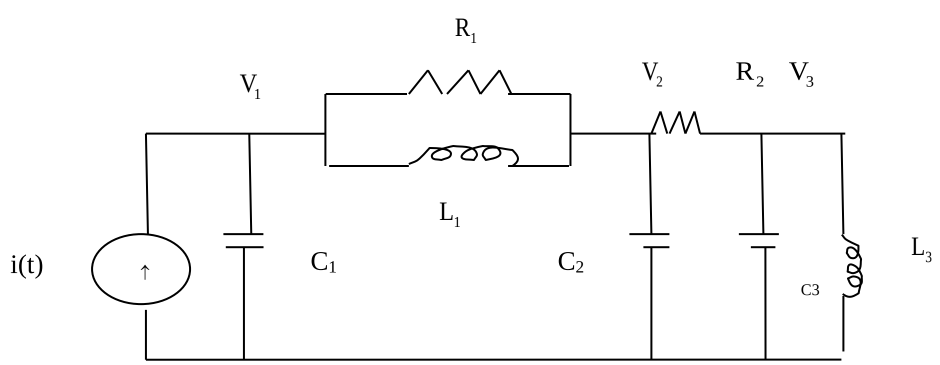
TORQUE – CURRENT ANALOGOUS CIRCUIT

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$ $\quad \omega_1 \rightarrow V_1$ $\quad J_1 \rightarrow C_1 \;\; B_1 \rightarrow 1/R_1$ $\quad K_1 \rightarrow 1/L_1$

$\omega_2 \rightarrow V_2$ $\quad J_2 \rightarrow C_2 \; B_2 \rightarrow 1/R_2$ $\quad K_2 \rightarrow 1/L_3$

$\omega_3 \rightarrow V_3$ $\quad J_3 \rightarrow C_3$



Applying KCL at node (1), (2) and (3)

$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1}\left(V_1 - V_2\right) + \frac{1}{L_1}\int \left(V_1 - V_2\right)dt = i(t) \qquad \rightarrow (10)$$

$$C_2 \frac{dV_2}{dt} + \frac{1}{R_1}\left(V_2 - V_1\right) + \frac{1}{R_2}\left(V_2 - V_3\right) + \frac{1}{L_1}\int \left(V_2 - V_1\right)dt = 0 \qquad \rightarrow (11)$$

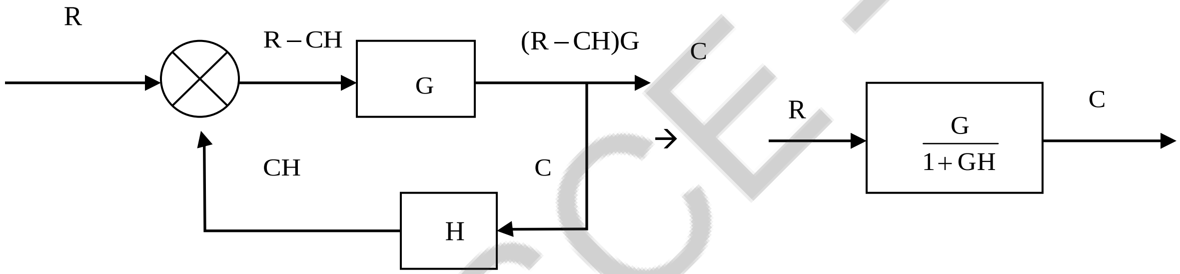
$$C_3 \frac{dV_3}{dt} + \frac{1}{R_2}\left(V_3 - V_2\right) + \frac{1}{L_3}\int V_3 dt = 0 \qquad \rightarrow (12)$$

It is observed that the node basis equations (10), (11), and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system

BLOCK DIAGRAMS

1. Write the rule for eliminating negative and positive feedback in block diagram reduction NOV/DEC 2015

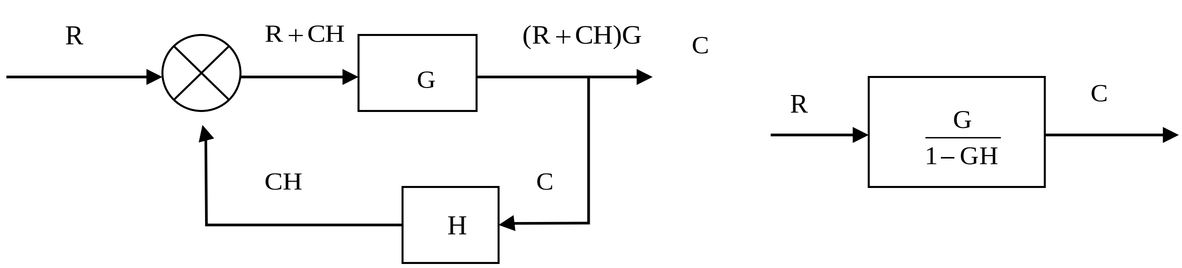
(A) Elimination of –ve feedback loop



Proof

$$C = (R - CH)G$$
$$C = RG - CHG$$
$$C + CHG = RG$$
$$C(1 + HG) = RG$$
$$\frac{C}{R} = \frac{G}{(1 + GH)}$$

(B) Elimination of positive feedback loop

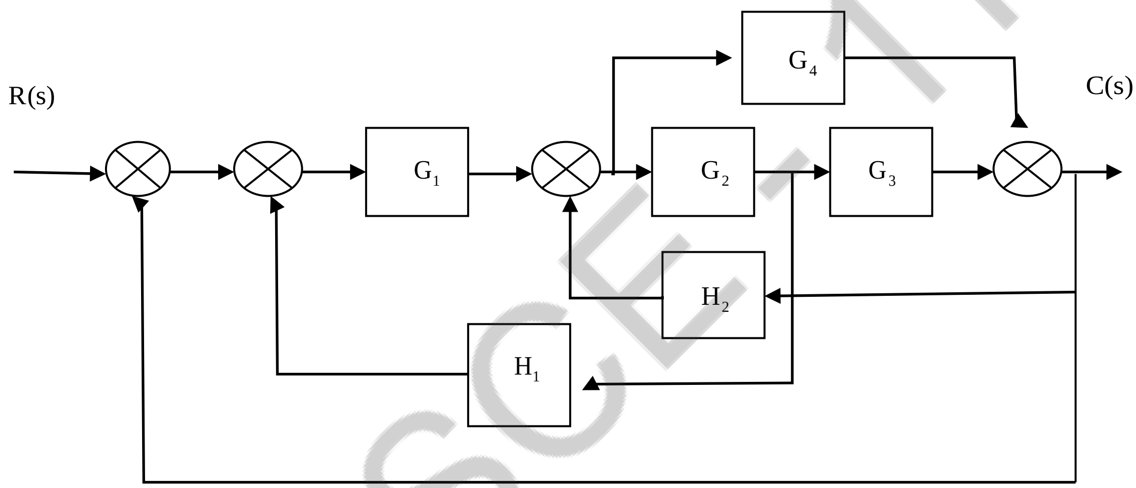


Proof

$$C = (R + CH)G$$
$$C = RG + CHG$$
$$C - CHG = RG$$
$$C(1 - GH) = RG$$
$$\frac{C}{R} = \frac{G}{1 - GH}$$

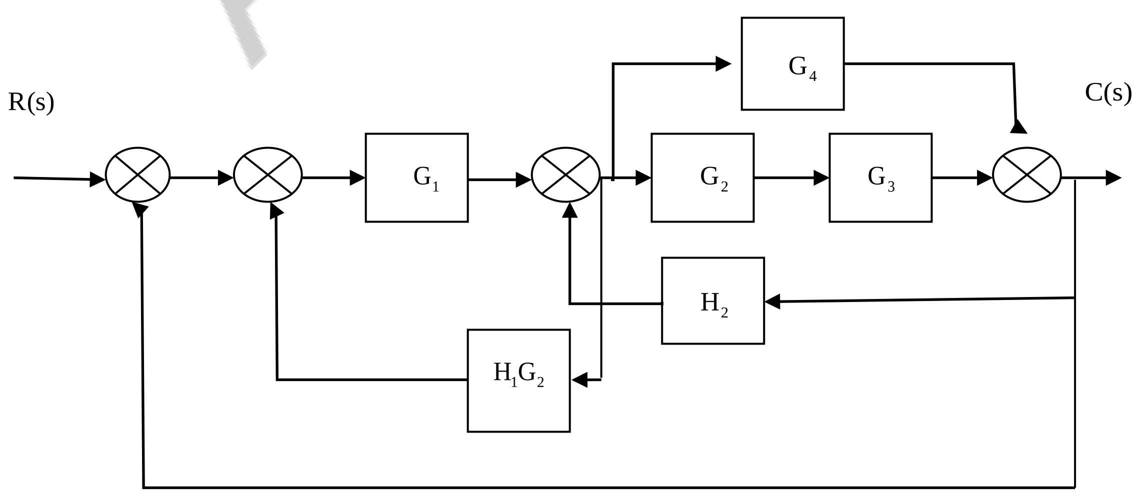
$$\frac{C(s)}{R(s)} = \frac{G_1G_3(G_2 + G_4)}{1 + G_3H_1 + G_1G_2G_3 + G_1G_3G_4}$$

2. Using the block diagram reduction technique. Find the closed loop transfer function of the system whose block diagram is shown in fig

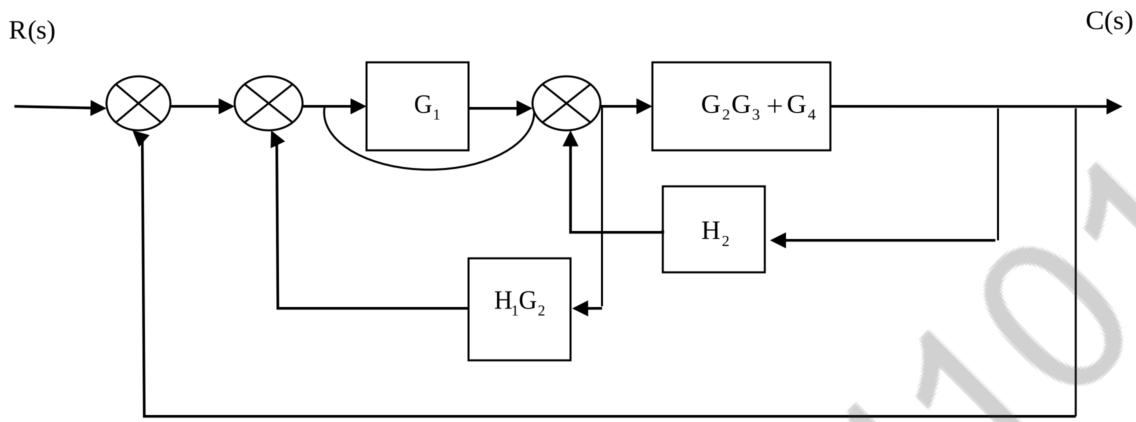


Sol

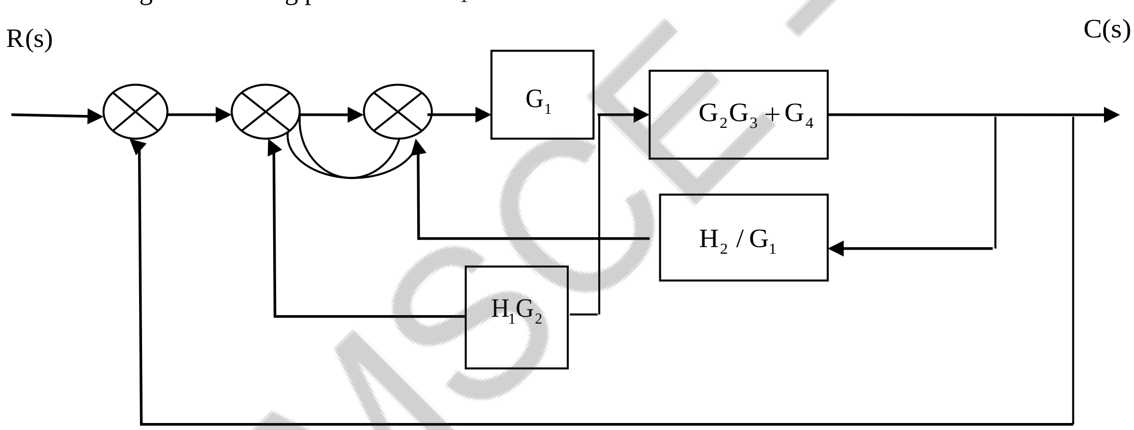
Moving the branch point before the block G_2



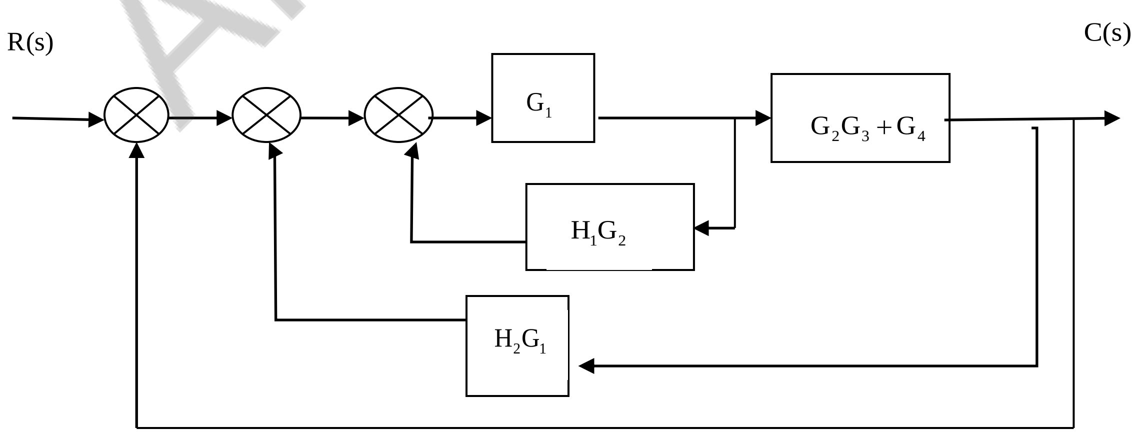
Combining the cascade blocks G_2 and G_3 parallel block G_1



Moving the summing point before G_1

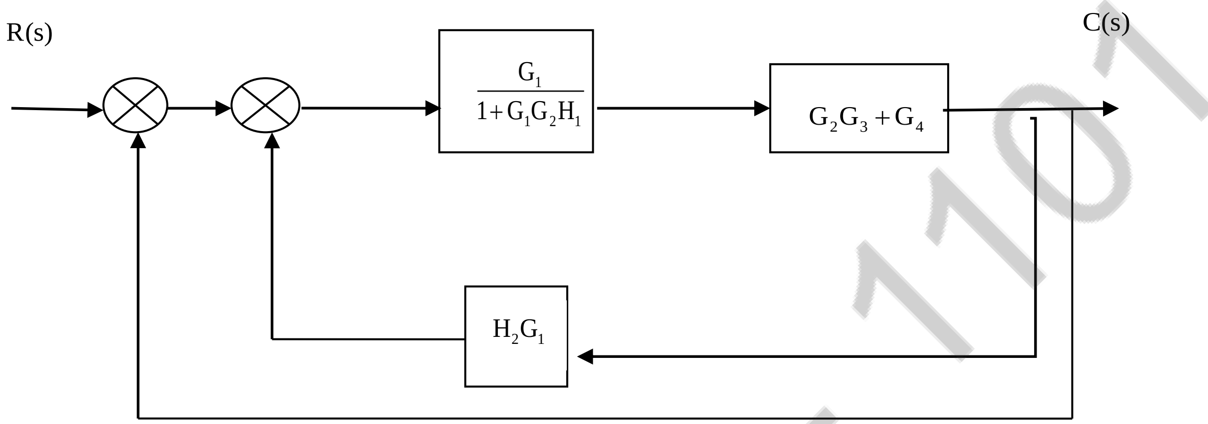


Interchanging the summing points

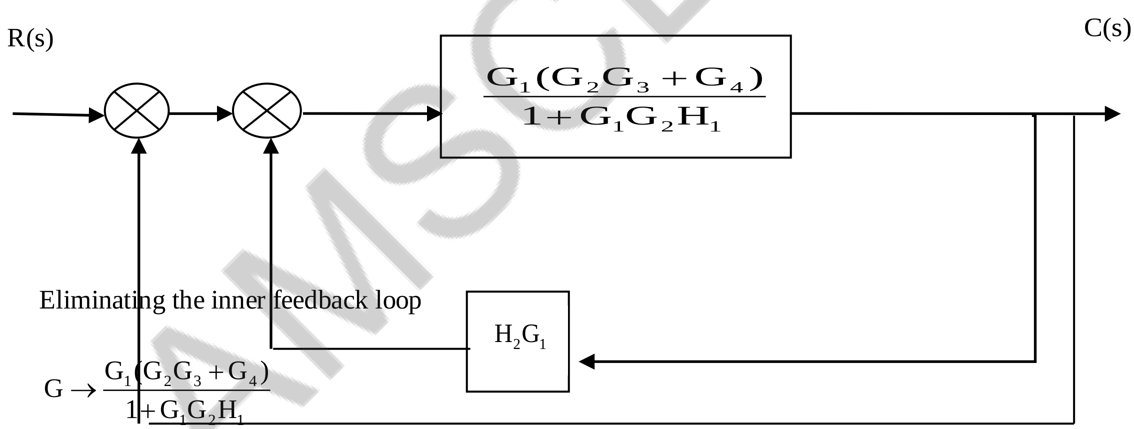


Eliminating the inner most negative feedback

$$G \rightarrow G_1$$
$$H \rightarrow H_1G_2$$
$$\frac{G}{1+GH} \rightarrow \frac{G_1}{1+G_1G_2H_1}$$



Combining cascade blocks

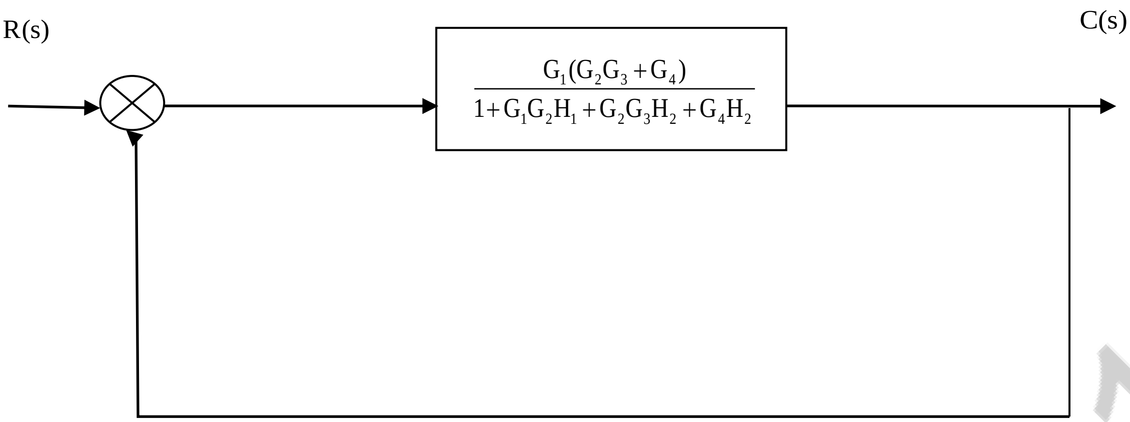


Eliminating the inner feedback loop

$$G \rightarrow \frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}$$
$$H \rightarrow H_2/G_1$$

$$\frac{G}{1+GH} = \frac{\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}}{1+\frac{\cancel{G_1}(G_2G_3+G_4)}{1+G_1G_2H_1}\frac{H_2}{\cancel{G_1}}} = \frac{\frac{G_1(G_2G_3+G_4)}{1+\cancel{G_1G_2H_1}}}{\frac{1+G_1G_2H_2+G_2G_3H_2+G_4H_2}{1+\cancel{G_1G_2H_1}}}$$

$$= \frac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}$$



Eliminating the negative, unity feedback

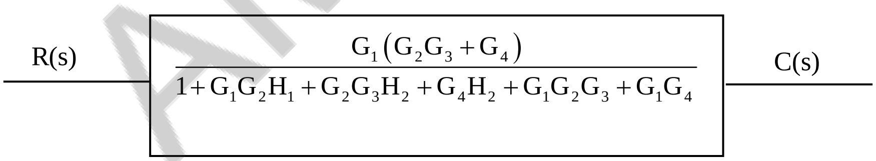
$$G \rightarrow \frac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}$$

$$H \rightarrow 1$$

$$\frac{G}{1+GH} = \frac{\dfrac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}}{1 + \dfrac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}}$$

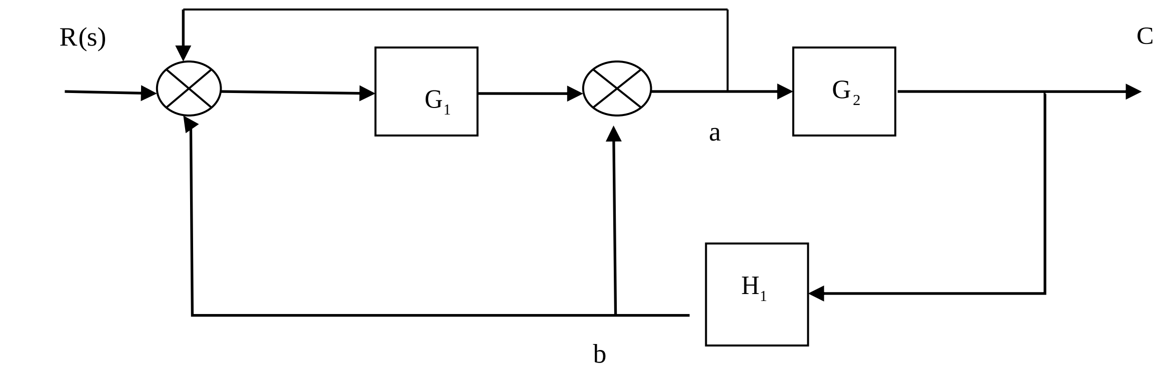
$$= \frac{\dfrac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + \cancel{G_2G_3H_2} + G_4H_2}}{1 + \dfrac{G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + \cancel{G_2G_3H_2} + G_4H_2}}$$

$$= \frac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_2G_3 + G_1G_4}$$

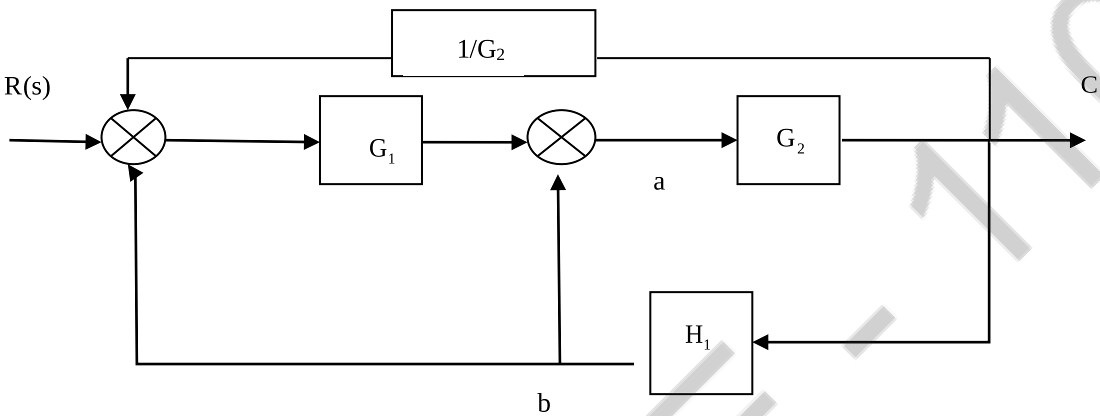


$$\frac{C(s)}{R(s)} = \frac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_2G_3 + G_1G_4}$$

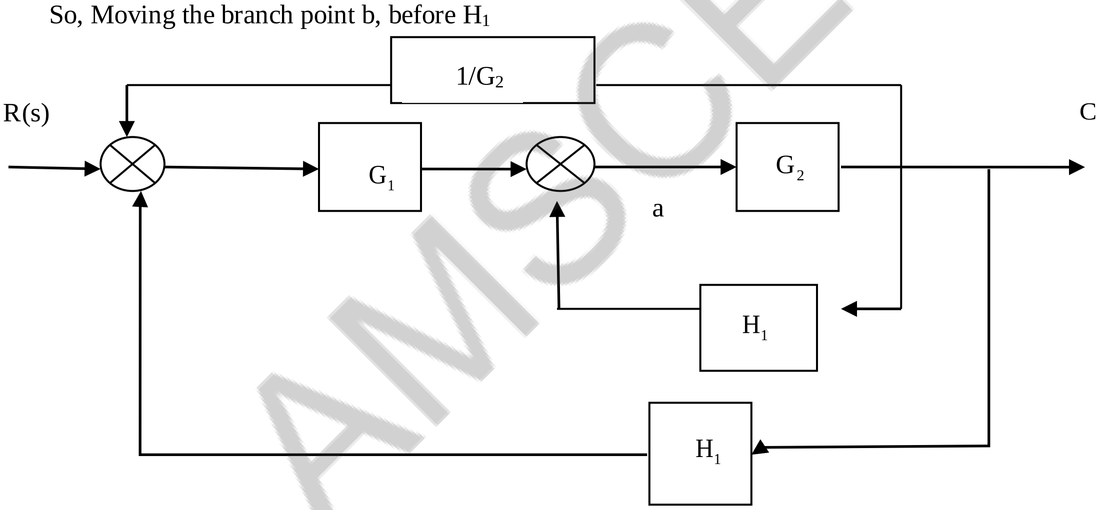
3.Using block diagram reduction technique, find C/R



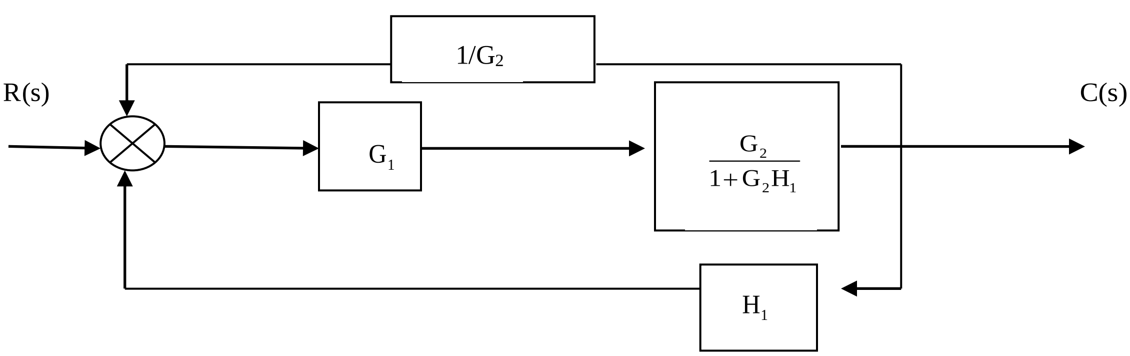
Sol : Moving th branch point a ahead of G_2,



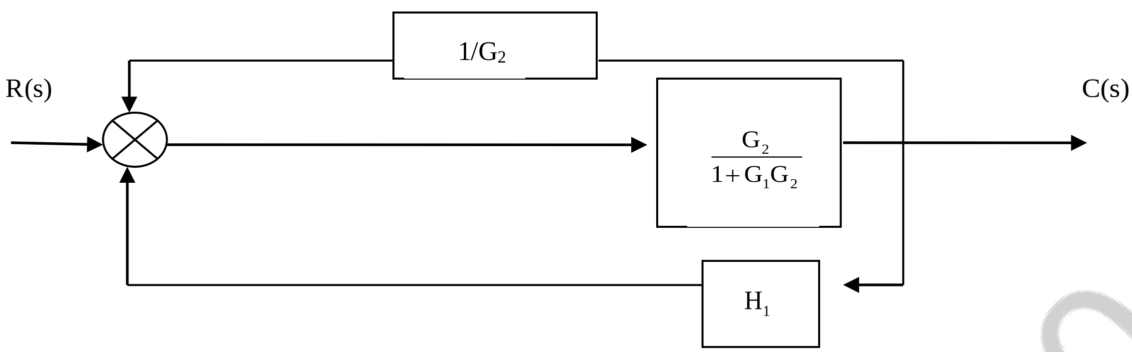
So, Moving the branch point b, before H_1



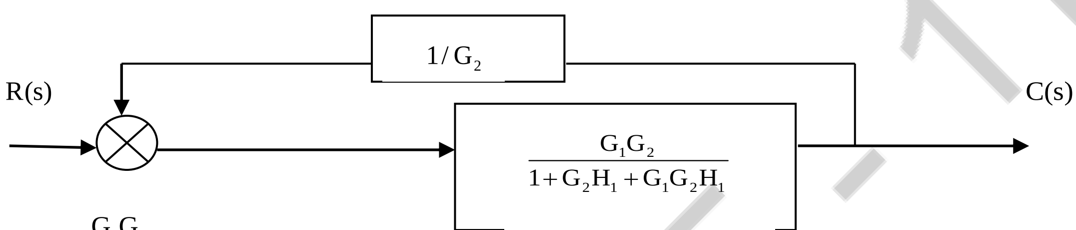
Eliminating the feedback H_1



Combining the cascade blocks



Eliminating the negative feedback, H_1



$$G = \frac{G_1G_2}{1+G_2H_1}$$

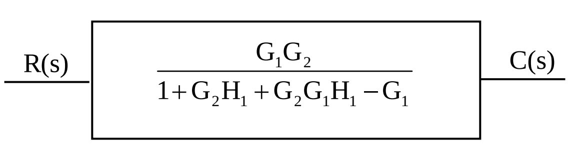
$$H = H_1$$

$$\frac{G}{1+GH} = \frac{\frac{G_1G_2}{1+G_2H_1}}{1+\frac{G_1G_2}{1+G_2H_1}H_1}$$

$$= \frac{G_1G_2}{1+G_2H_1+G_1G_2H_1}$$

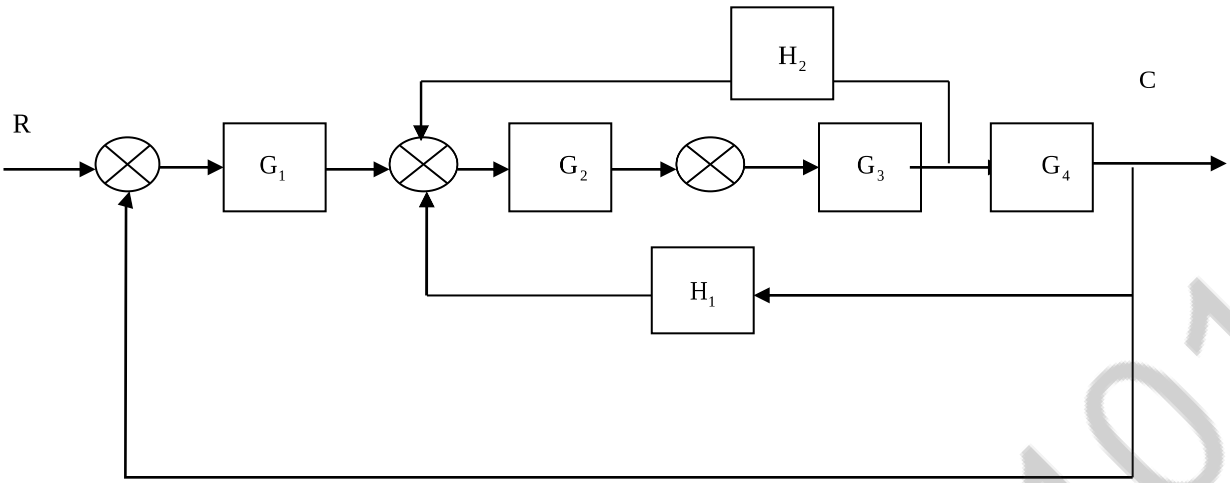
$$G = \frac{G_1G_2}{1+G_2H_1+G_1G_2H_1},\ H \to 1/G_2$$

$$\frac{G}{1-GH} = \frac{\frac{G_1G_2}{1+G_2H_1+G_1G_2H_1}}{1-\frac{G_1G_2}{1+G_2+G_1G_2H_1}\frac{1}{G_2}} = \frac{G_1G_2}{1+G_2H_1+G_1G_2H_1-G_1}$$

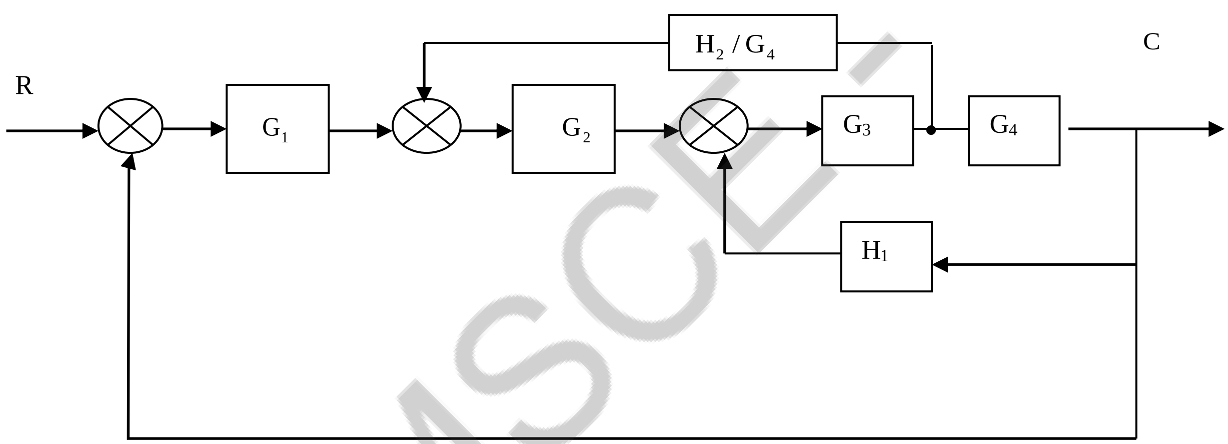


$$\frac{C(S)}{R(S)} = \frac{G_1G_2}{1+G_2H_1+G_1G_2H_1-G_1}$$

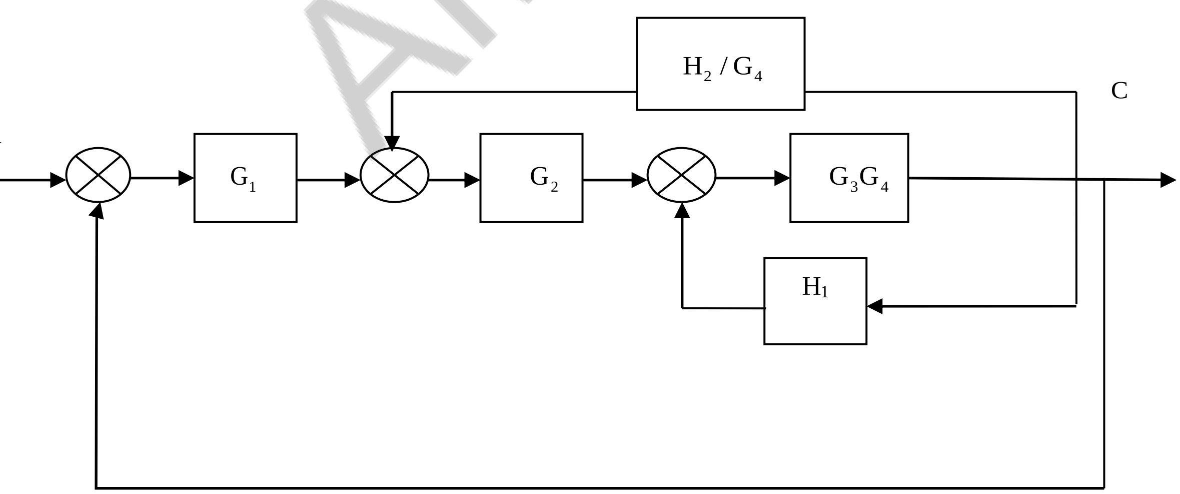
4.Using block diagram reduction technique find C/R NOV/DEC 2016



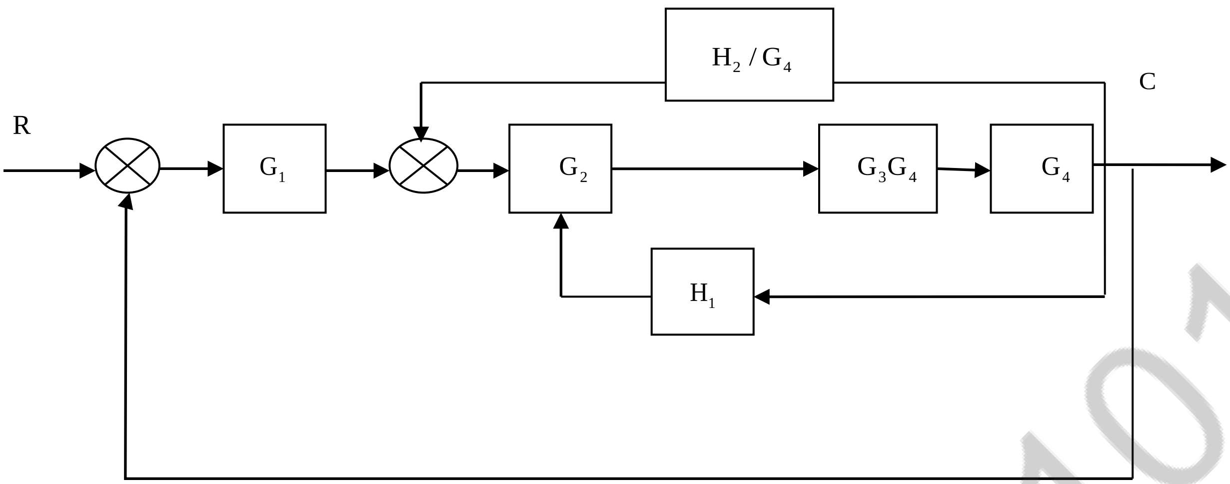
Sol: Moving the branch a ahead of G_4



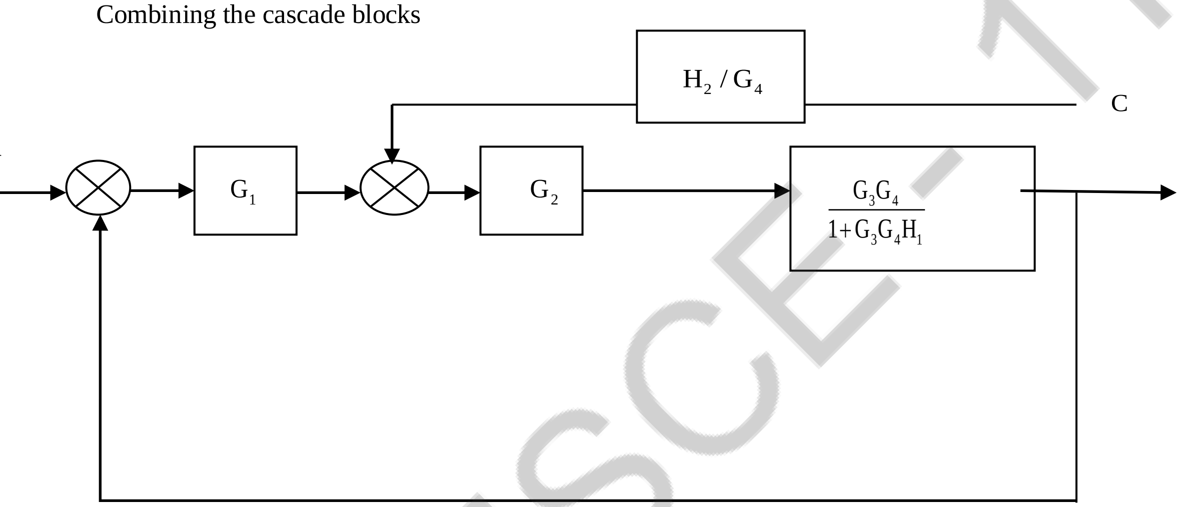
Combining the cascade blocks G_3 and G_4



Eliminating the feedback H_1



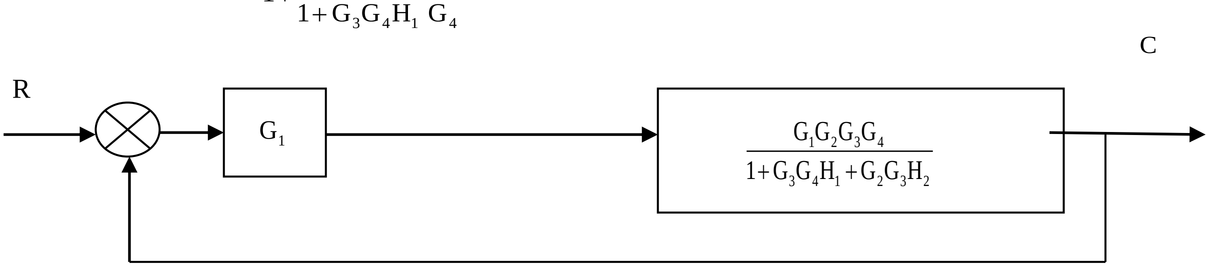
Combining the cascade blocks



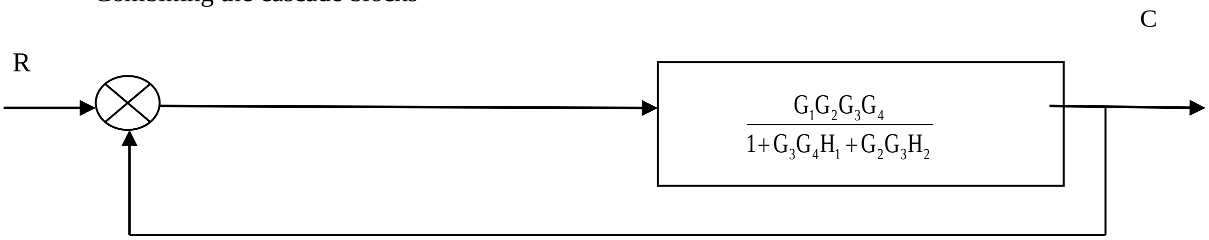
Eliminating the feedback, H_2 / G_4

$$G \rightarrow \frac{G_2G_3G_4}{1+G_3G_4H_1} \qquad H \rightarrow H_2 / G_4$$

$$\frac{G}{1+GH} = \frac{\frac{G_2G_3G_4}{1+G_3G_4H_1}}{1+\frac{G_2G_3 \not{G_4}}{1+G_3G_4H_1}\frac{H_2}{G_4}} \Rightarrow \frac{G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2}$$



Combining the cascade blocks



Eliminating the feedback (unity)

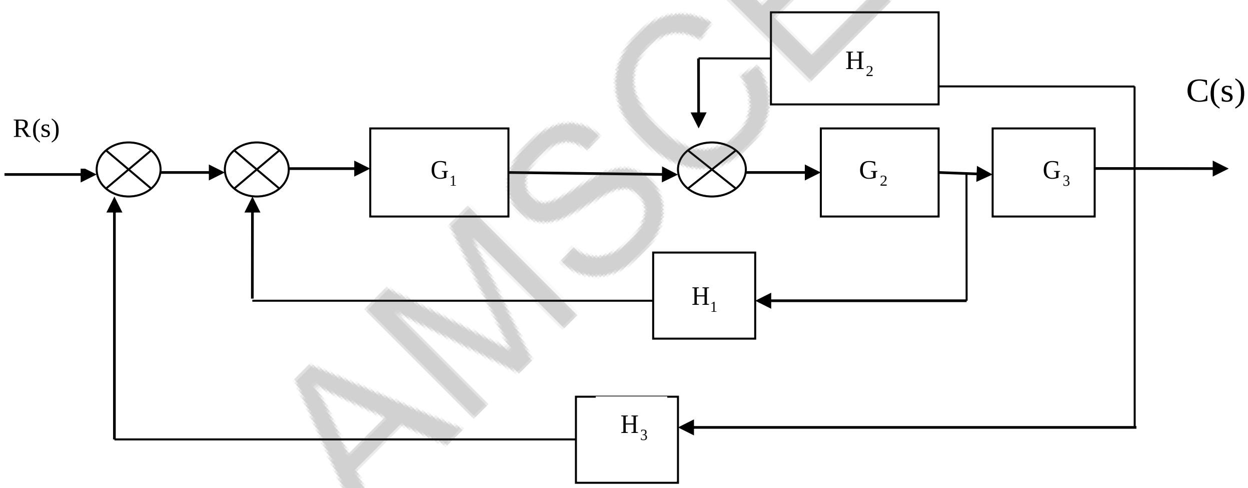
$$G \rightarrow \frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2} \qquad H \rightarrow 1$$

$$\frac{G}{1+GH} = \frac{\frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2}}{1+\frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2}} \Rightarrow \frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

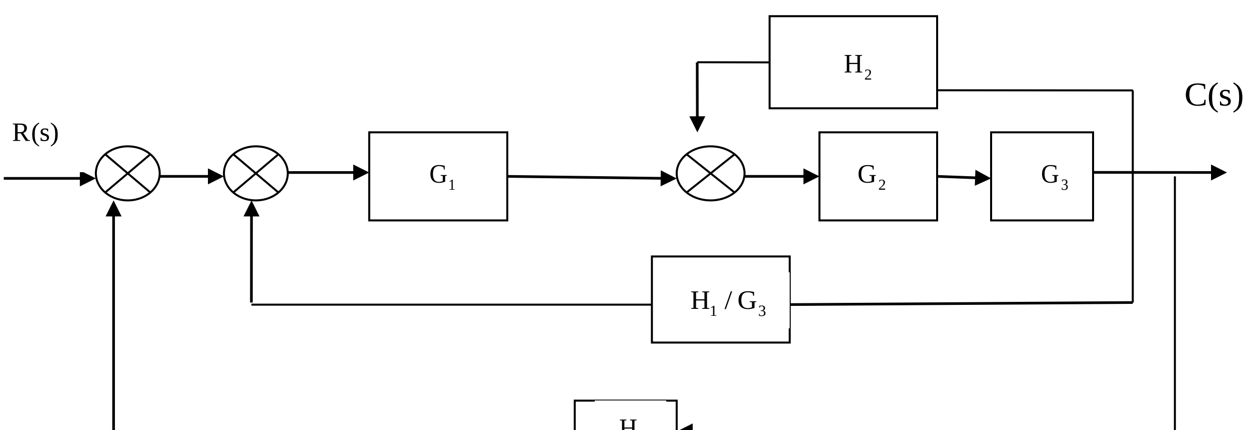
$$\underset{\rightarrow}{R}\boxed{\frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}}\underset{\rightarrow}{C}$$

$$\frac{C}{R} = \frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

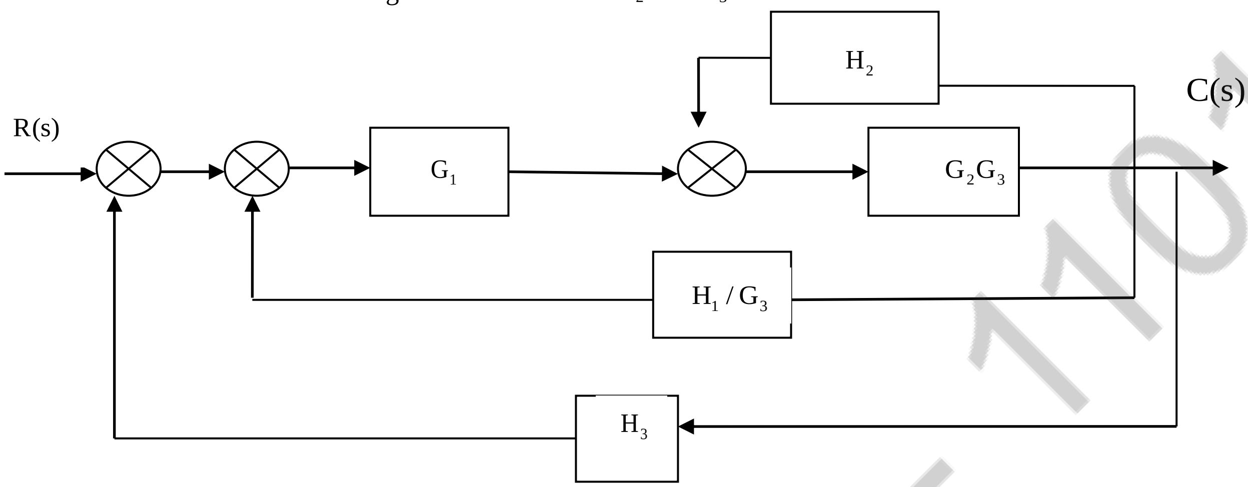
5.Using block diagram reduction technique, find C/R



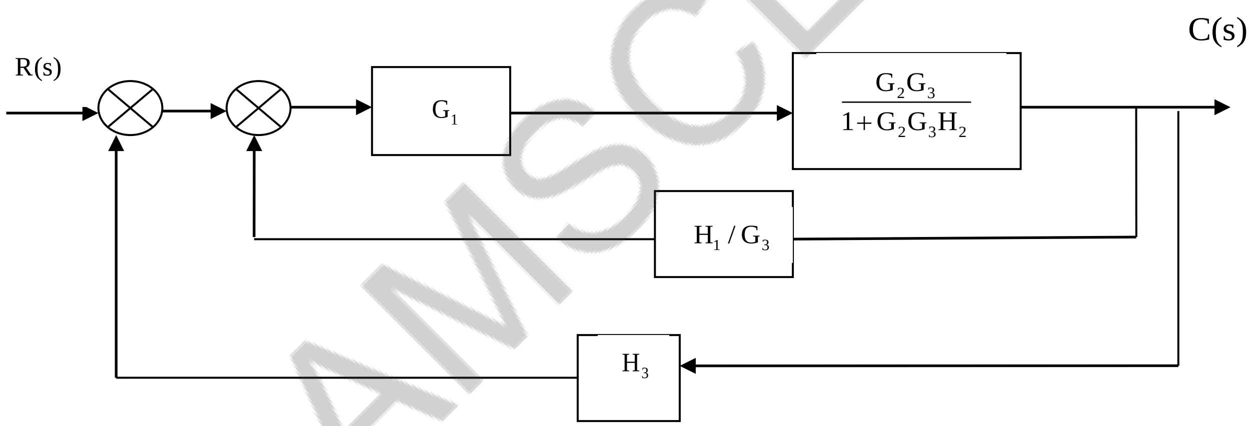
Sol: Moving the branch point a, ahead of G_3



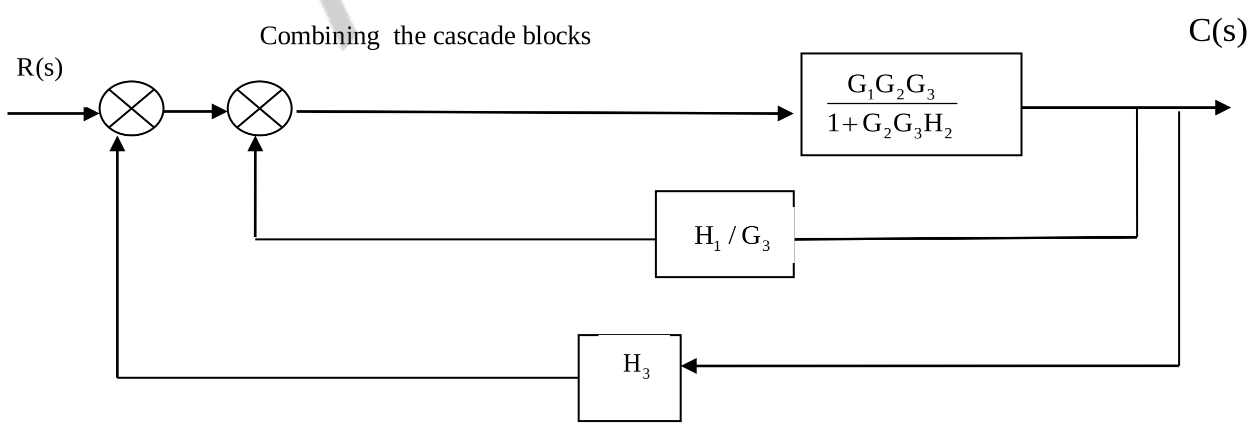
Combining the cascade blocks G_2 and G_3



Eliminating the feedback H_2

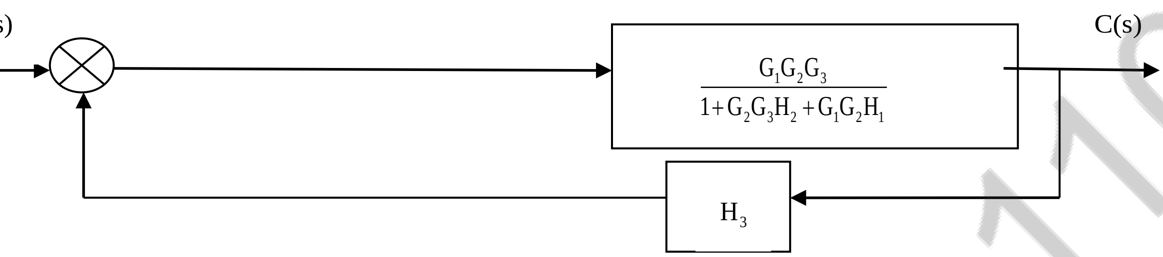


Combining the cascade blocks



Eliminating the feedback parts H_1 / G_3

$$G \rightarrow \frac{\dfrac{G_1G_2G_3}{1+G_2G_3H_2}}{1+\dfrac{G_1G_2\cancel{G_3}}{1+G_2G_3H_2}\dfrac{H_1}{\cancel{G_3}}} \Rightarrow \frac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1}$$



Eliminating the feedback path H_3,

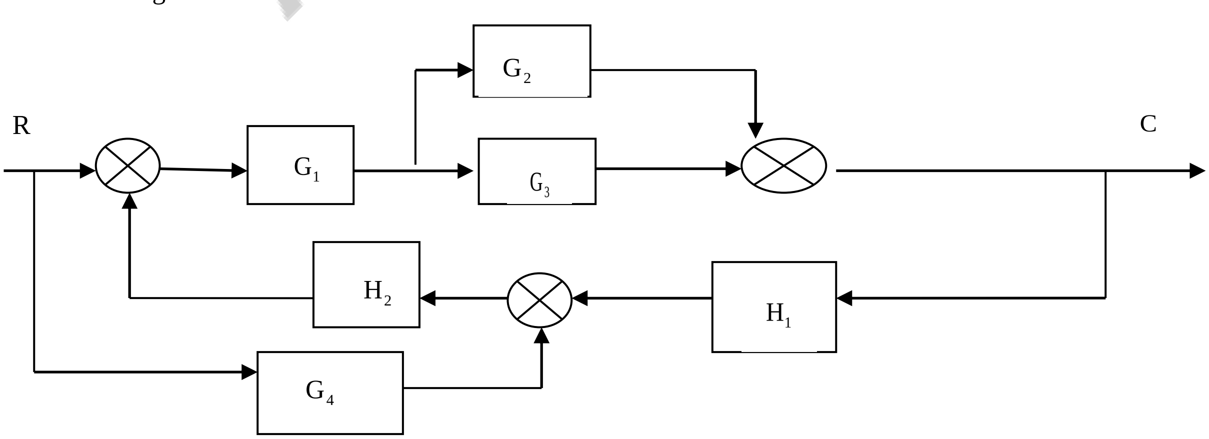
$$G \rightarrow \frac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1},\ H \rightarrow H_3$$

$$\frac{G}{1+GH} = \frac{\dfrac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1}}{1+\dfrac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1}H_3} \Rightarrow \frac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1+G_1G_2G_3H_3}$$

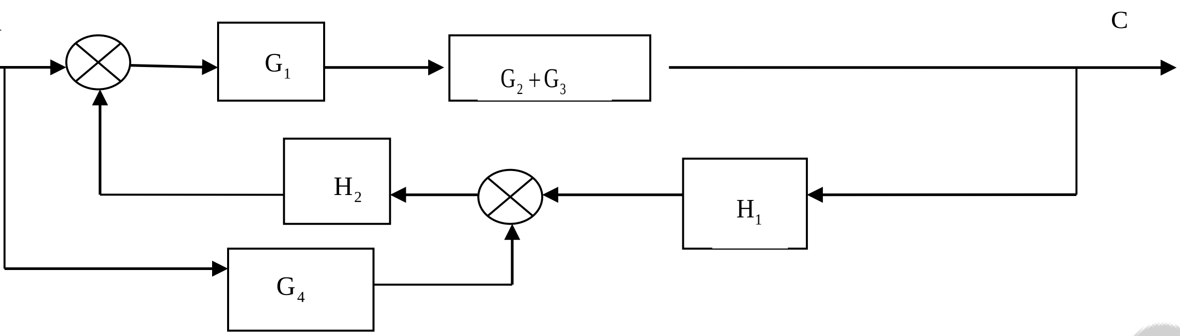
$$R(S) \rightarrow \boxed{\frac{G_1G_2G_3}{1+G_2G_3H_2+G_2G_1H_1+G_1G_2G_3H_3}} \rightarrow C(S)$$

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1+G_1G_2G_3H_3}$$

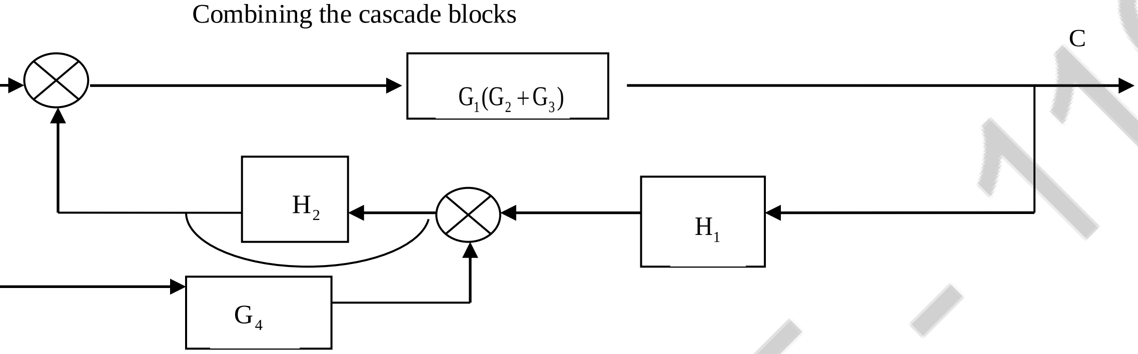
6.Using block diagram reduction technique, find the closed loop transfer function of a system, whose block diagram is shown in fig



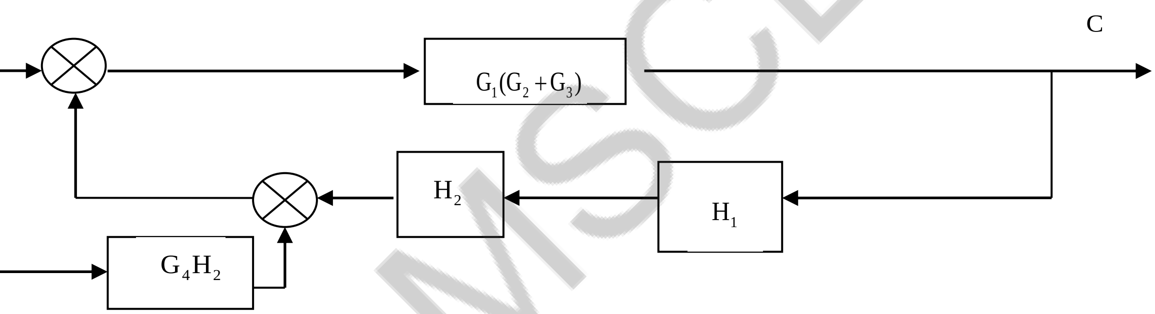
Sol: Combining the parallel blocks



Combining the cascade blocks

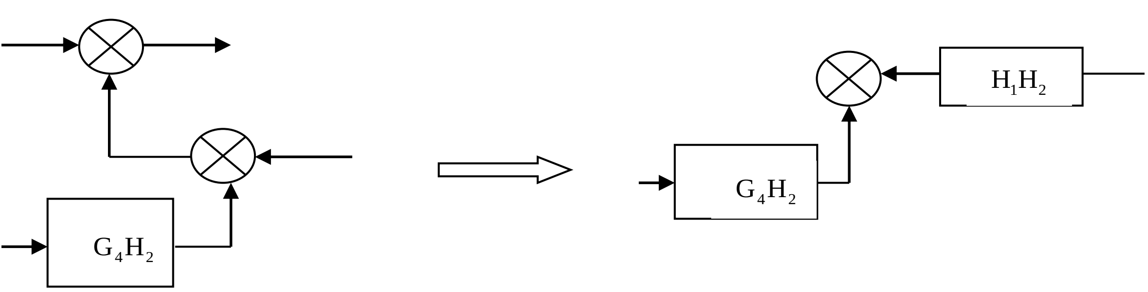


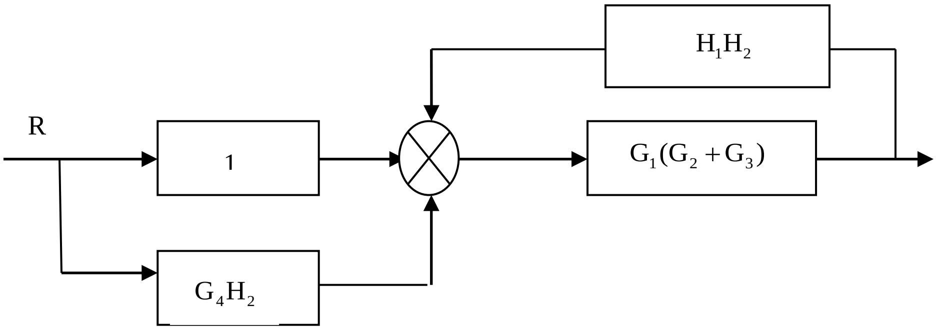
Moving the summing point ahead of H_2



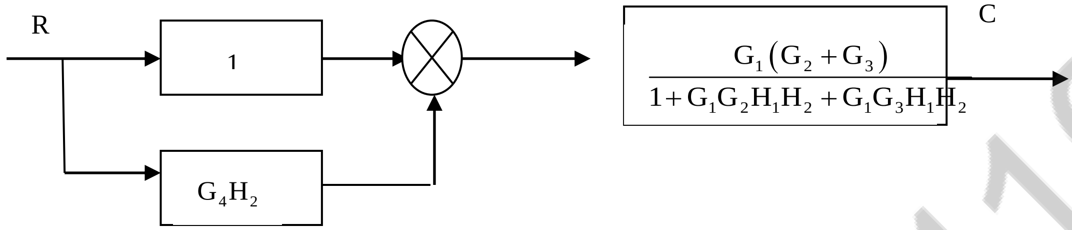
Cascading H_1 and H_2 combining the summing points.

Elimination of summing point by multiply signs

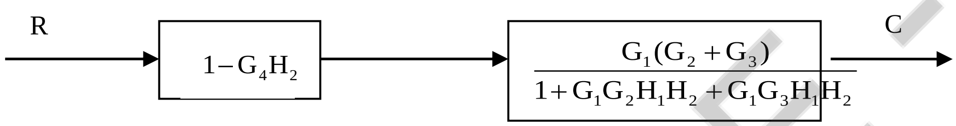




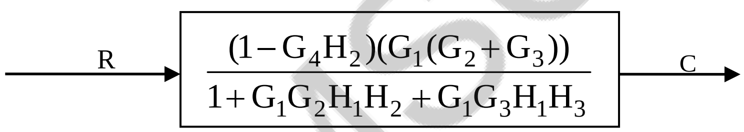
Feedback path H_1, H_2 elimination



Combining parallel path

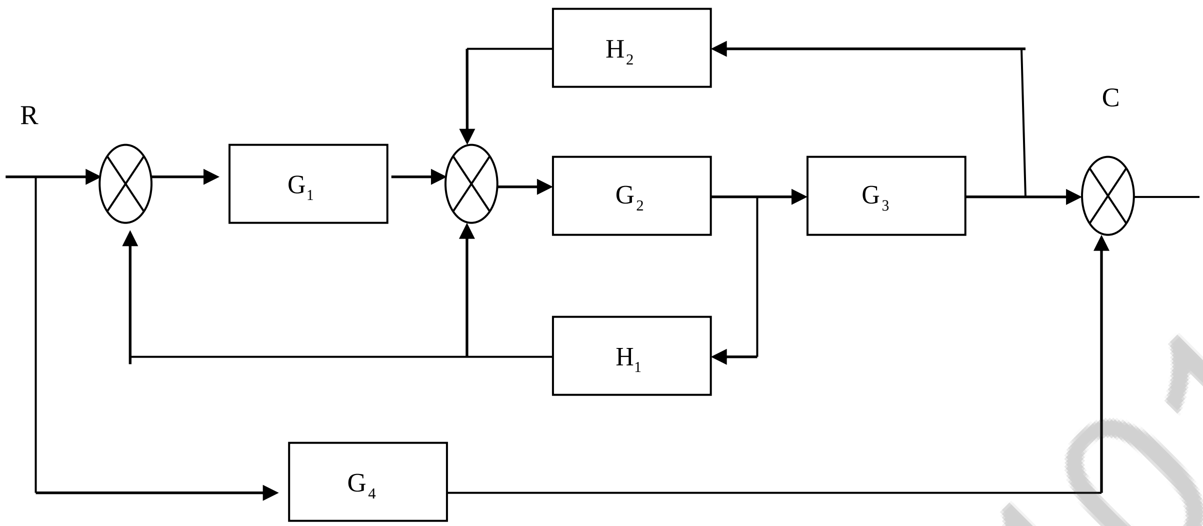


Combining cascade blocks

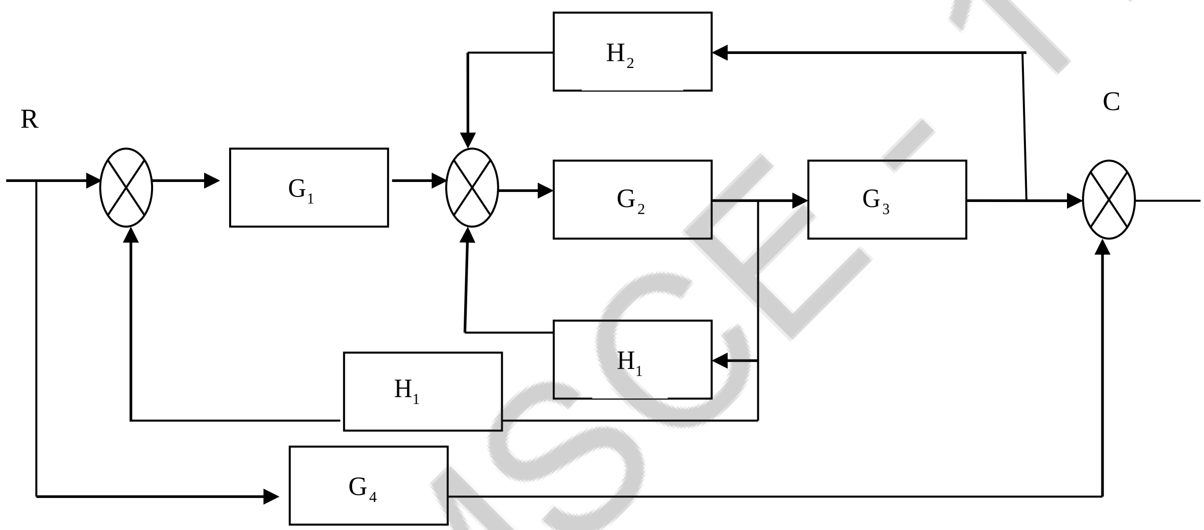


$$\frac{C}{R} = \frac{G_1(G_2 + G_3)(1 - G_4H_2)}{1 + G_1G_2H_1H_2 + G_1G_3H_1H_2}$$

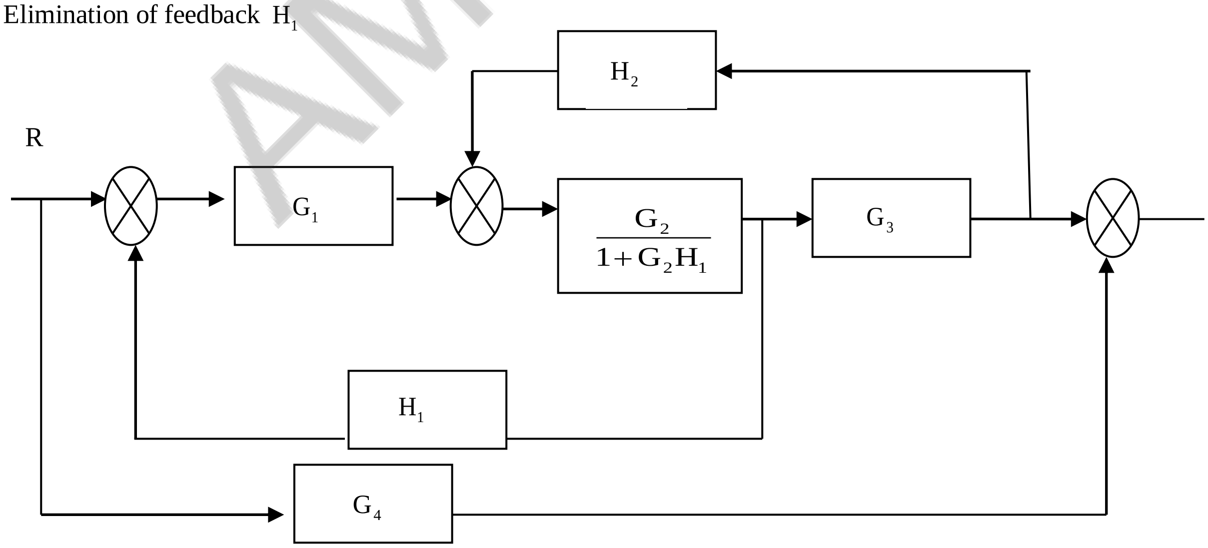
7.Using block diagram reduction technique, find C/R



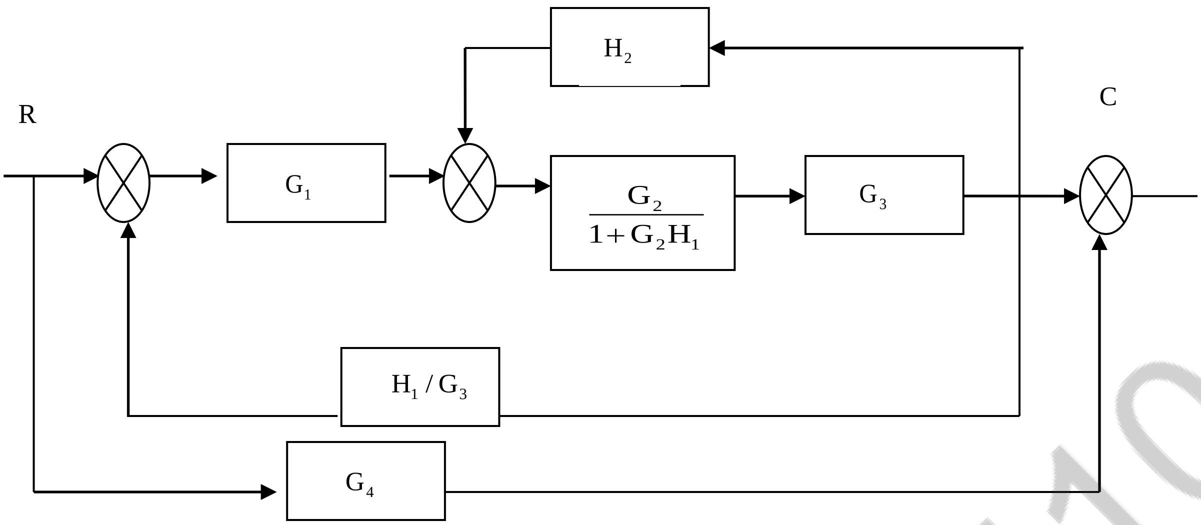
Sol: Moving the branch point a before H_1



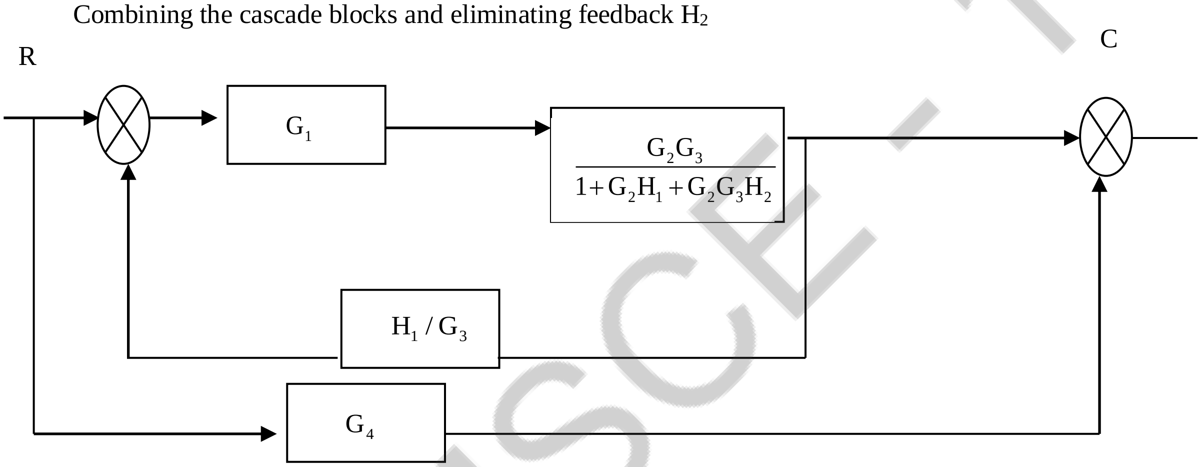
Elimination of feedback H_1



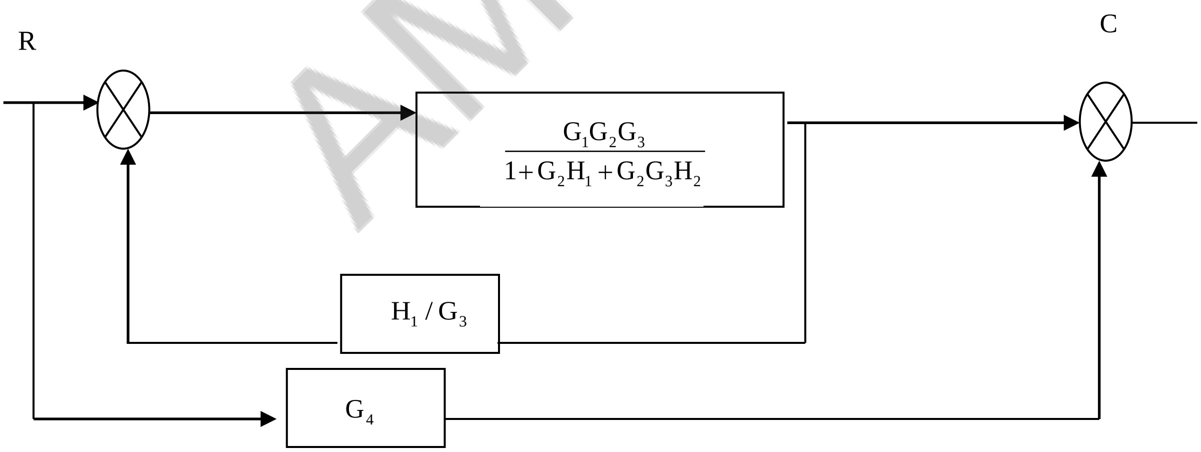
Moving the branch point ahead of G_3



Combining the cascade blocks and eliminating feedback H_2



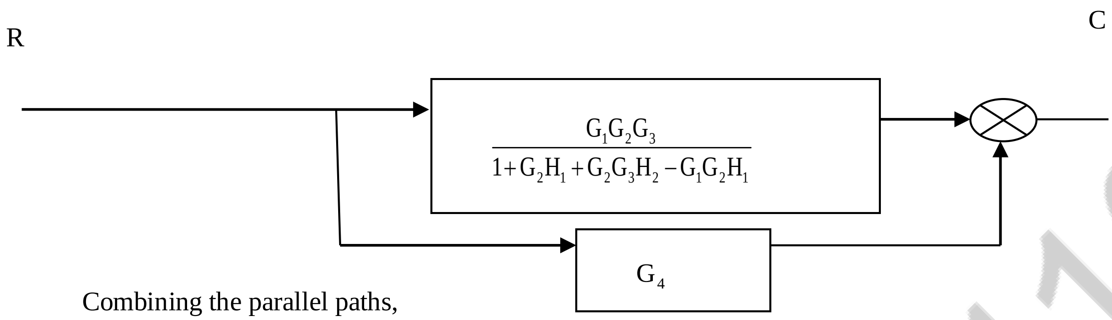
Combining the cascade blocks



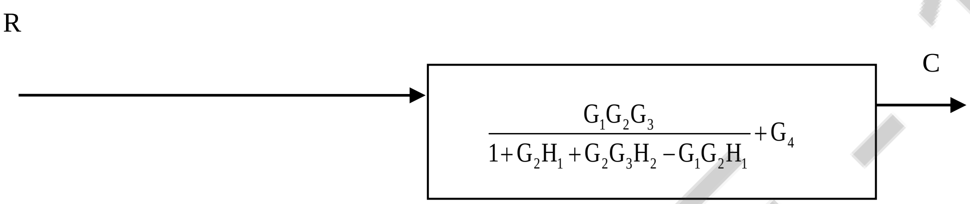
Eliminating the feedback path,

$$G \rightarrow \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2}; \quad H \rightarrow \frac{H_1}{G_3}$$

$$\frac{G}{1-GH} = \frac{\dfrac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2}}{1-\dfrac{G_1G_2\cancel{G_3}}{1+G_2H_1+G_2G_3H_2}\dfrac{H_1}{G_3}} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1}$$

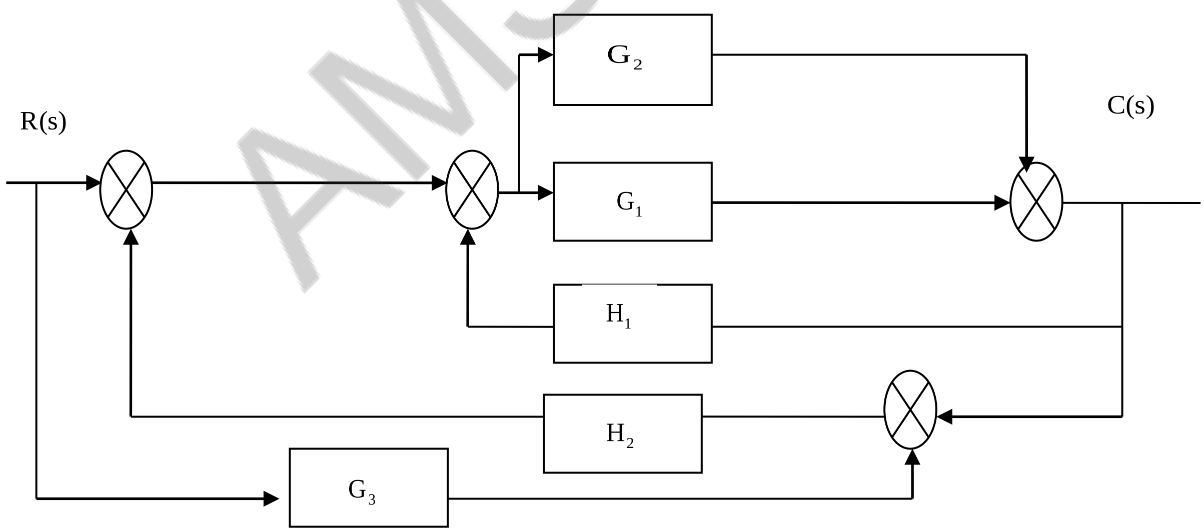


Combining the parallel paths,

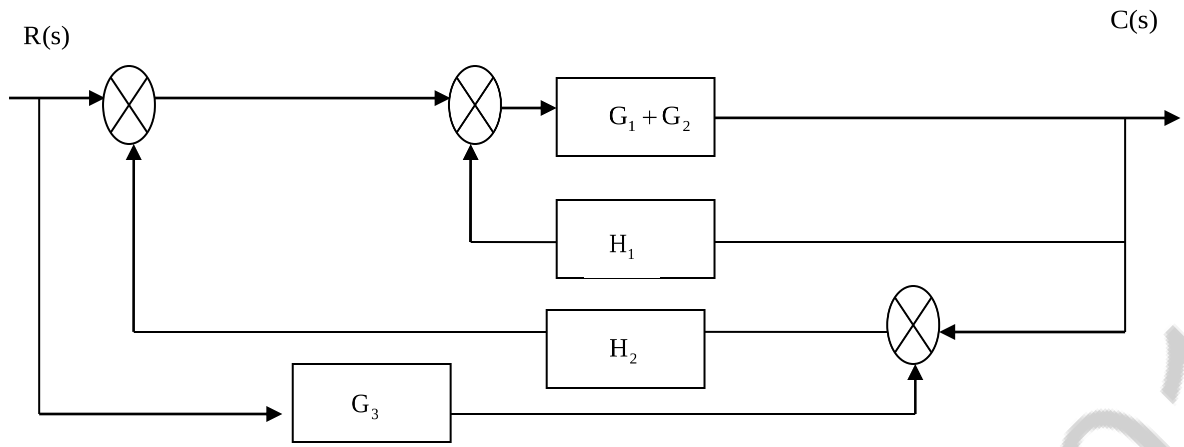


$$\frac{C}{R} = \frac{G_1G_2G_3+G_4(1+G_2H_1+G_2G_3H_2-G_1G_2H_1)}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1}$$

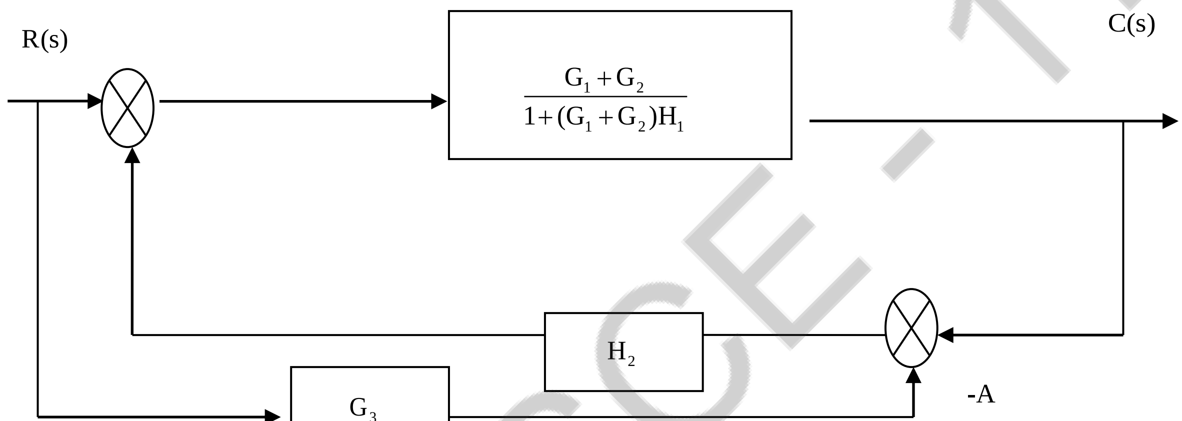
8.Find C(s)/R(s) of the system shown in fig using block diagram reduction technique



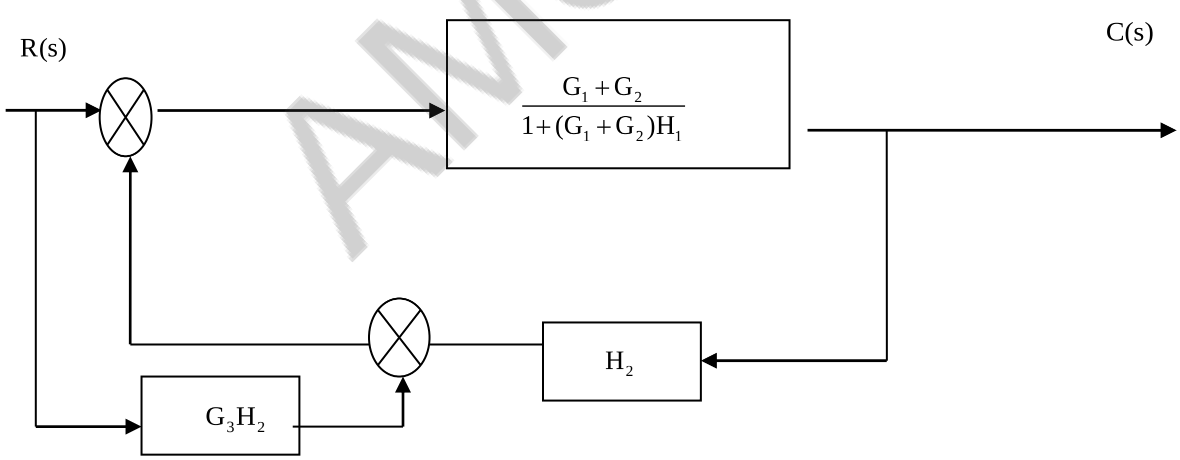
Sol: Combining the parallel blocks G_2 and G_1



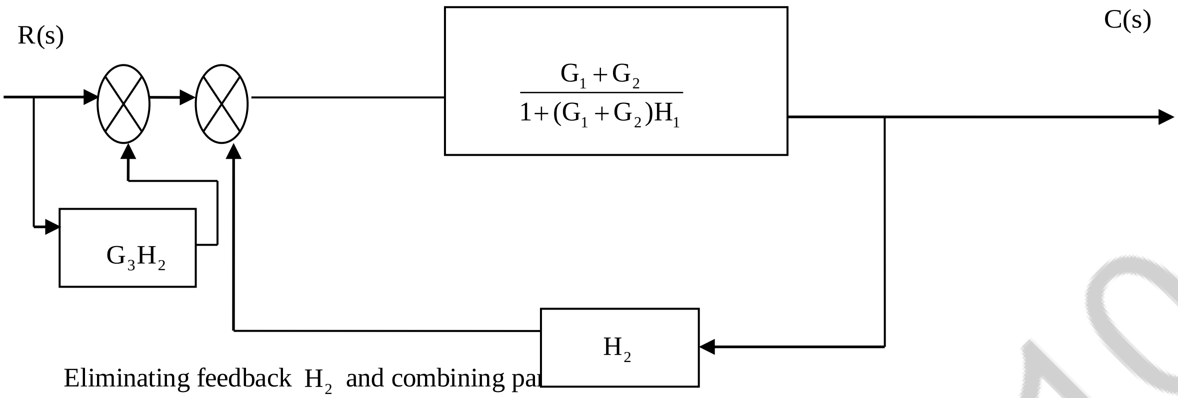
Eliminating the feedback H_1



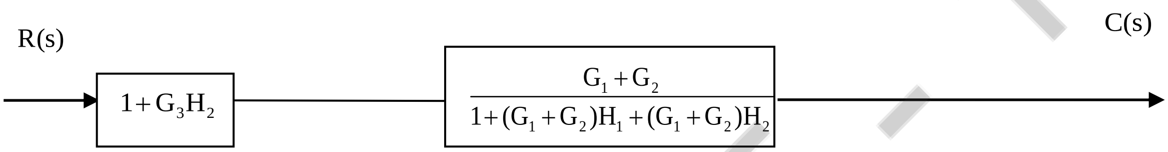
Moving the summing point (a) ahead of H_2



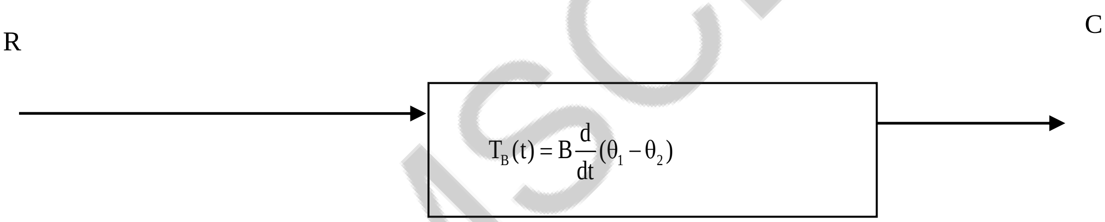
Eliminating the summing point by multiply signs



Eliminating feedback H_2 and combining par

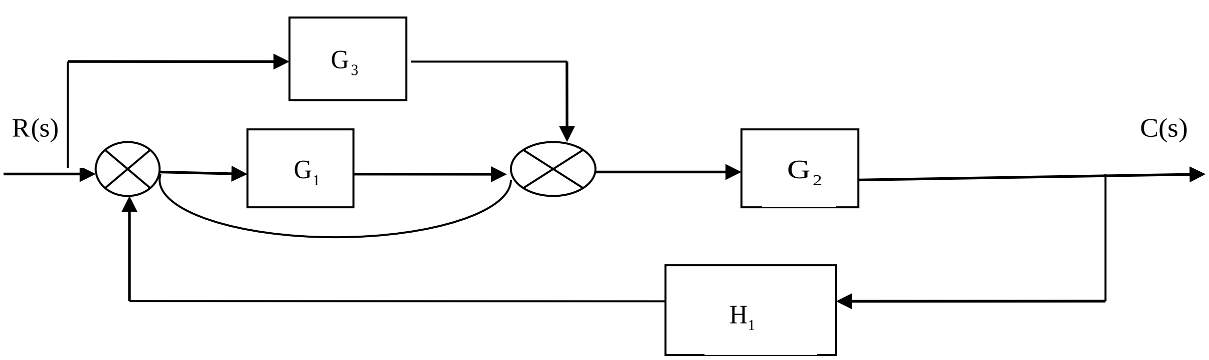


Combining the cascade blocks

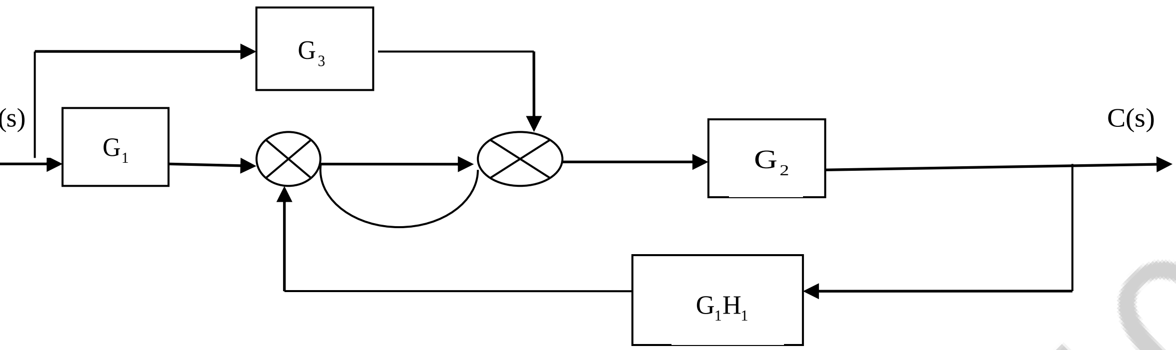


$$\frac{C(s)}{R(s)} = \frac{(1+G_3H_2)(G_1+G_2)}{1+(G_1+G_2)H_1+(G_1+G_2)H_2}$$

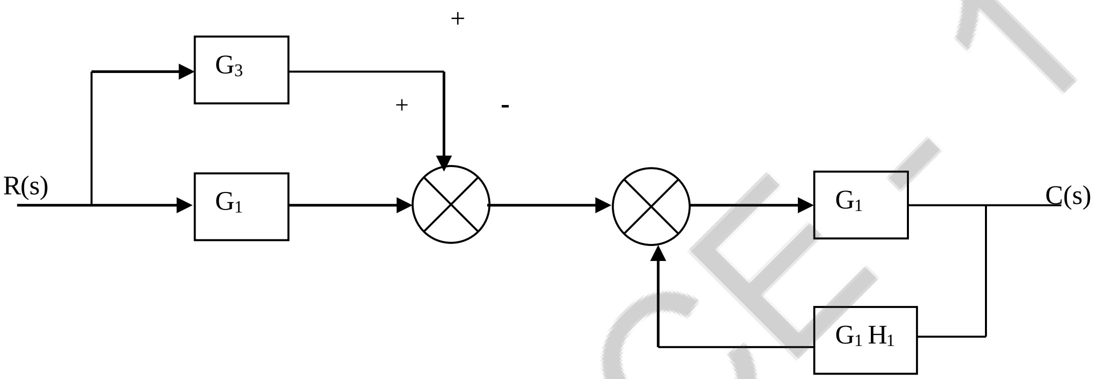
9. Find C(s)/R(s) of the system shown in fig. using block diagram reduction technique.



Sol: Moving the summing point a ahead of G_1

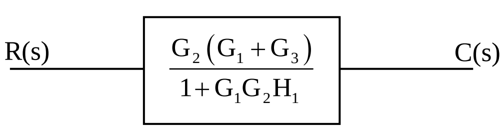


Inter changing the summing points,



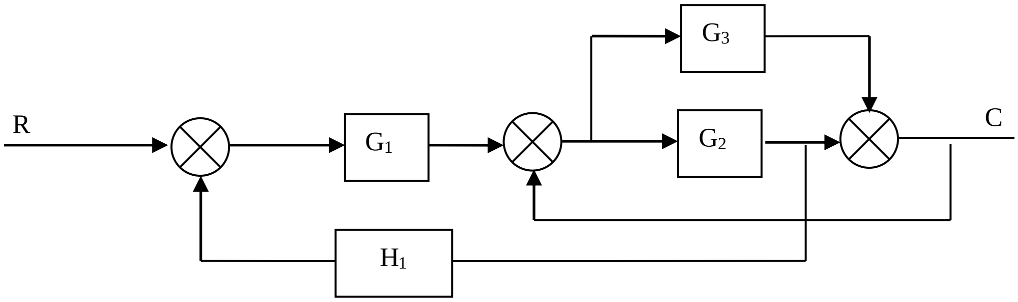
Combining the parallel blocks & Eliminating the feedback path G_1H_1.

Combining the cascade blocks

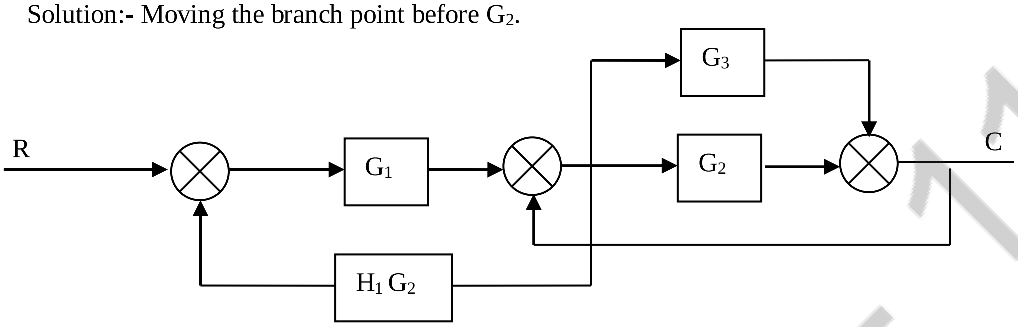


$$\frac{C(s)}{R(s)} = \frac{G_2(G_1+G_3)}{1+G_1G_2H_1}$$

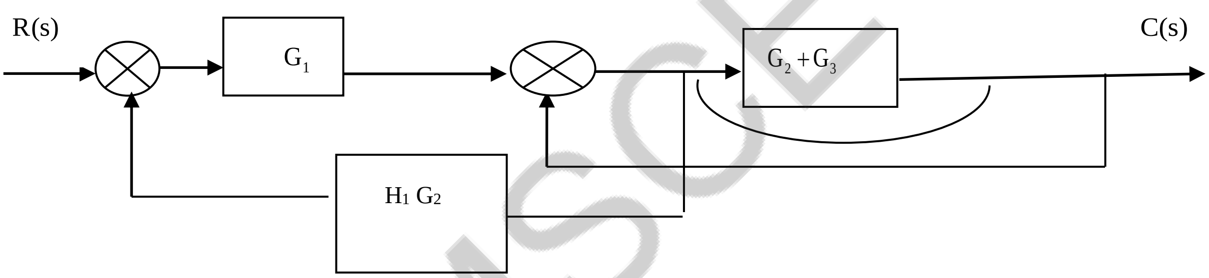
10. Determine the transfer function $\frac{C(s)}{R(s)}$ for the following block diagram



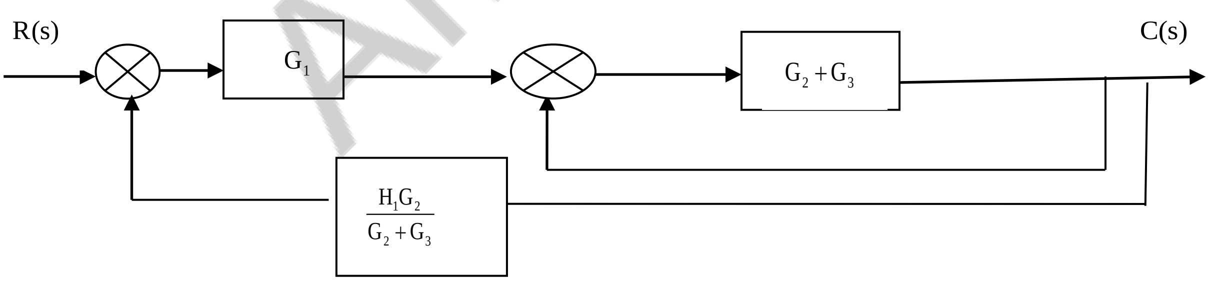
Solution:- Moving the branch point before G_2.



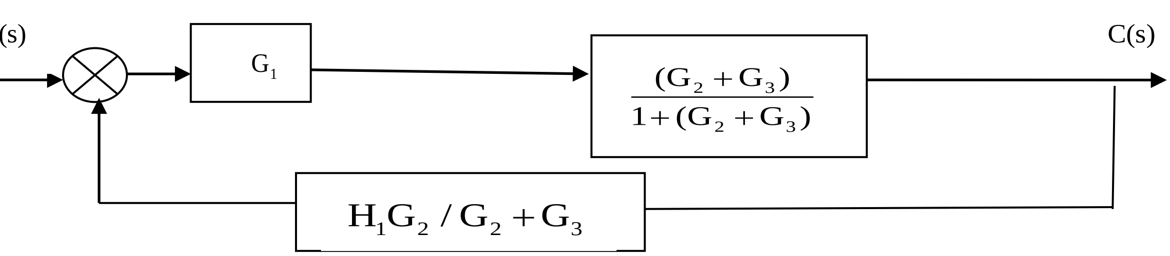
Combining the parallel blocks



Moving the branch point ahead of $(G_2 \text{and} G_3)$



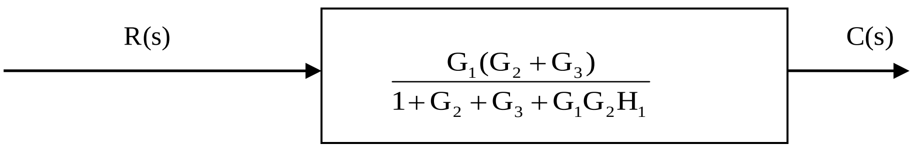
Eliminating the unity feedback path,



Combining the cascade blocks and eliminating the feedback

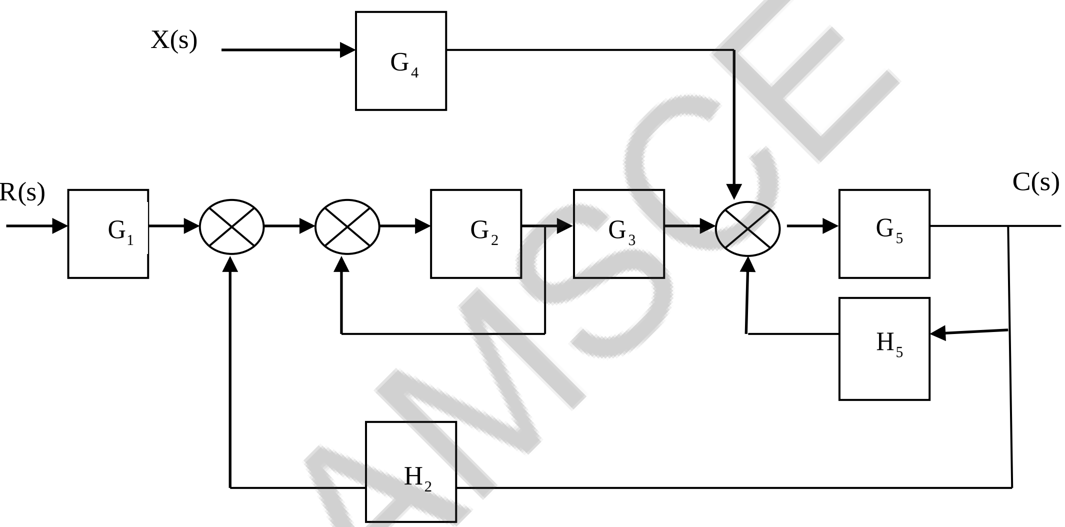
$$G \rightarrow \frac{G_1(G_2+G_3)}{1+(G_2+G_3)}, \quad H \rightarrow \frac{H_1G_2}{G_2+G_3}$$

$$\frac{G}{1+GH} = \frac{\dfrac{G_1(G_2+G_3)}{1+(G_2+G_3)}}{1+\dfrac{G_1(\cancel{G_2+G_3})}{1+(G_2+G_3)}\cdot\dfrac{H_1G_2}{\cancel{G_2+G_3}}} = \frac{G_1(G_2+G_3)}{1+G_2+G_3+G_1G_2H_1}$$

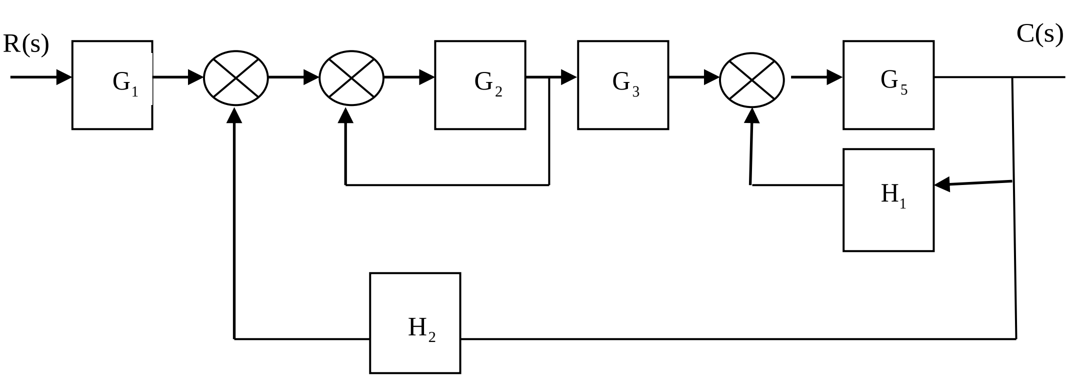


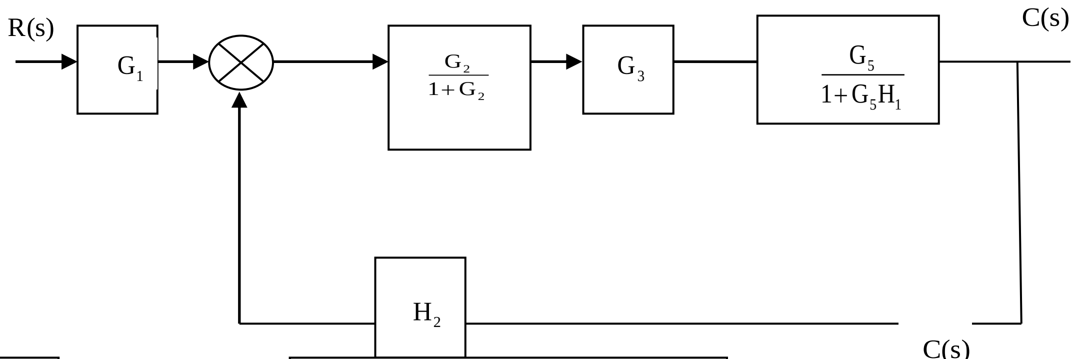
$$\frac{C(S)}{R(S)} = \frac{G_1(G_2+G_3)}{1+G_2+G_3+G_1G_2H_1}$$

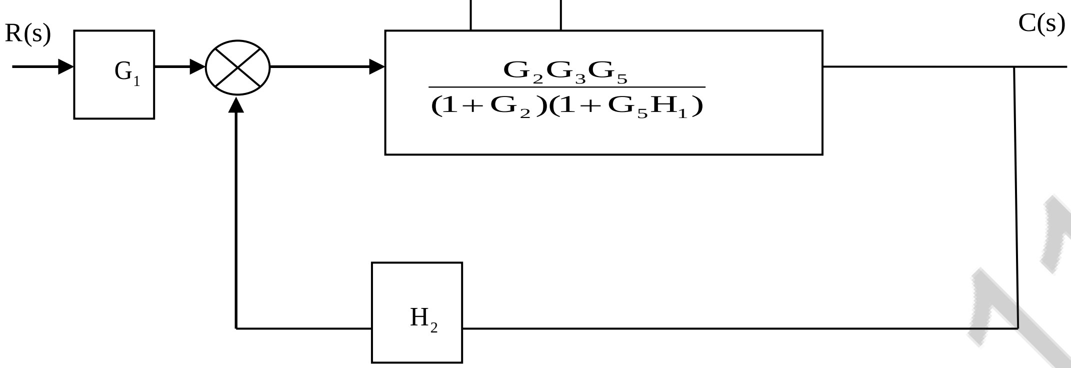
11. Using block diagram reduction technique find the transfer function from each input to the output C for the system shown in fig.

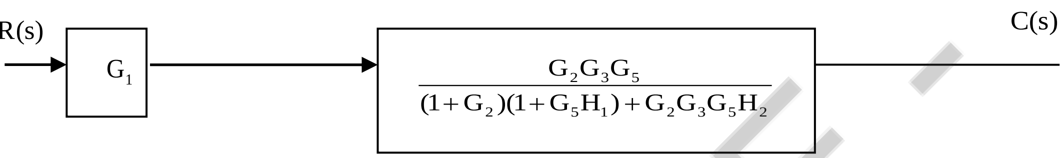


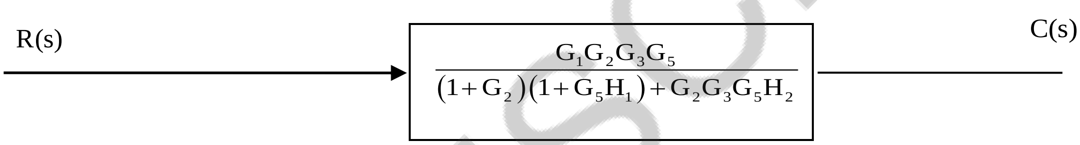
Sol: To find $\frac{C(s)}{R(s)} \rightarrow$ put $X(s) = 0$





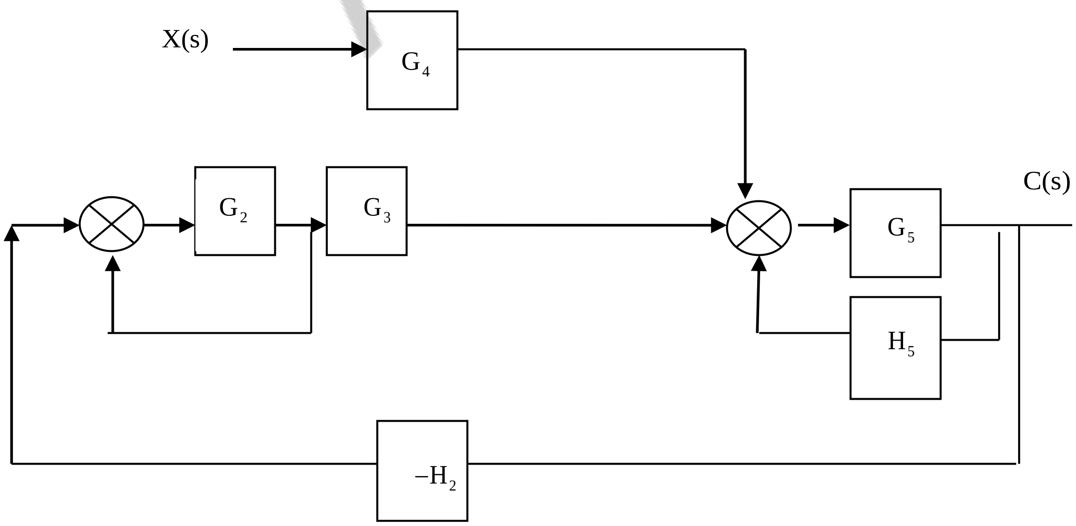


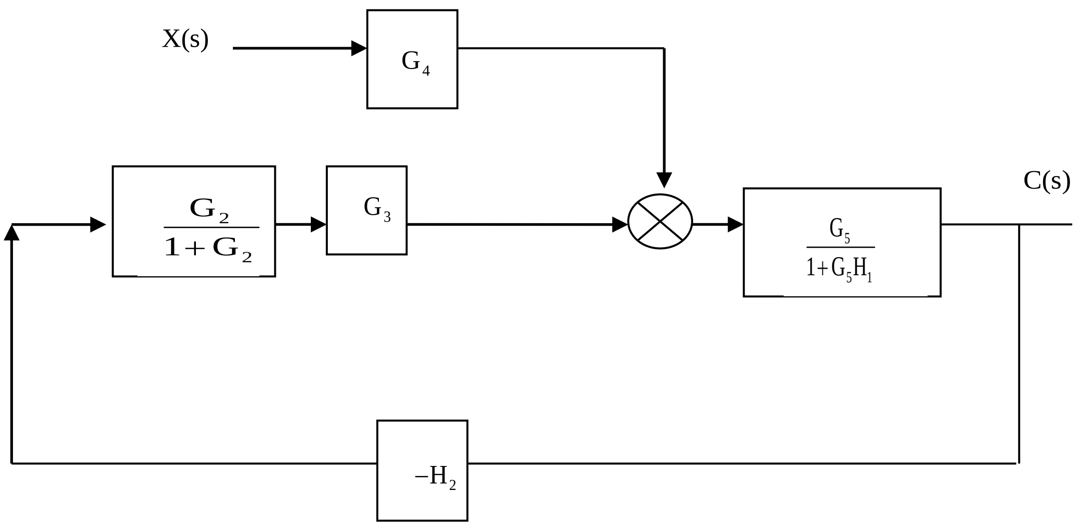


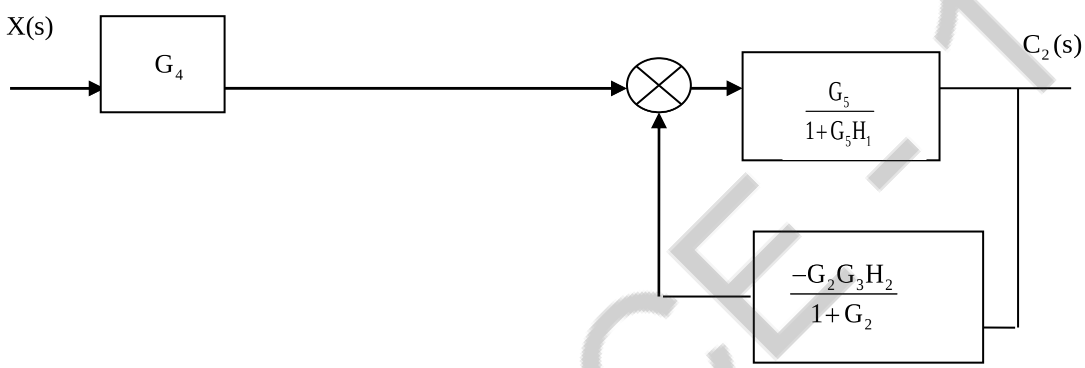


$$\frac{C(S)}{R(S)} = \frac{G_1G_2G_3G_5}{(1+G_2)(1+G_5H_1)+G_2G_3G_5H_2}$$

To find $\frac{C_2(s)}{X(s)}$ put R(s)=0,

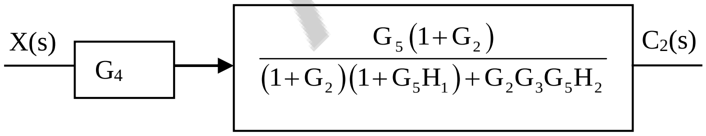


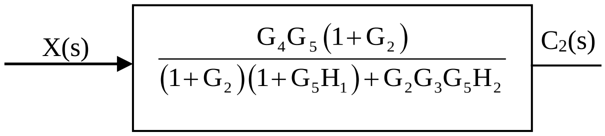




$$G \rightarrow \frac{G_5}{1+G_5H_1}, \quad H \rightarrow -\frac{G_2G_3H_2}{1+G_2}$$

$$\frac{G}{1+GH} = \frac{\frac{G_5}{1+G_5H_1}}{1+\frac{G_5}{1+G_5H_1}\frac{G_2G_3H_2}{(1+G_2)}} = \frac{G_5(1+G_2)}{(1+G_2)(1+G_5H_1)+G_2G_3G_5H_2}$$





$$\frac{C_2(s)}{X(s)} = \frac{G_4G_5(1+G_2)}{(1+G_2)(1+G_5H_1)+G_2G_3G_5H_2}$$

When both R(s) and X(s) are simultaneously present, the output

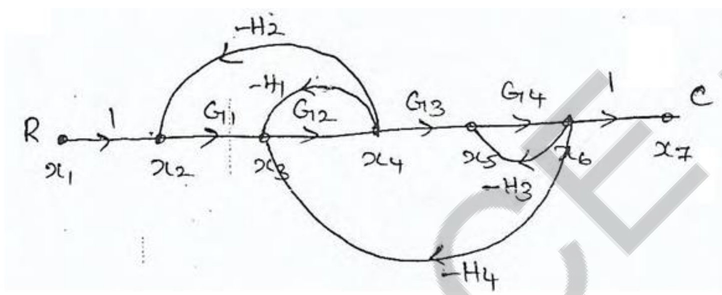
$C(s) = C_1(s) + C_2(s)$ as per superposition theorem

Hence $C(s) = \dfrac{R(s)G_1G_2G_3G_5 + X(s)G_4G_5(1+G_2)}{(1+G_2)(1+G_5H_1)+G_2G_3G_5H_2}$

SIGNAL FLOW GRAPH

PROBLEMS

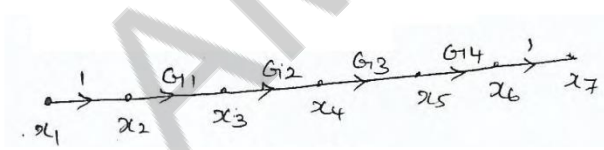
1. Obtain the overall transfer function of the following signal flow graph using Mason's gain formula



Sol:

Step 1:Forward path gains No. of forward path K=1

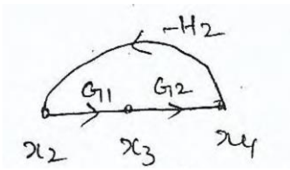
Forward path gain path $\rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5 \rightarrow x_6 \rightarrow x_7$

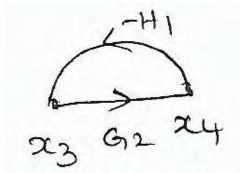


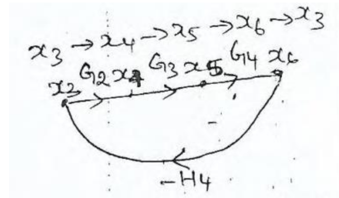
$P_1 = G_1G_2G_3G_4$

Step 2: Individual loop gains

$x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_2$ $\quad$ $x_3 \rightarrow x_4 \rightarrow x_3$ $\quad$ $x_3 \rightarrow x_2 \rightarrow x_5 \rightarrow x_6 \rightarrow x_3$





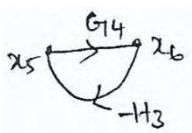


$P_{11} = -G_1G_2H_2$

$P_{21} = -G_2H_1$

$P_{31} = -G_2G_3G_4H_4$

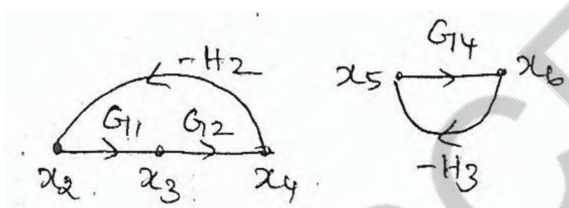
$x_5 \rightarrow x_6 \rightarrow x_5$



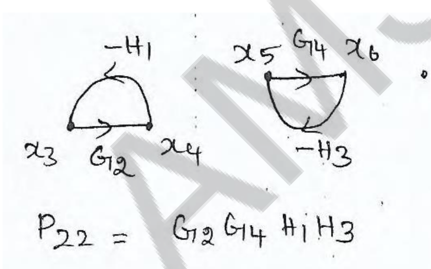
$P_{41} = -G_4H_3$

Step 3: Non touching loops →Gain products

There are two pairs of non touching loops



$P_{12} = G_1G_2G_4H_2H_3$



$P_{22} = G_2G_4H_1H_3$

Step 4: To find Δ and Δk

$$\Delta = 1 - [P_{11} + P_{21} + P_{31} + P_{41}] + [P_{12} + P_{22}]$$
$$= 1 + G_1G_2H_2 + G_2H_1 + G_4H_3 + G_2G_3G_4H_4 + G_1G_2G_4H_2H_3 + G_2G_4H_1H_3$$

$\Delta_1 = 1$ Since there in no part of the graph is not touching with first forward path

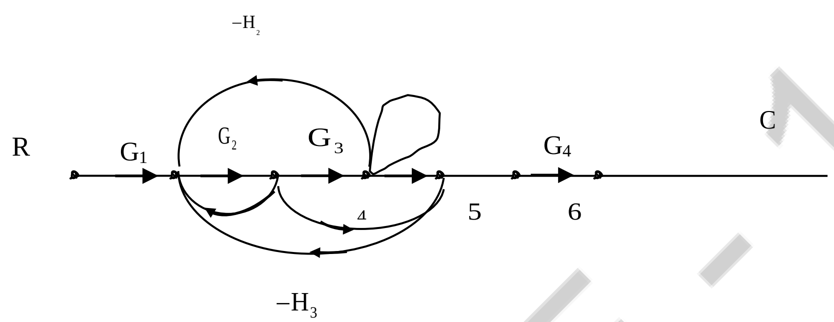
Step 5: Transfer function

By Mason's gain formula

$$T(s) = \frac{1}{\Delta}\sum_{k=1}^{n} P_k\Delta_k = \frac{1}{\Delta}\left(P_1\Delta_1\right)$$

$$T(s) = \frac{G_1G_2G_3G_4}{1+G_1G_2H_2+G_2H_1+G_4H_3+G_2G_3G_4H_4+G_1G_2G_4H_2H_3+G_2G_4H_1H_3}$$

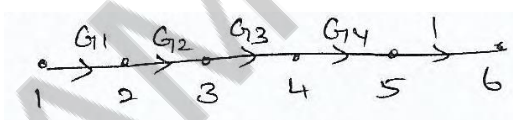
2.Find the overall gain of the system whose signal flow graph is shown in fig. Nov/Dec 2017



Sol: Step 1: Forward path gain No. Of forward path K=2

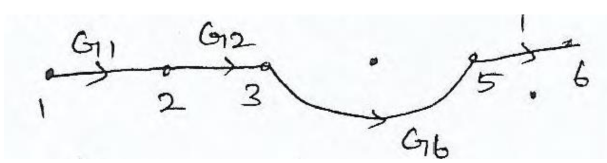
Forward path gain

Path 1 : 1→2→3→4→5→6



$P_1 = G_1G_2G_3G_4$

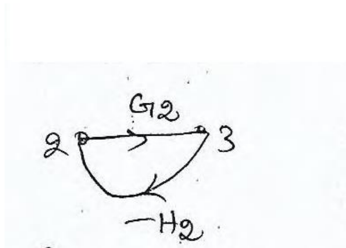
Path 2: 1→2→3→5→6

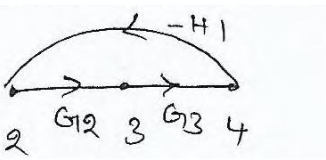


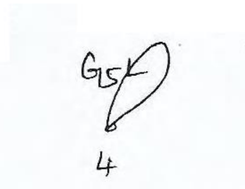
$P_2 = G_1G_2G_6$

Step 2: Individual loop gain

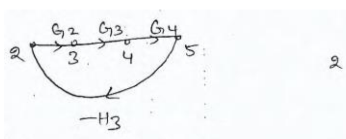
$2 \to 3 \to 2$ $\qquad$ $2 \to 3 \to 4 \to 2$ $\qquad$ $4 \to 4$

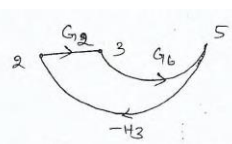






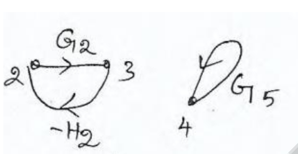
$P_{11} = -G_2H_2$ $\qquad$ $P_2 = -G_2G_3H_1$ $\qquad$ $P_{31} = G_5$

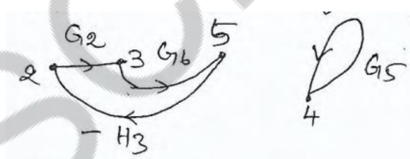




$P_{41} = -G_2G_3G_4H_3$ $\qquad$ $P_{51} = -G_2G_6H_3$

Step 3: Non touching loops – gain products. There are two pairs of non touching loops





$P_{12} = -G_2G_5H_2$ $\qquad$ $P_{22} = -G_2G_5G_6H_3$

Step 4: To find Δ and Δk

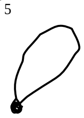
Δ=1-[$P_{11} + P_{21} + P_{31} + P_{41} + P_{51}$]+[$P_{12} + P_{22}$]

$$\Delta = 1 + G_2H_2 + G_2G_3H_1 - G_5 + G_2G_3G_4H_3 + G_2G_6H_3 - G_2G_5H_2 - G_2G_5G_6H_3$$

$\Delta_1 = 1 - 0 = 1$ Since there is no part of the graph is not touching with first forward path

$\Delta_2 = 1 - G_5 \to$ Since when forward path 2 being removed remaining part of the graph is as shown

G_5



3 4 5

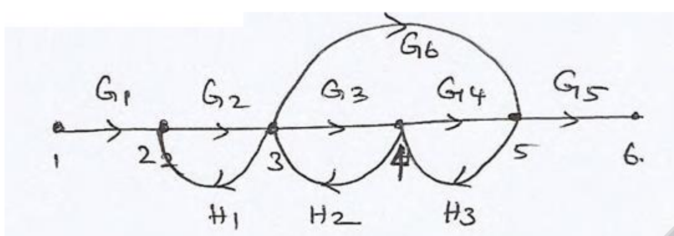
STEP 5: Transfer function:

By Mason's gain formula

$$T(S) = \frac{1}{\Delta}\sum_{K} P_K \Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$$

$$\therefore T(S) = \frac{G_1G_2G_3G_4 + G_1G_2G_6(1-G_5)}{1+G_2H_2 + G_2G_3H_1 - G_5 + G_2G_3G_4H_3 + G_2G_6H_3 - G_2G_5H_2 - G_2G_5G_6H_3}$$

3.The signal flow graph for a feedback control system is shown in fig. Determine the closed loop transfer function C(s)/R(s). Nov/Dec 2015

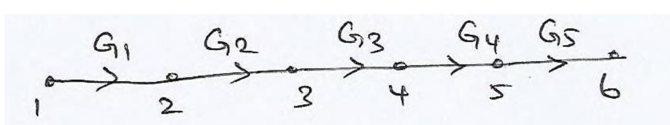


Sol:

Step : 1 forward path gains

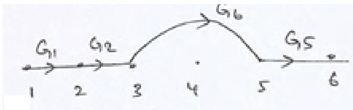
No. of forward path K=2

Path 1 1→2→3→4→5→6



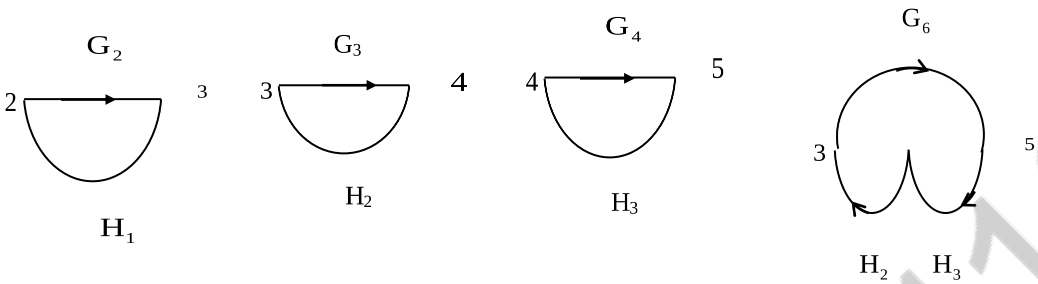
$$P_1 = G_1G_2G_3G_4G_5$$

Path 2 $\quad 1\rightarrow2\rightarrow3\rightarrow4\rightarrow5\rightarrow6$



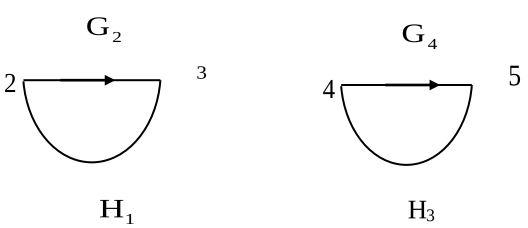
$$P_2 = G_1G_2G_5G_6$$

Step 2: Individual loop gains



$P_{11} = G_2H_1$ $\quad$ $P_{21} = G_3H_2$ $\quad$ $P_{31} = G_4H_3$ $\quad$ $P_{41} = G_6H_2H_3$

Step 3: Non touching loops gain products



$$P_{12} = G_2G_4H_1H_2$$

Step : 4 To find $\Delta + \Delta_k$

$$\Delta = 1 - [P_{11} + P_{21} + P_{31} + P_{41}] + [P_{12}]$$
$$= 1 - G_2H_1 - G_3H_2 - G_4H_3 - G_6H_2H_3 + G_2G_4H_1H_3$$
$$\Delta_1 = 1 - 0 = 1$$
$$\Delta_2 = 1 - 0 = 1$$

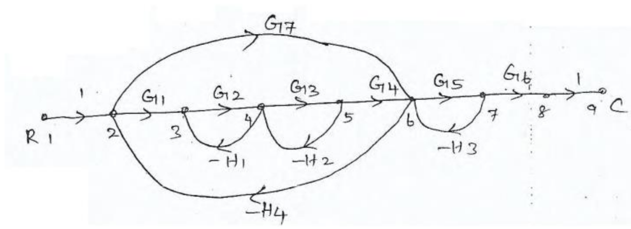
Since there is no part of the graph is not touching with first and second forward path respectively.

Step 5: Transfer function by Mason's gain formula

$$T(s) = \frac{1}{\Delta}\sum_k P_k\Delta_k - \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$$

$$\therefore T(s) = \frac{G_1G_2G_3G_4G_5 + G_1G_2G_5G_6}{1 - G_2H_1 - G_3H_2 - G_4H_3 - G_6H_2H_3 + G_2G_4H_1H_3}$$

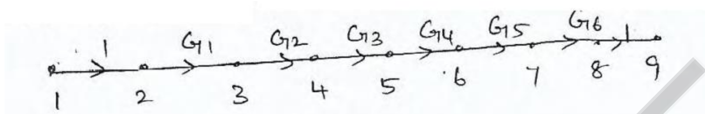
4.Obtain the overall transfer function of the following signal flow graph using Masons gain formula.



Sol: Step 1 forward path gain

No. of forward path K=2

Path 1→1→2→3→4→5→6→7→8→9



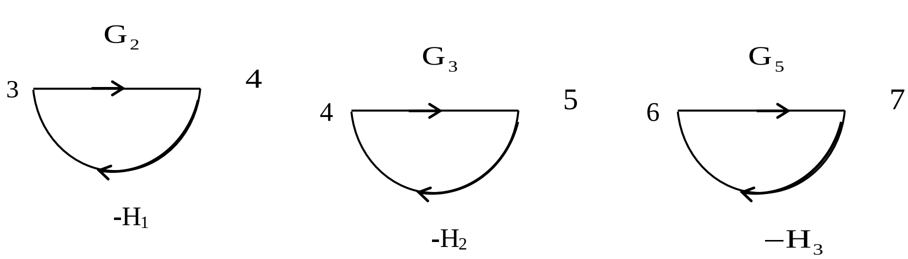
$P_1 = G_1G_2G_3G_4G_5G_6$

Path 2 →1→2→6→7→8→9

$P_2 = G_5G_6G_7$

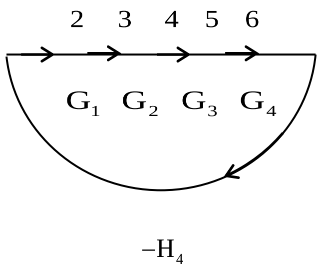
STEP:2 individual loop gains

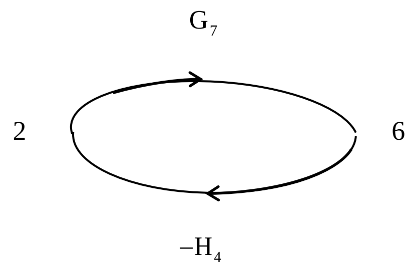
3→4→3 4→5→4 6→7→6



$P_{11} = -G_2H_1$ $P_{21} = -G_3H_2$ $P_{31} = -G_5H_3$

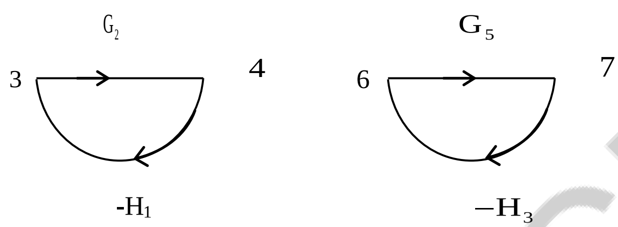
$2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 2$ $2 \rightarrow 6 \rightarrow 2$



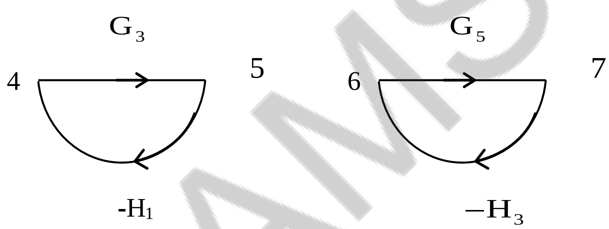


$P_{41} = -G_1G_2G_3G_4H_4$ $P_{51} = -G_7H_4$

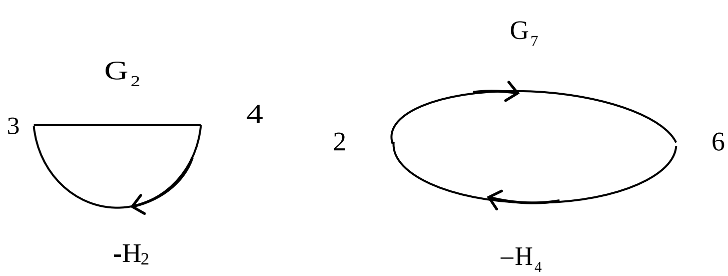
Step:3 Gain productions of non touching loops



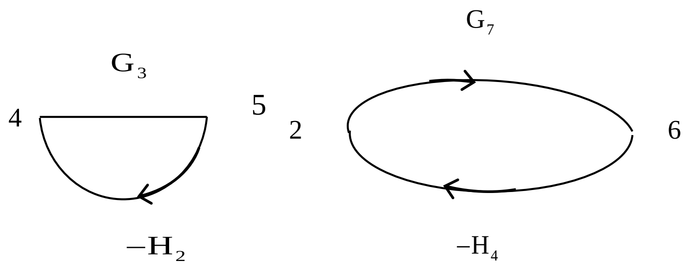
$P_{12} = G_2G_5H_1H_3$



$P_{22} = G_3G_5H_2H_3$



$$P_{32} = G_2G_7H_1H_4$$



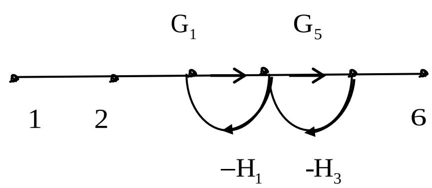
$$P_{42} = G_3G_7H_2H_4$$

Step:4 Determination of Δ & Δk

Δ_1=1; Since there is no part of the graph is not touching with first forward path

Δ_1=1-[L_1+ L_2]

$=1+ G_1H_1 + G_5H_3$



Individual loops

$L_1 = -G_1H_1$

$L_2 = -G_5H_3$

$$\Delta = 1 - [P_{11} + P_{21} + P_{31} + P_{41} + P_{51}] + [P_{12} + P_{22} + P_{32} + P_{42}]$$

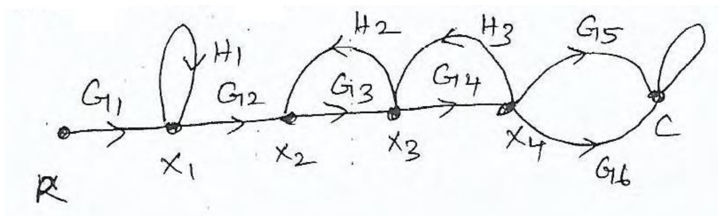
$$= 1 + G_2H_1 + G_3H_2 + G_5H_3 + G_1G_2G_3G_4H_4 + G_7H_4 + G_2G_5H_1H_3 + G_3G_5H_2H_3 + G_2G_7H_1H_4 + G_3G_7H_2H_4$$

Step:5 Transfer function by Hason's gain formula

$$T(s)=\sum_k \frac{P_k\Delta_k}{\Delta}=\frac{P_1\Delta_1+P_2\Delta_2}{\Delta}$$

$$=\frac{G_1G_2G_3G_4G_5G_6+G_5G_6G_7\left(1+G_1H_1+G_5H_3\right)}{1+G_2H_1+G_3H_2+G_5H_3+G_1G_2G_3G_4H_4+G_7H_4+G_2G_5H_1H_3+G_3G_5H_2H_3+G_2G_7H_1H_4+G_3G_7H_2H_4}$$

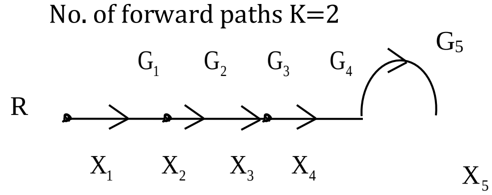
5.Determine the overall transfer function of SFG using Mason's gain formula



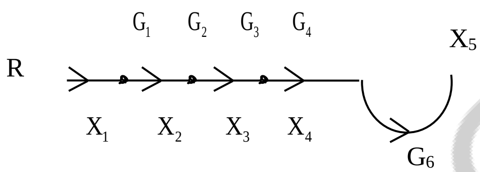
Sol

Step1: forward path gain

No. of forward paths K=2

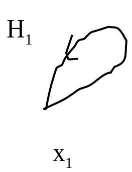


$P_1 = G_1G_2G_3G_4G_5$

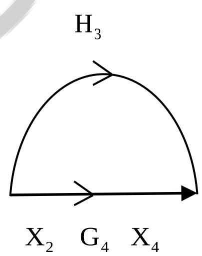


$P_1 = G_1G_2G_3G_4G_6$

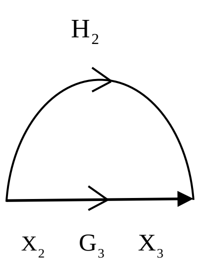
Step2: Individual loop gains



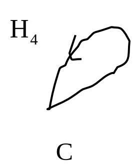
$P_{11} = H_1$



$P_{31} = G_4H_3$



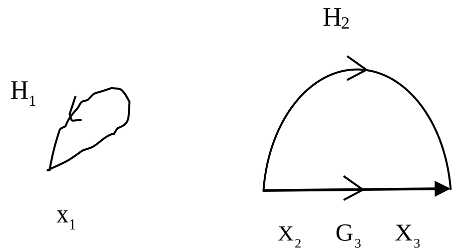
$P_{21} = G_3\,H_2$



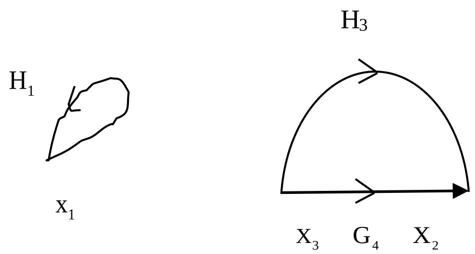
$P_{41} = H_4$

Step:3 Gain products of non touching loops Two non touching loops

Two non touching loops



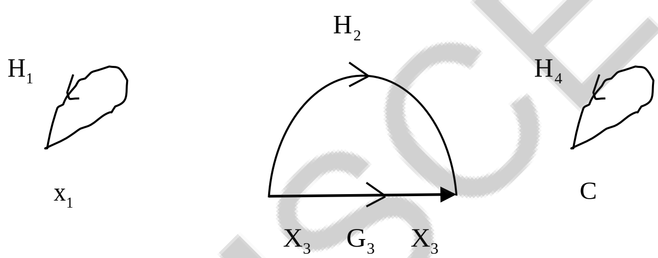
$P_{12} = H_1H_2G_3$



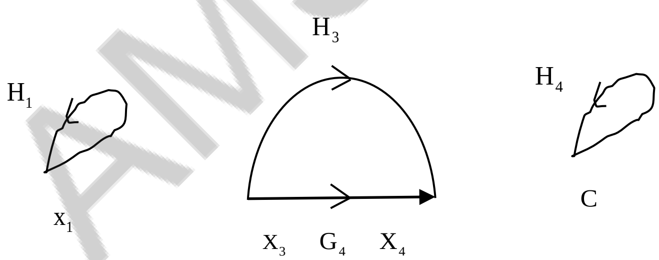
$P_{22} = H_1H_3G_4$

$P_{32} = H_1H_4$

Three non touching loops



$P_{13} = G_3H_1H_2H_4$



$P_{23} = G_4H_1H_3H_4$

Step :4 Determination of Δ and Δk

$$= 1 - [P_{11} + P_{21} + P_{31} + P_{41}] + [P_{12} + P_{22} + P_{32}] - [P_{13} + P_{23}]$$
$$= 1 - H_1 - G_3H_2 - G_4H_3 - H_4 + H_1H_2G_3 + H_1H_3G_4 + H_1H_4 - G_3H_1H_2H_4 - G_4H_1H_3H_4$$

$\Delta_1 = \Delta_2 = 1$ Since there is no part of the graph is not touching with forward path s

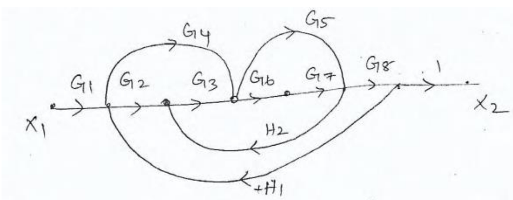
Step:5 Transfer function

By Mason's gain formula $T(s) = \sum_k \frac{p_k \Delta_k}{\Delta}$

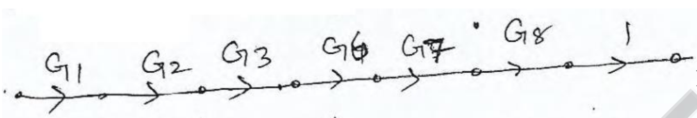
$$T(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$T(s) = \frac{G_1G_2G_3G_4G_5 + G_1G_2G_3G_4G_6}{1 - H_1 - G_3H_2 - G_4H_3 - H_4 + H_1H_2G_3 + H_1H_3G_4 + H_1H_4 - G_3H_1H_2H_4 - G_4H_1H_3H_4}$$

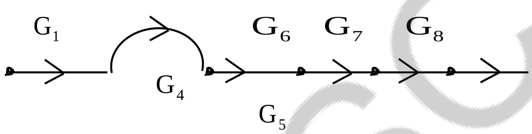
6. Using Mason's gain formula to find $\frac{x_2}{x_1}$



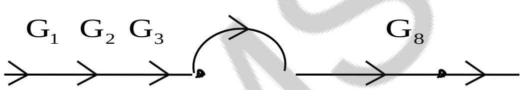
1. No. of forward paths k=4



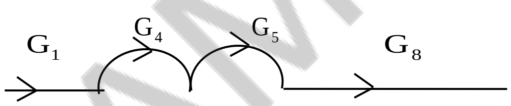
$P_1 = G_1G_2G_3G_6G_7G_8$



$P_2 = G_1G_4G_6G_7G_8$

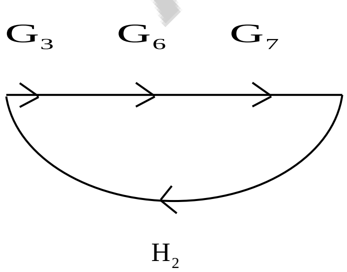


$P_3 = G_1G_2G_3G_5G_8$

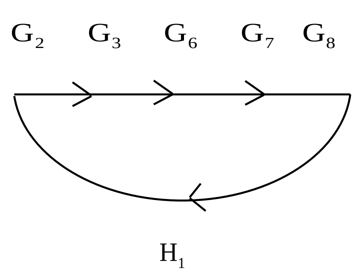


$P_4 = G_1G_4G_5G_8$

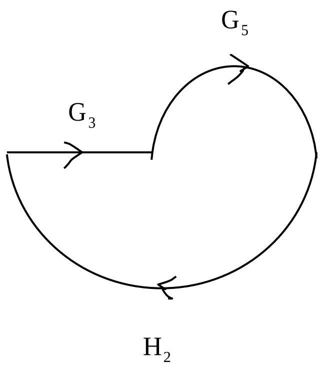
2. Individual loops and gains



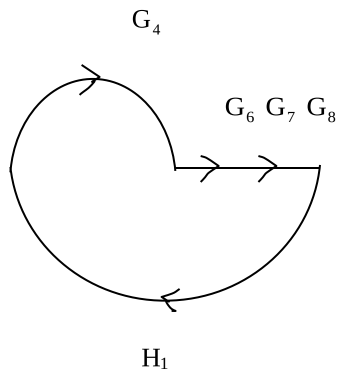
$P_{11} = G_2G_6G_7H_2$



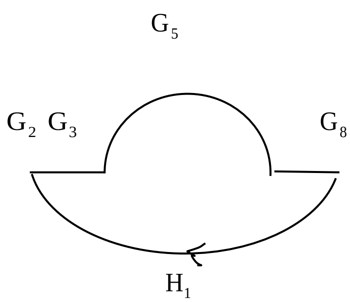
$P_{21} = G_2G_3G_6G_7G_8H_1$



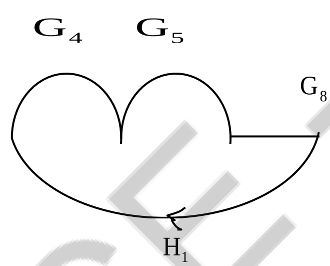
$P_{31} = G_3G_5H_2$



$P_{41} = G_4G_6G_7G_8H_1$



$P_{51} = G_2G_3G_5G_8H_1$



$P_{61} = G_4G_5G_8H_1$

Non touching loops →Nil

Δ and Δk

Δ=1-[$P_{11} + P_{21} + P_{31} + P_{41} + P_{51} + P_{61}$]

=1-$G_2G_6G_7H_2 + G_2G_3G_6G_8H_1 - G_3G_5H_2 - G_4G_6G_7G_8H_1 - G_2G_3G_5G_8H_1 - G_4G_5G_8H_1$

$\Delta_1 = 1$

$\Delta_2 = 1$

$\Delta_3 = 1$

$\Delta_4 = 1$

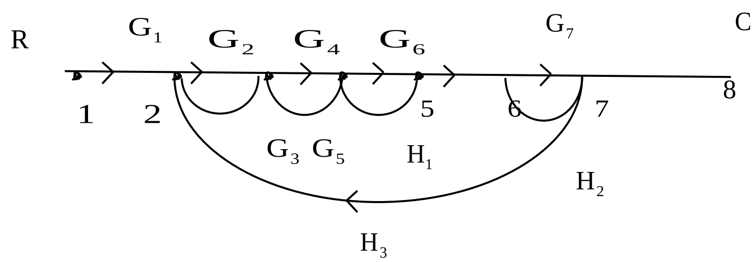
Since there is no part of the graph is not touching with forward paths

Transfer function by Mason's gain formula $T(s) = \sum_k \frac{P_k \Delta_k}{\Delta}$

$$T(s) = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4}{\Delta}$$

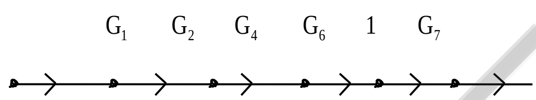
$$T(s) = \frac{G_1G_2G_3G_6G_7G_8 + G_1G_4G_6G_7G_8 + G_1G_2G_3G_5G_8 + G_1G_4G_5G_8}{1 - G_2G_6G_7H_2 - G_2G_3G_6G_7G_8H_1 - G_3G_5H_2 - G_4G_6G_7G_8H_1 - G_2G_3G_5G_8H_1 - G_4G_5G_8H_1}$$

7.Find L/R using Mason's gain formula

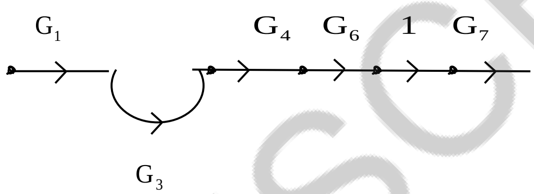


Sol :

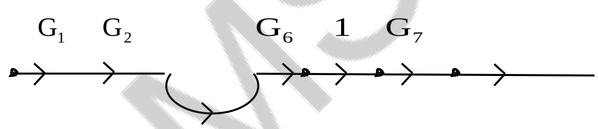
1. No. of forward path and forward path gains k=4.



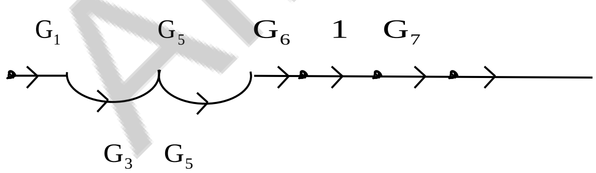
$P_1 = G_1G_2G_4G_6G_7$



$P_2 = G_1G_3G_4G_6G_7$

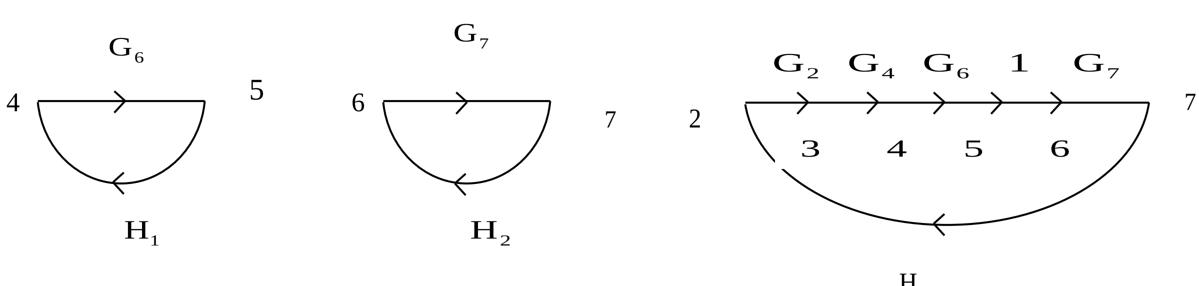


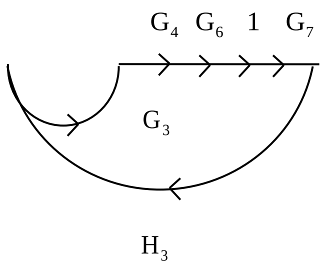
$P_3 = G_1G_2G_5G_6G_7$



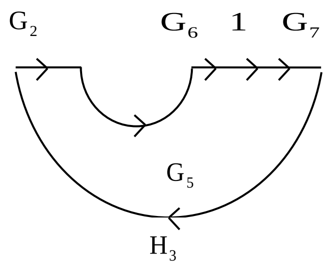
$P_4 = G_1G_3G_5G_6G_7$

2.Individual loops and gain

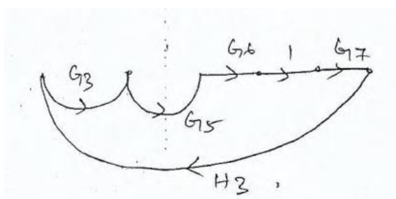




$P_{41} = G_3G_4G_6G_7H_3$

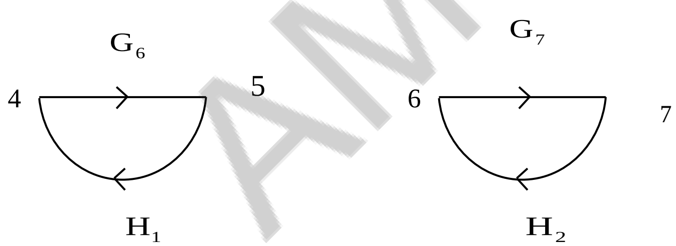


$P_{51} = G_2G_5G_6G_7H_3$



$P_{51} = G_3G_5G_6G_7H_3$

3.Non touching loops



$P_{12} = G_6G_7H_1H_2$

4. Δ and Δ_k

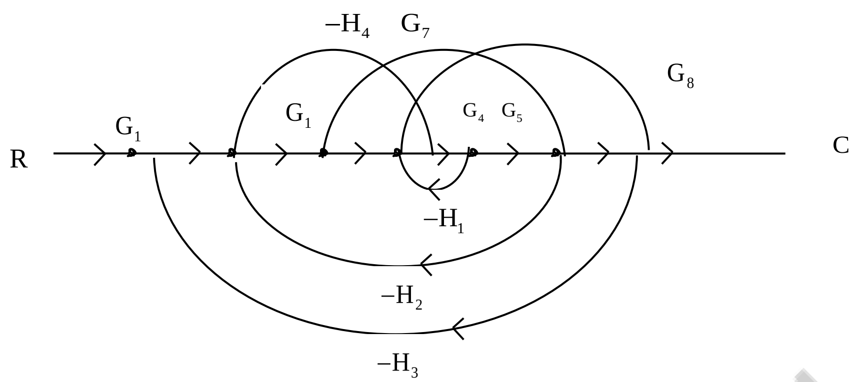
$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$ [Since there is no part of the graph is not touching the forward paths]

5.Transfer function by Mason's gain formula

$$T(S) = \sum_K \frac{p_K \Delta_K}{\Delta} = \frac{p_1\Delta_1 + p_2\Delta_2 + p_3\Delta_3 + p_4\Delta_4}{\Delta}$$

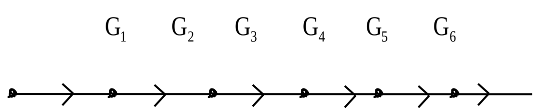
$$= \frac{G_1G_2G_4G_6G_7 + G_1G_3G_4G_6G_7 + G_1G_2G_5G_6G_7 + G_1G_3G_5G_6G_7}{1 - G_6H_1 - G_7H_2 - G_2G_4G_6G_7H_3 - G_3G_4G_6G_7H_3 - G_2G_5G_6G_7H_3 - G_3G_5G_6G_7H_3 + G_6G_7H_1H_2}$$

8. Find L/R using Mason's gain formula

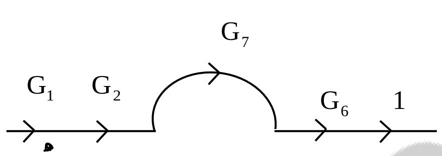


Sol

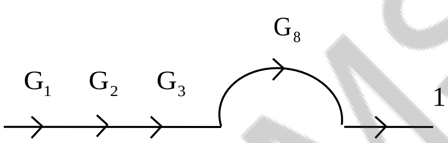
1. No. of forward path and gain K=3



$P_1 = G_1G_2G_3G_4G_5G_6$

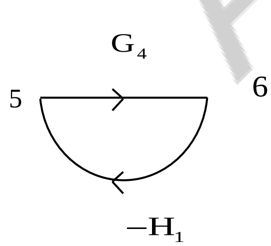


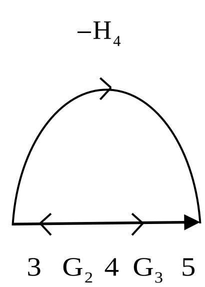
$P_2 = G_1G_2G_7G_6$

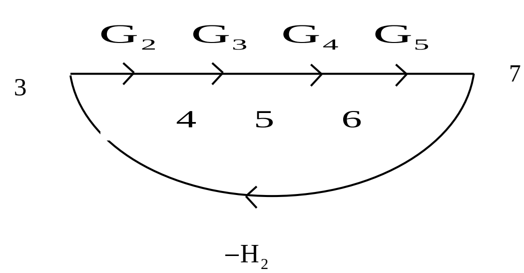


$P_3 = G_1G_2G_3G_8$

2. Individual loops and gain



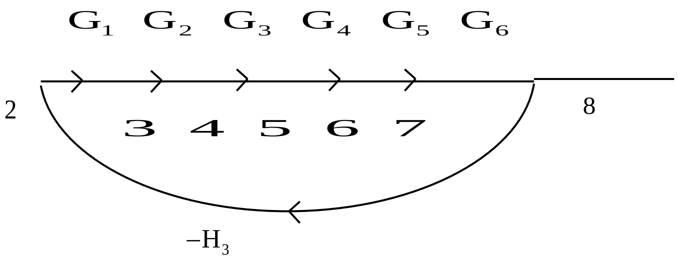




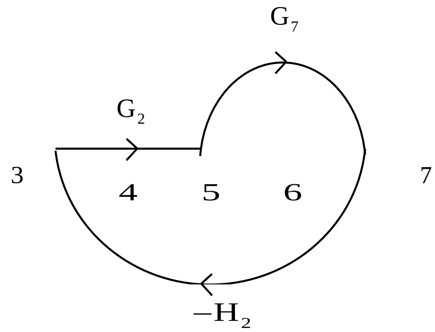
$P_{11} = -G_4H_1$

$P_{21} = -G_2G_3H_4$

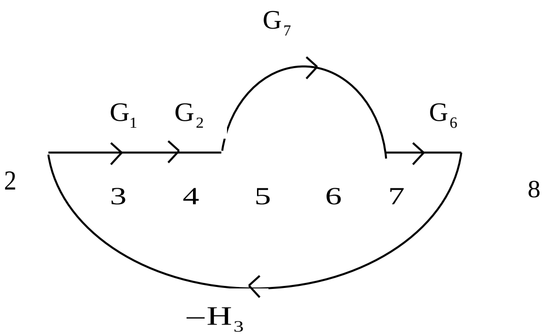
$P_{31} = -G_2G_3G_4G_5H_2$



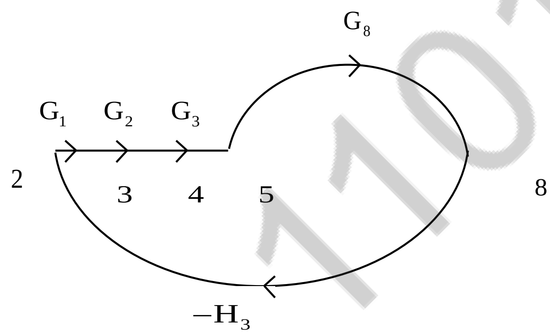
$P_{41} = -G_1G_2G_3G_4G_5G_6H_3$



$P_{51} = -G_2G_7H_2$

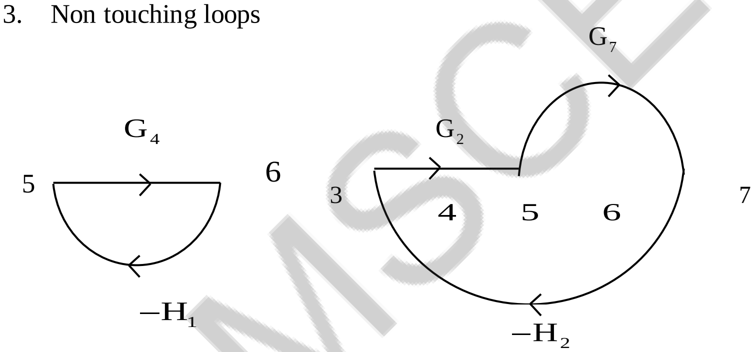


$P_{61} = -G_1G_2G_7G_6H_3$

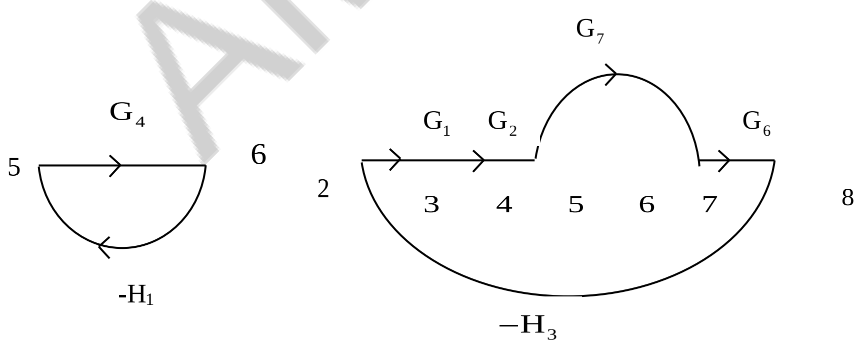


$P_{71} = -G_1G_2G_3G_8H_3$

3. Non touching loops



$P_{12} = G_4G_2G_7H_1H_2$



$P_{22} = G_1G_2G_4G_6G_7H_1H_3$

4. Δ and Δ_k

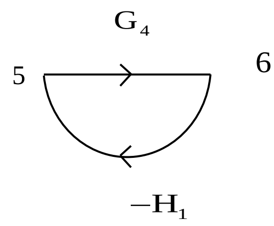
$\Delta = 1-[P_{11}+P_{21}+P_{31}+P_{41}+P_{51}+P_{61}+P_{71}]+[P_{12}+P_{22}\}$

$\Delta = 1+G_4H_1+G_2G_3H_4+G_2G_3G_4G_5H_2+G_1G_2G_3G_4G_5G_6H_3+G_2G_7H_2+G_1G_2G_7G_6H_3+G_1G_2G_3G_8H_3+G_2G_4G_7H_1H_2+G_1G_2G$

$\Delta_1 = 1;$

$\Delta_3 = 1;$ since there is no part of the graph is not touching with forward path 1 and 3

$\Delta_2 = 1+G_4H_1;$ when forward path 2 being removed, remaining part of the graph is as shown



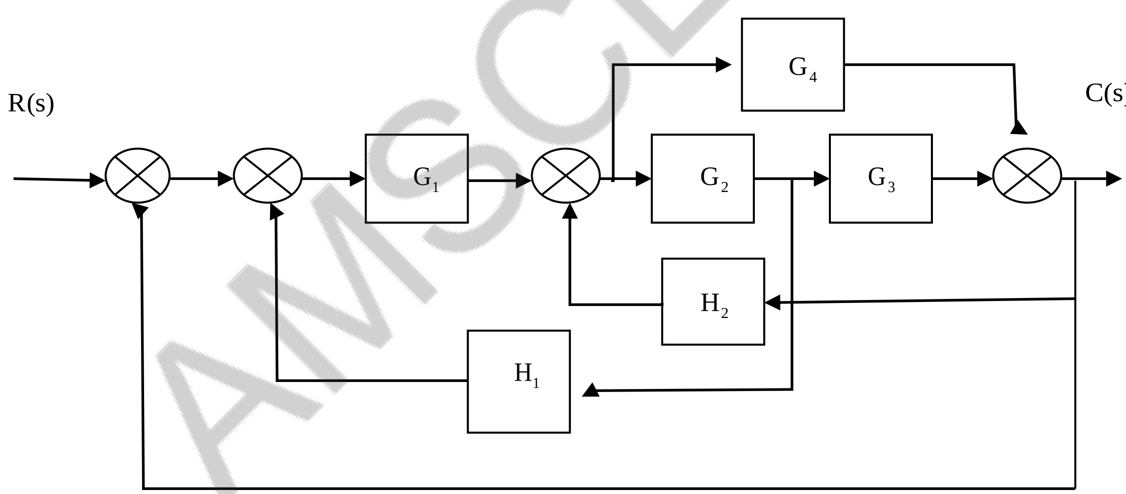
5.Transfer function By Mason's gain formula

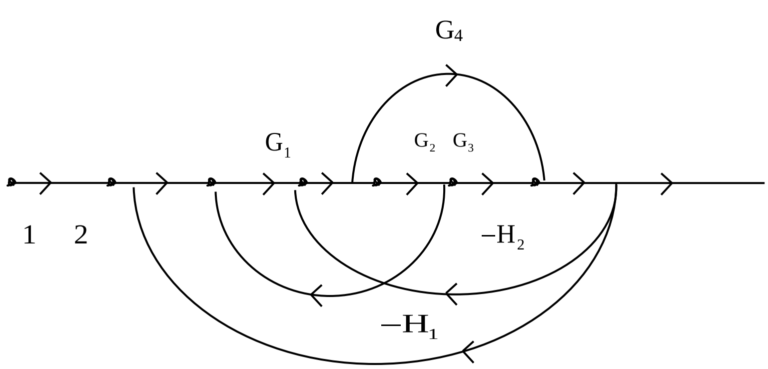
$$T(S) = \sum_K \frac{p_K\Delta_K}{\Delta} = \frac{p_1\Delta_1+p_2\Delta_2+p_3\Delta_3}{\Delta}$$

$$= \frac{G_1G_2G_3G_4G_5G_6+G_1G_2G_6G_7(1+G_4H_1)+G_1G_2G_3G_8}{1+G_4H_1+G_2G_3H_4+G_2G_3G_4G_5H_2+G_2G_7H_2+G_1G_2G_3G_4G_5G_6H_3+G_1G_2G_3G_8H_8+G_2G_4G_7H_1H_2+G_1G_2G_4G_6G_7H_1H_3}$$

CONVERSION OF BLOCK DIAGRAM TO SIGNAL FLOW GRAPH

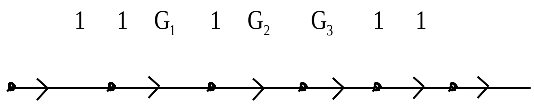
1. Convert the block diagram into signal flow graph and find the transfer function using Masons gain formula
   Procedure:
   1.Select a node for every branch point and summing point, input and output signal in block diagram
   2.for each block have a line segment on which its gain is written with direction



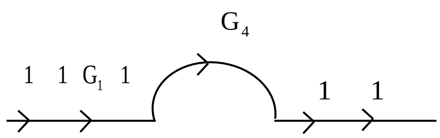


Sol :

1. No. of forward path and forward path gains k=2.

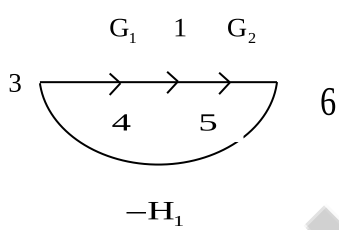


$P_1 = G_1G_2G_3$

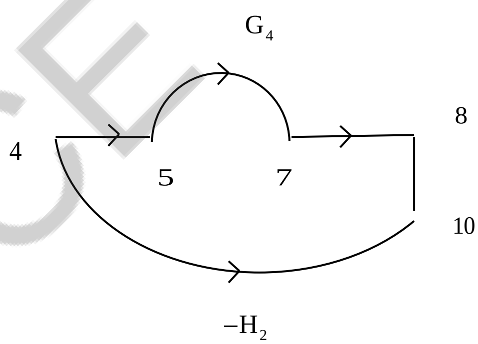


$P_2 = G_1G_4$

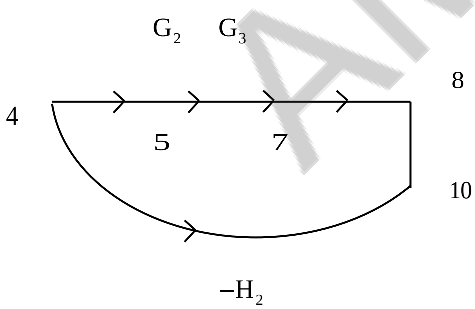
2.Individual loops and gain



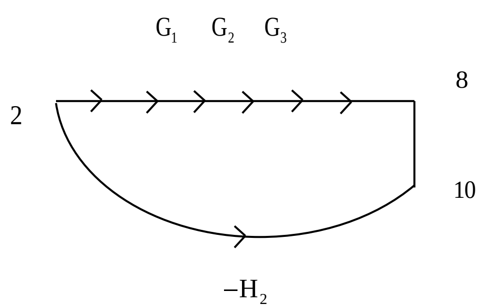
$P_{11} = -G_1G_2H_1$



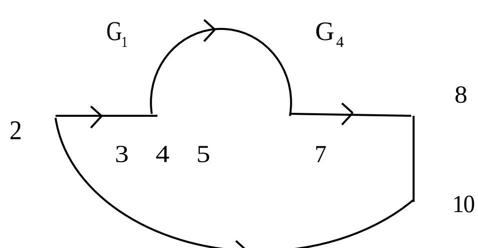
$P_{21} = -G_4H_2$



$p_{31} = -G_2G_3H_2$



$P_{41} = -G_1G_2G_3$



$P_{51} = -G_1G_4$

3.Non touching loops - NIL

Δ and Δk

4. $\Delta = 1 - [P_{11} + P_{21} + P_{31} + P_{41} + P_{51}]$

$= 1 + G_1G_2H_1 + G_4H_2 + G_2G_3H_2 + G_1G_2G_3 + G_1G_4$

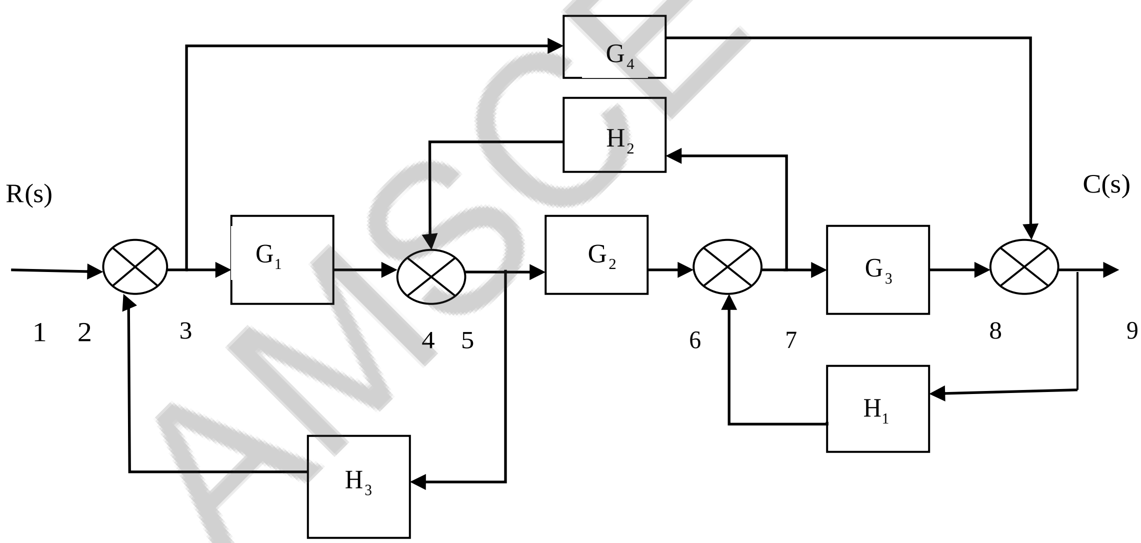
$\Delta_1 = \Delta_2 = 1$ [Since there is no part of the graph is not touching with the forward paths]

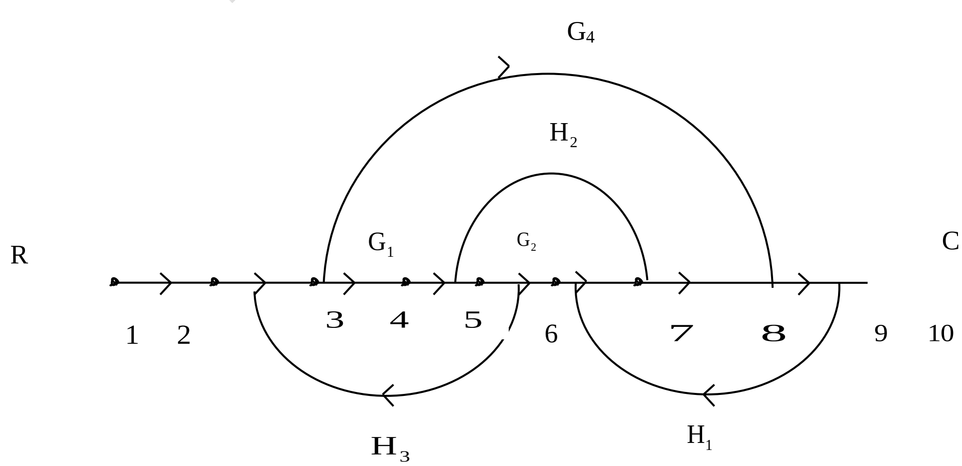
5. Transfer function by Mason's gain formula

$$T(s) = \sum_k \frac{P_k\Delta_k}{\Delta} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_4 + G_1G_2G_3 + G_4H_2}$$

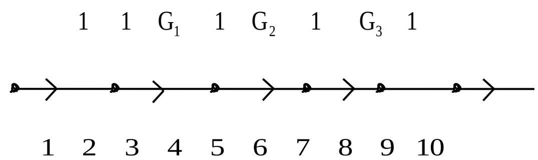
2.Convert block diagram to signal flow graph and find the transfer function using Mason's gain formula



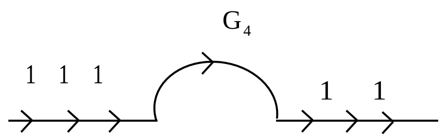


SOL:

1. No. of forward path and forward path gains k=2.

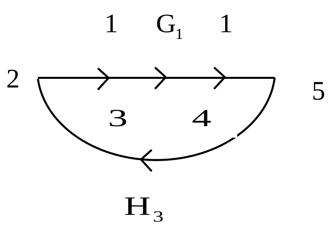


$P_1 = G_1G_2G_3$

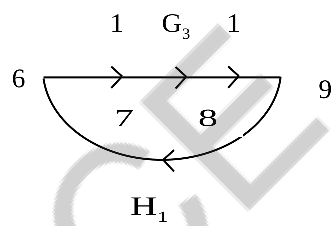


$P_2 = G_4$

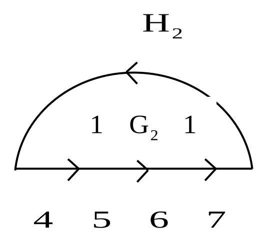
2.Individual loops and gain



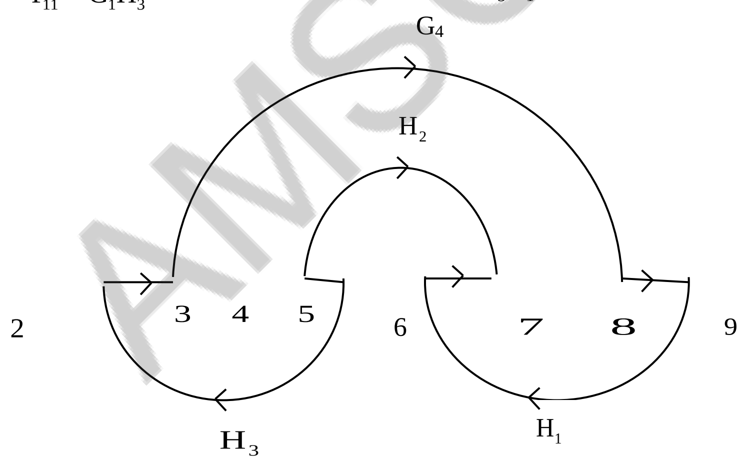
$P_{11} = G_1H_3$



$P_{21} = G_3H_1$

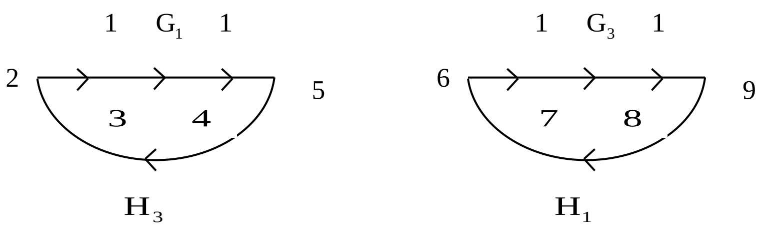


$P_{31} = G_2H_2$



$P_{41} = G_4H_1H_2H_3$

3.Non touching loops



$P_{12} = G_1G_3H_1H_3$

4. Δ and Δ_{ki}

$$\Delta = 1 - [P_{11} + P_{21} + P_{31} + P_{41}] + [P_{12}]$$
$$= 1 - G_1H_3 - G_3H_1 - G_2H_2 - G_4H_1H_2H_3G_1G_3H_1H_3$$

$\Delta_1 = 1$ [Since there is no part of the graph is not touching with the forward paths 1]

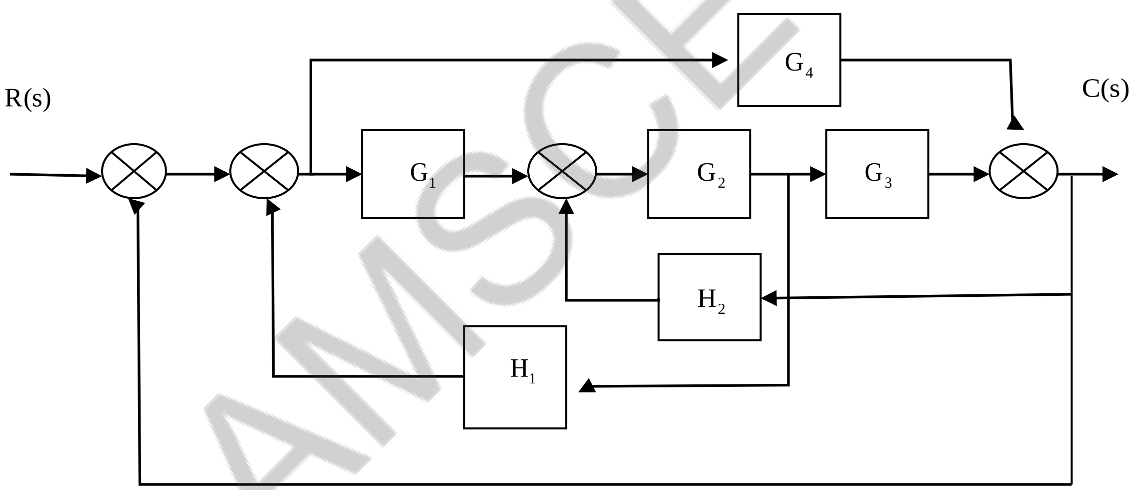
$\Delta_2 = 1 - G_2H_2$; When forward path 2 being removed, remaining part of the graph is as shown u Diagram

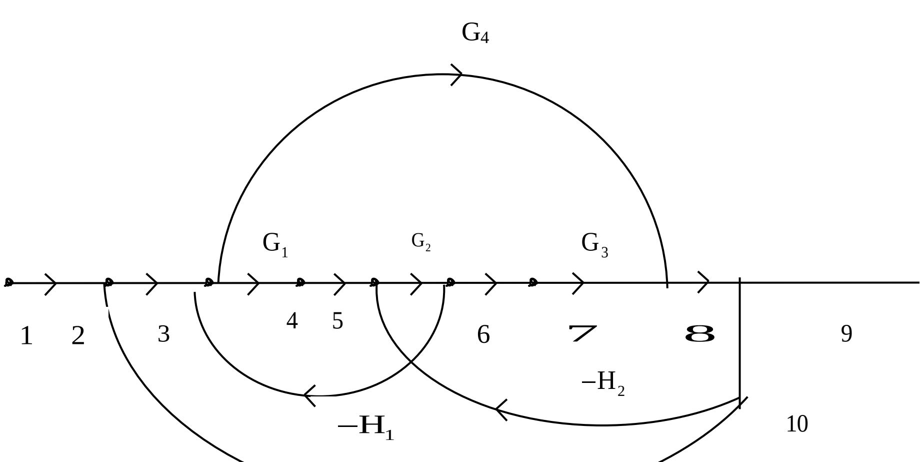
5. Transfer function By Masons gain formula

$$T(s) = \sum_k \frac{P_k\Delta_k}{\Delta} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3 + G_4(1 - G_2H_2)}{1 + G_1H_3 + G_3H_1 - G_4H_1H_2H_3 + G_1G_3H_1H_3}$$

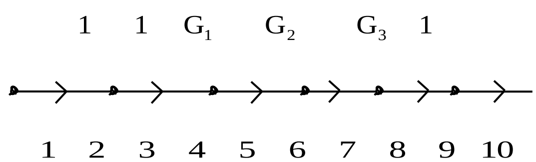
3.Construct an equivalent signal flow graph for the block diagram shown in fig and evaluate the transfer function



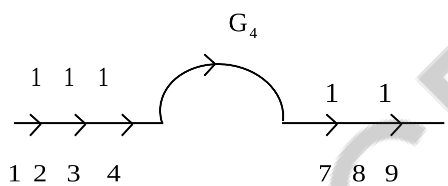


Sol

1. No. of forward path and forward path gains k=2.

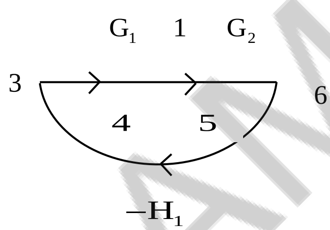


$P_1 = G_1G_2G_3$

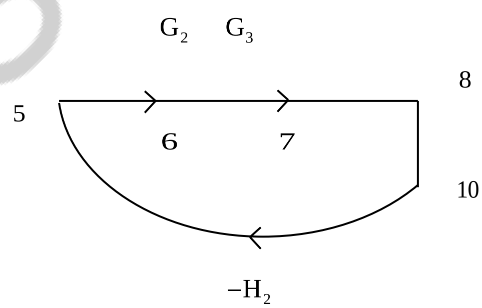


$P_2 = G_4$

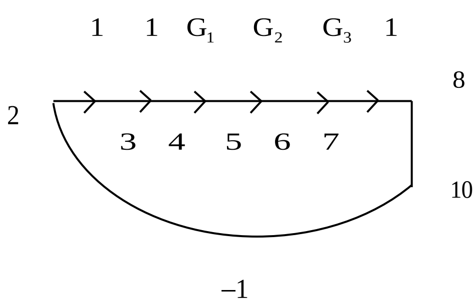
2. .Individual loops and gain



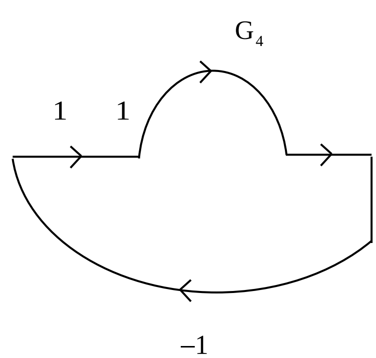
$P_{11} = -G_1H_1$



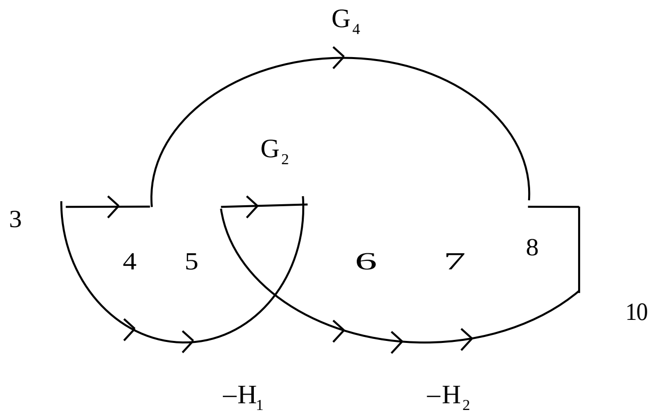
$P_{21} = -G_2G_3H_2$



$P_{31} = -G_1G_2G_3$



$P_{41} = -G_4$



$$P_{51} = G_2G_4H_1H_2$$

3 . Non touching loops - Nil

Δ and Δk

4. $\Delta = 1 - [P_{11} + P_{21} + P_{31} + P_{41} + P_{51}]$

$= 1 + G_1H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4 - G_2G_4H_1H_2$

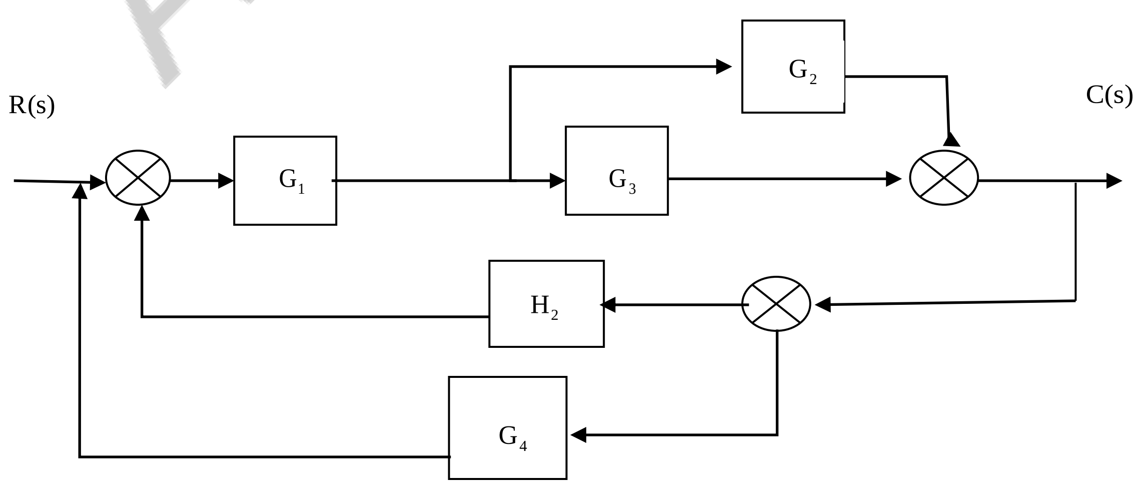
$\Delta_1 = \Delta_2 = 1$ [All the loops are touching the two forward paths]

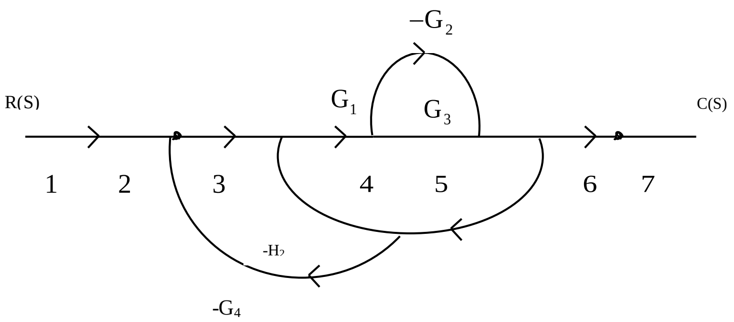
5.Transfer function By Masons gain formula

$$T(s) = \frac{1}{\Delta}\sum_k P_k\Delta_k = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

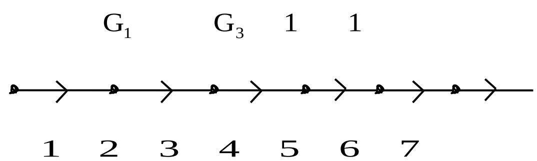
$$= \frac{G_1G_2G_3 + G_4}{1 + G_1G_2H_1 + G_2G_3H_2 - G_1G_2G_3 + G_4 - G_2G_4H_1H_2}$$

4.Draw the signal flow graph and evaluate the closed loop transfer of a system whose block diagram is shown in fig

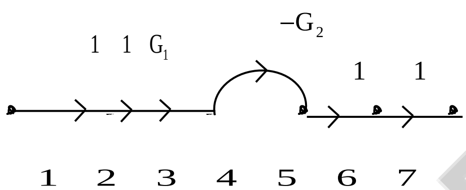




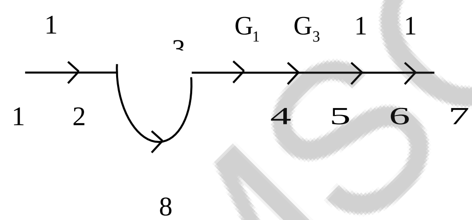
1. No. of forward path and forward path gains k=4.



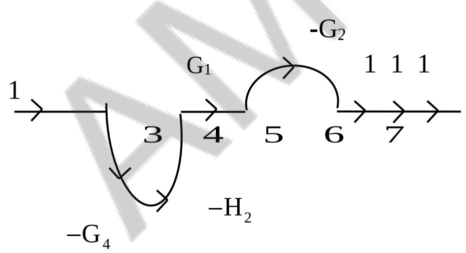
$P_1 = G_1G_3$



$P_2 = -G_1G_2$

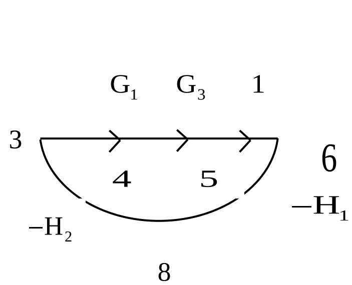


$P_3 = G_1G_3G_4H_2$

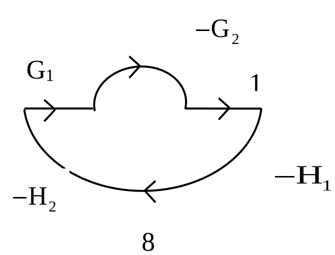


$P_4 = -G_1G_2G_4H_2$

2.Individual loops and gain



$P_{11} = G_1G_3H_2H_1$



$P_{21} = -G_1G_2H_1H_2$

3 . Non touching loops - Nil

4. Δ and Δk

$$\Delta = 1 - [P_{11} + P_{21}] = 1 + G_1G_2H_1H_2 - G_1G_3H_1H_2$$

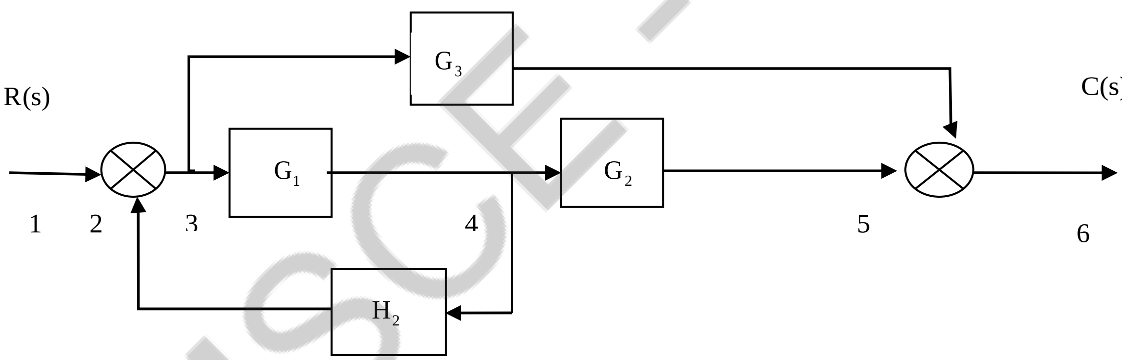
$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$ [All the loops are touching the two forward paths]

5.Transfer function By Masons gain formula

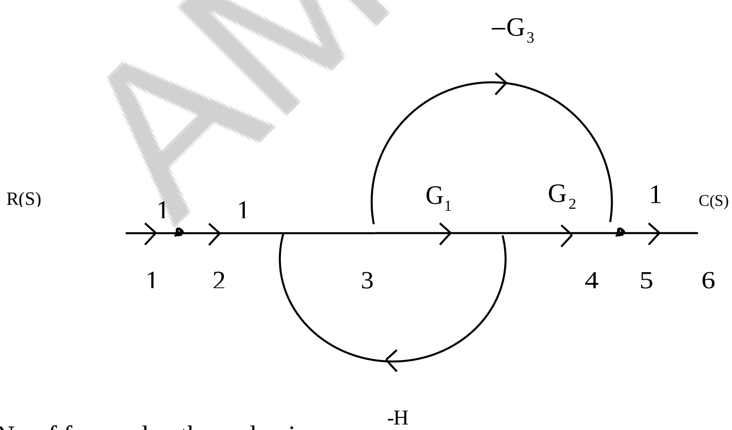
$$T(s) = \frac{1}{\Delta}\sum_k P_k\Delta_k = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4}{\Delta}$$

$$= \frac{G_1G_3 + G_1G_3G_4H_2 - G_1G_2 - G_1G_2G_4H_2}{1 + G_1G_2H_1H_2 - G_1G_3H_1H_2}$$

5.Convert the block diagram shown in fig, to signal flow graph and find the transfer function, using Mason's gain formula. Verify with block diagram approach MAY/JUNE 2016

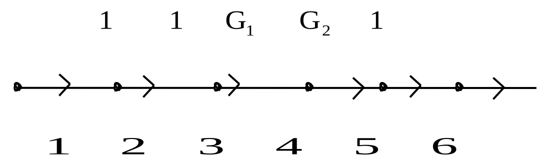


SOL

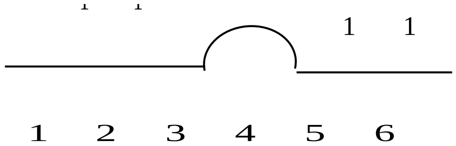


Sol

1. No of forward paths and gain

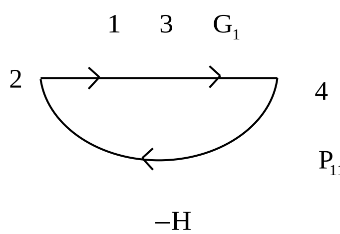


$P_1 = G_1 G_2$



$P_2 = -G_3$

2. Individual loops and loop gain



$P_{11} = -G_1 H_1$

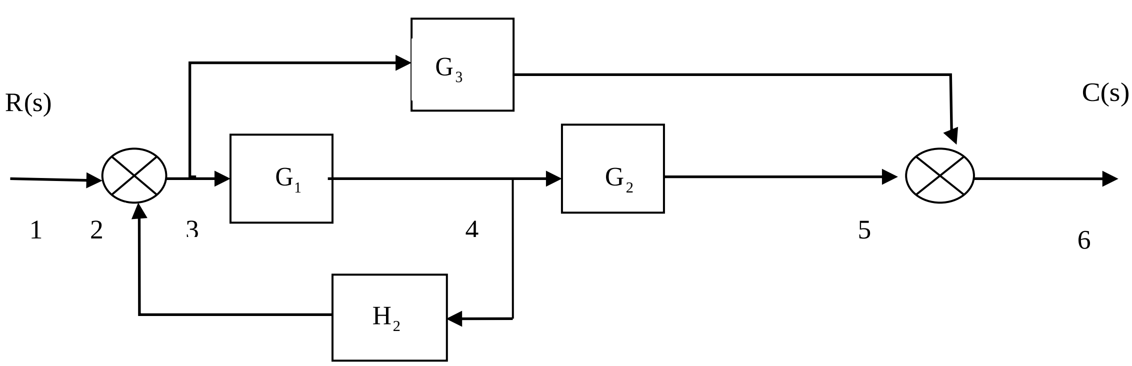
3. Non touching loops – Nil
   Δ and Δk
4. $\Delta = 1 - P_{11} = 1 + G_1 H$
   $\Delta_1 = \Delta_2 = 1$
   As all the loops are touching the forward paths

5.Transfer function By Masons gain formula

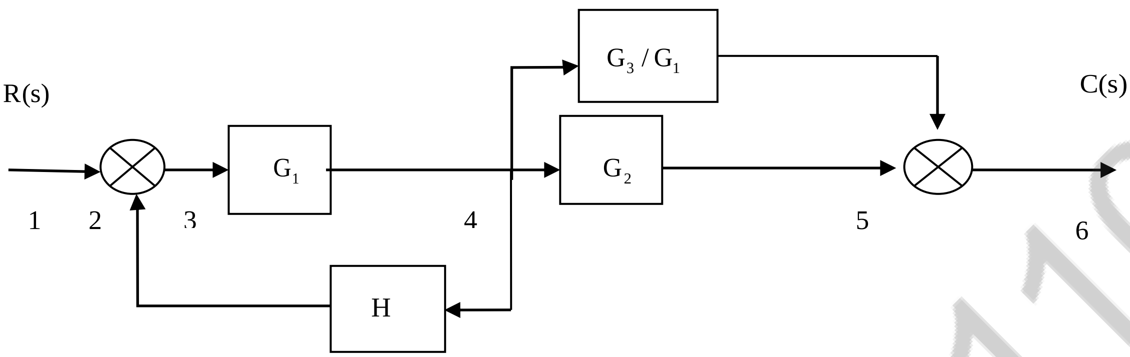
$$T(s) = \frac{1}{\Delta} \sum_k P_k \Delta_k = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

$$= \frac{G_1 G_2 - G_3}{1 + G_1 H_1}$$

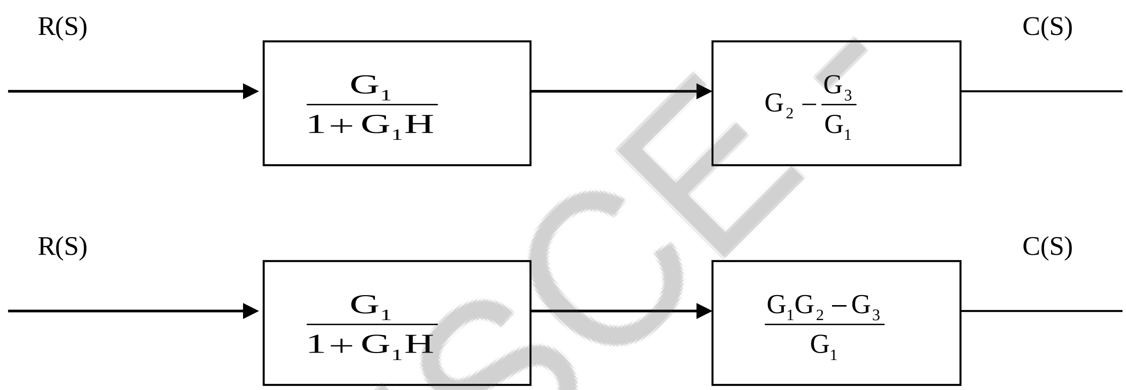
Verification by block diagram reduction technique



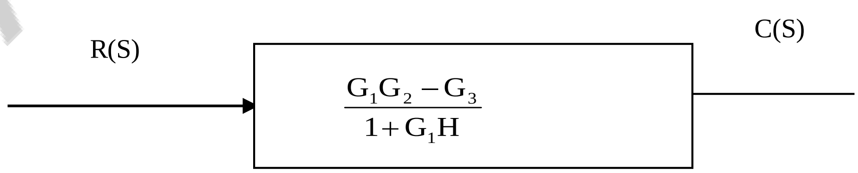
Step :1 Moving a branch point ahead of block G_1



Step:2 Eliminating –ve feedback and combining the parallel path



Step:3 Combining the cascade blocks

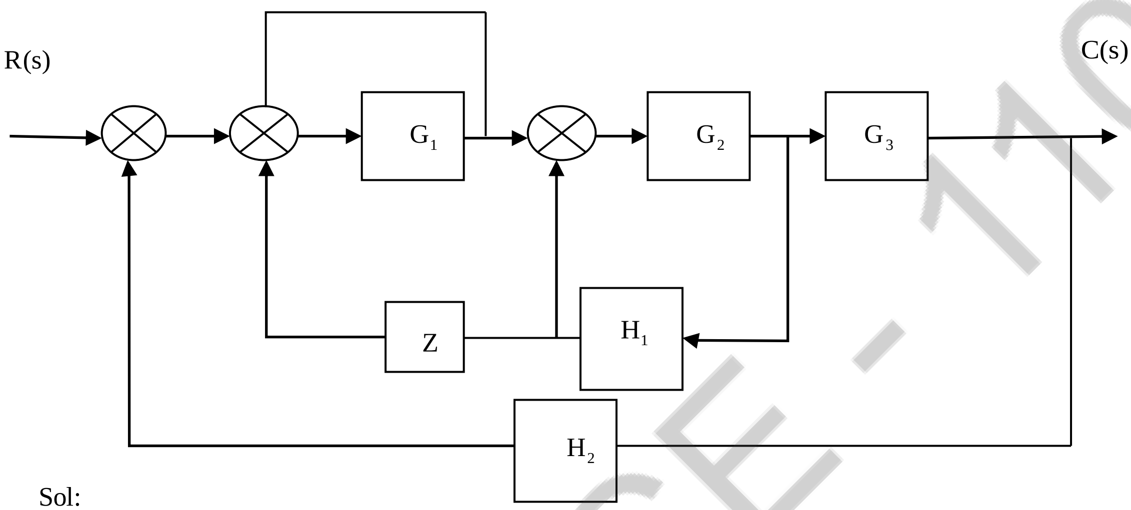


$$\therefore \frac{C(s)}{R(s)} = \frac{G_1G_2 - G_3}{1+G_1H} \qquad (2)$$

Eqn (1) and (2) are equal. Hence Verified.

6.Find the transfer function of the system shown un fig. by block diagram reduction technique and signal flow graph technique

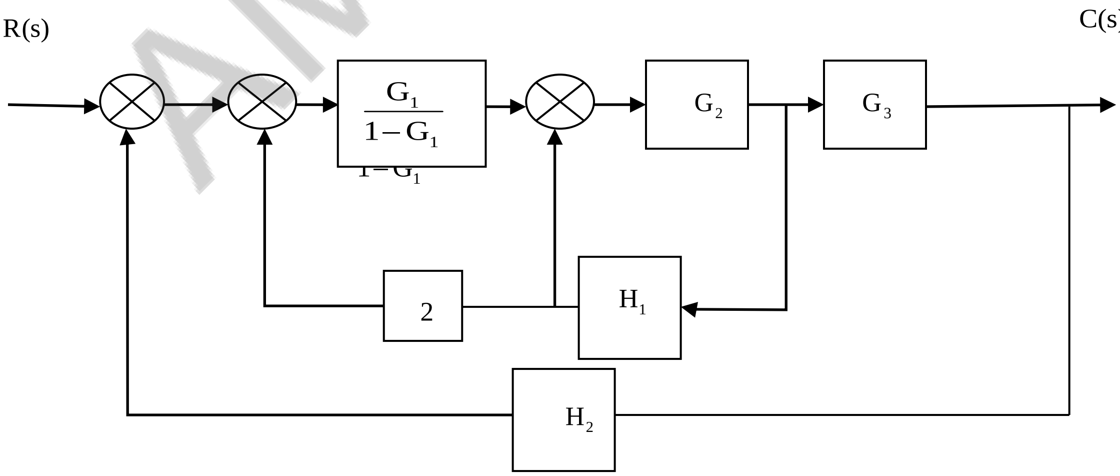
April/ May 2015



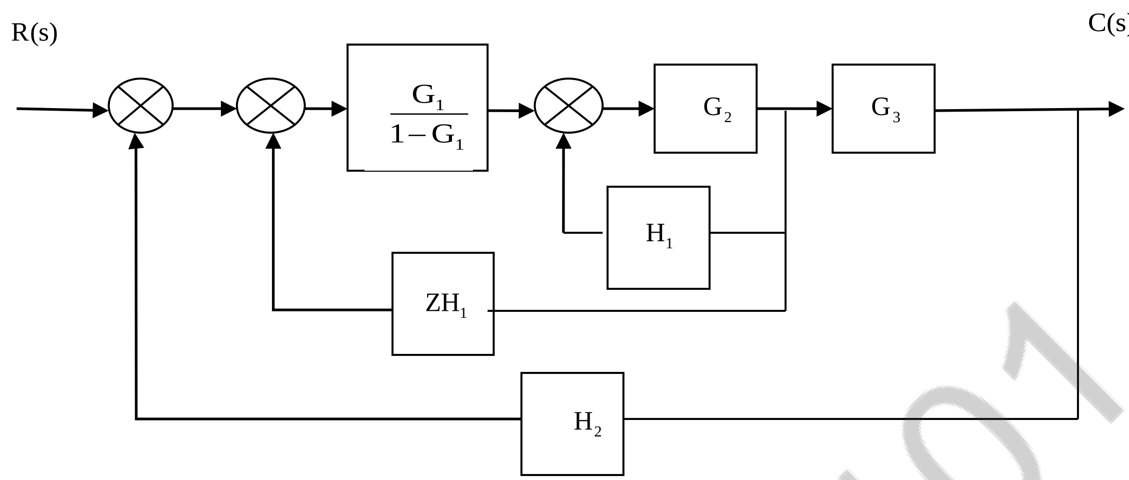
Sol:

By block diagram reduction technique

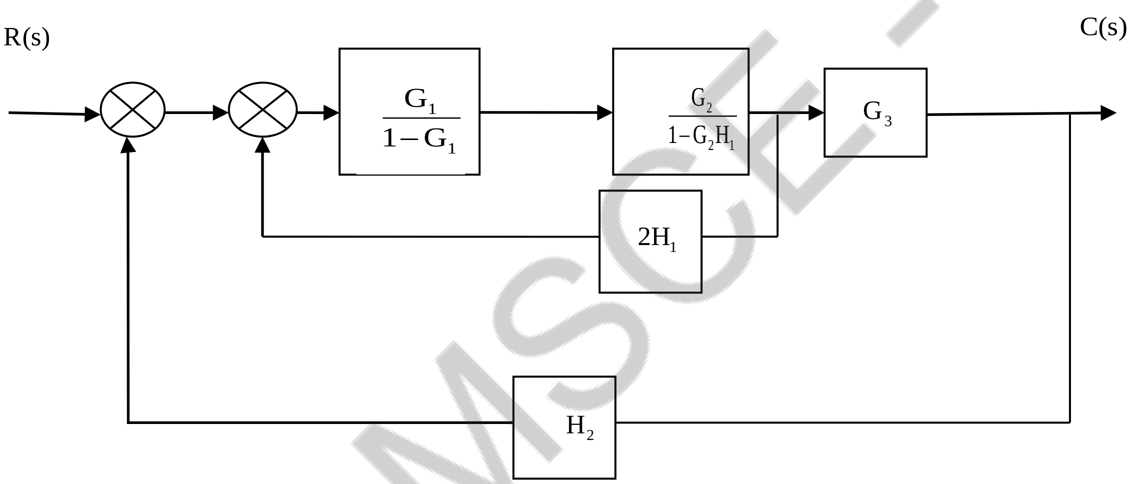
Step:1 Removing Unity feedback



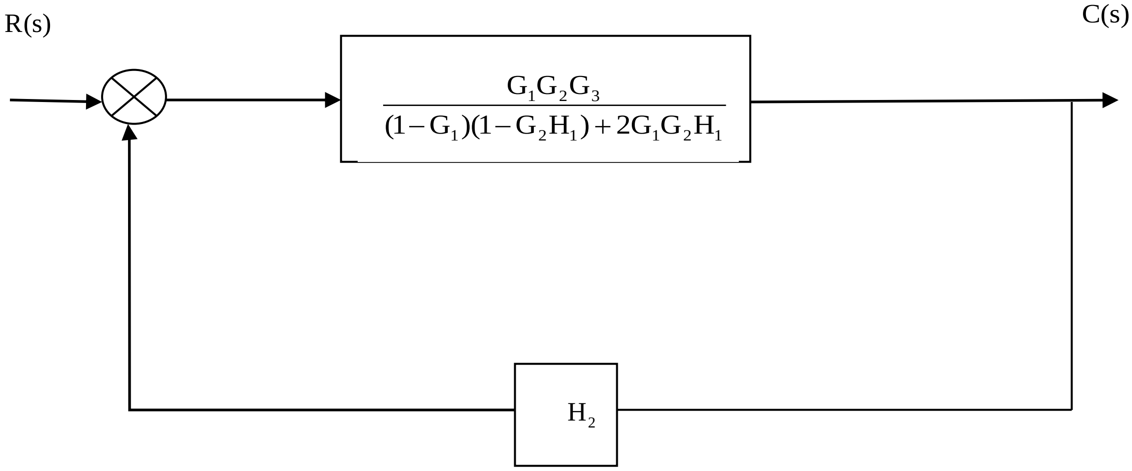
Step:2 Moving the branch point before H_1



Eliminating of feedback H_1



Combining the cascade blocks combining the cascade blocks



Eliminating the feedback path H_2

$$G \rightarrow \frac{G_1G_2G_3}{(1-G_1)(1+G_2H_1)+2G_1G_2H_1} \qquad H \rightarrow H_2$$

$$\frac{G}{1+GH} \rightarrow \frac{\dfrac{G_1G_2G_3}{(1-G_1)(1+G_2H_1)+2G_1G_2H_1}}{1+\dfrac{G_1G_2G_3H_2}{(1-G_1)(1+G_2H_1)+2G_1G_2H_1}}$$

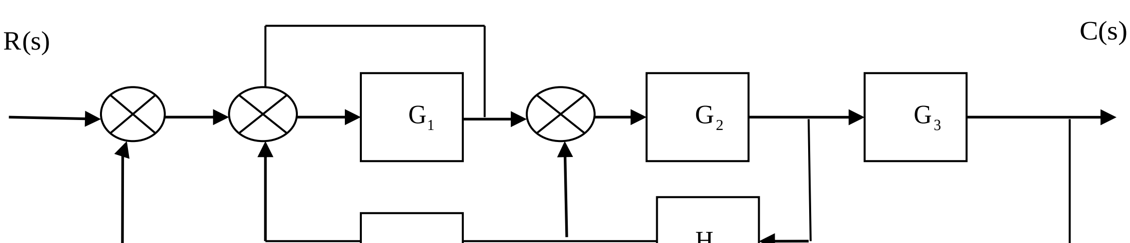
$$= \frac{G_1G_2G_3}{(1-G_1)(1+G_2H_1)+2G_1G_2H_1+G_1G_2G_3H_2}$$

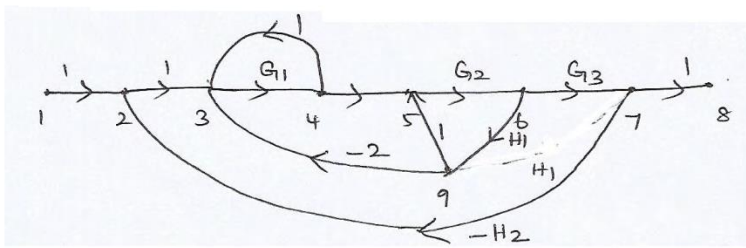
$$= \frac{G_1G_2G_3}{1-G_1-G_2H_1+3G_1G_2H_1+G_1G_2G_3H_2}$$

$$R(s) \rightarrow \boxed{\frac{G_1G_2G_3}{1-G_1-G_2H_1+3G_1G_2H_1+G_1G_2G_3H_2}} \rightarrow C(s)$$

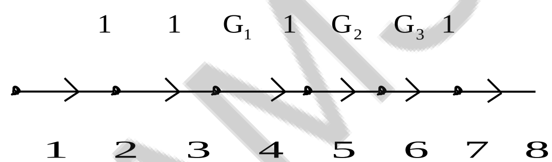
$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1-G_1-G_2H_1+3G_1G_2H_1+G_1G_2G_3H_2}$$

(ii) by signal flow graph technique



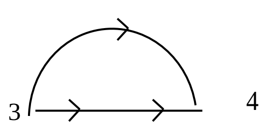


Step:1No of forward path and forward path gain K=1

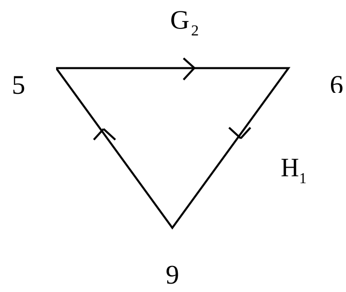


$P_1 = G_1G_2G_3$

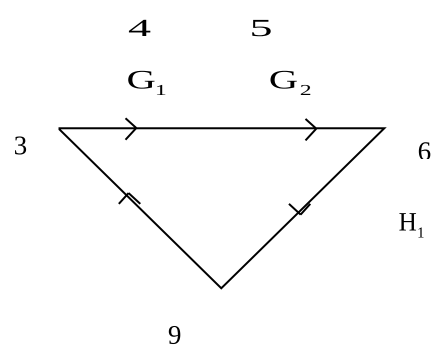
Step:2 Individual loops and individual loop gains



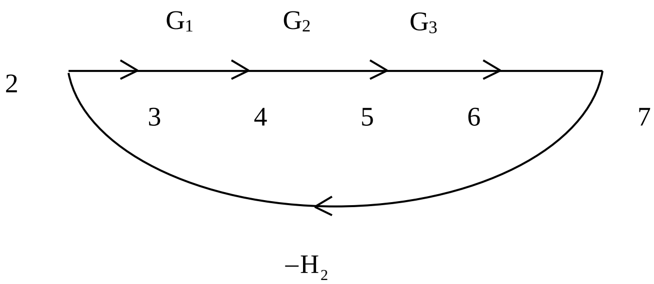
$P_{11} = G_1$



$P_{21} = G_2H_1$

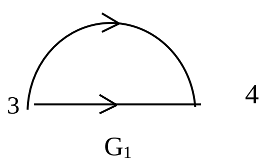


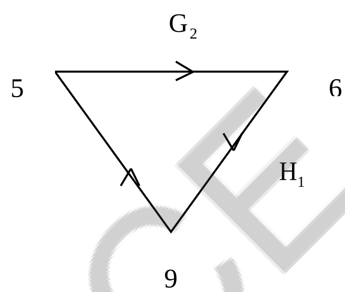
$P_{31} = 2G_1G_2H_1$



$P_{41} = -G_1G_2G_3H_2$

Step:3 Non touching loops – gain products





$P_{12} = G_1G_2H_1$

Step:4 To find

Δ and Δk

$\Delta = 1 - \left[P_{11} + P_{21} + P_{31} + P_{41}\right] + P_{12}$

$= 1 - G_1 - G_2H_1 + 2G_1G_2H_1 + G_1G_2G_3H_2 + G_1G_2H_1$

$= 1 - G_1 - G_2H_1 + 3G_1G_2H_1 + G_1G_2G_3H_2$

$\Delta_1 =$ as all the loops are touching the forward paths

Step:5 Transfer function by Mason's gain formula,

$$T(S) = \frac{1}{\Delta}\sum_{K} p_K \Delta_K = \frac{p_1\Delta_1}{\Delta}$$

$$= \frac{G_1G_2G_3}{1 - G_1 - G_2H_1 + 3G_1G_2H_1 + G_1G_2G_3H_2}$$

$$\therefore T(s) = \frac{G_1G_2G_3}{1 - G_1 - G_2H_1 + 3G_1G_2H_1 + G_1G_2G_2H_2}$$ Hence verified

CONSTRUCTION OF SIGNAL FLOW GRAPH FROM THE SYSTEM EQUATIONS

STEPS:

1.Obtain the system equations by writing differential equations governing the system

2.Represent each variable by a separate node.

3.Use the property that value of the variable represented by a node is an variable represented by a node is an algebraic sum of all the signals entering at that node, to simulate the equations

4. Coeffcients of the variables in the equations are to be represented as the brancg gain, joining the nodes in signal flow graph.

5.Show the input and output variables separately to completely signal flow graph

1. Construct signal flow graph for the set of linear equations and determine overall transfer for using Mason's gain formula

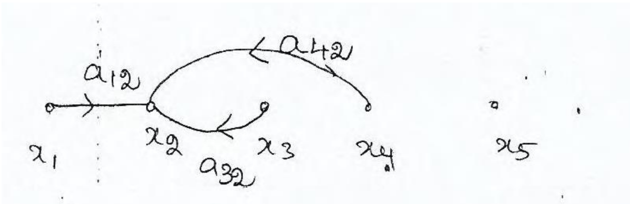
$$x_2 = a_{12}x_1 + a_{32}x_3 + a_{42}x_4 \qquad (1)$$
$$x_3 = a_{23}x_2 \qquad (2)$$
$$x_4 = a_{24}x_2 + a_{34}x_3 + a_{44}x_4 \qquad (3)$$
$$x_5 = a_{25}x_2 + a_{45}x_4 \qquad (4)$$

Step:1

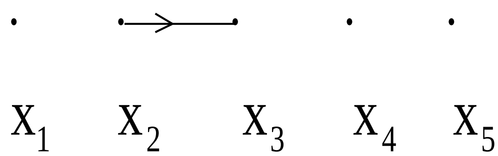
. . . . .

x_1 x_2 x_3 x_4 x_5

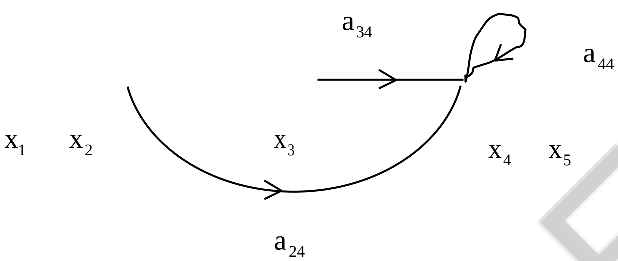
Eqn 1



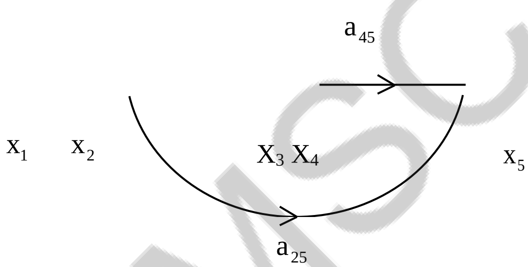
Eqn 2



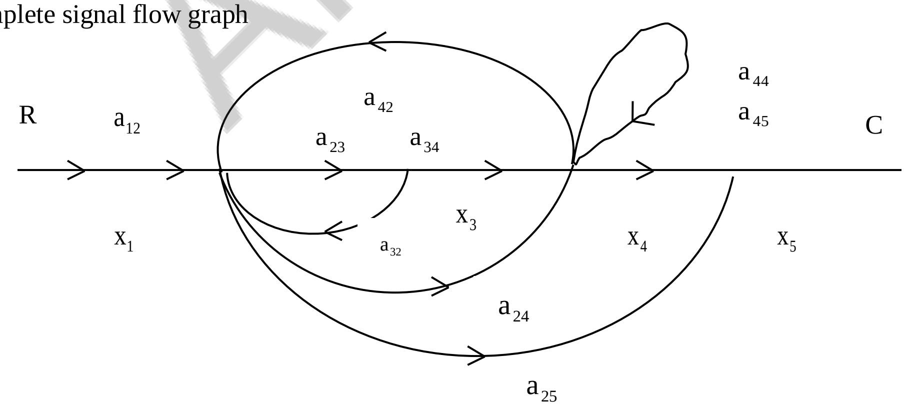
Eqn 3



Eqn 4



Complete signal flow graph



1. No. of forward paths and forward path gain k=3

$P_1 = a_{12}a_{23}a_{34}a_{45}\left[x_1 \to x_2 \to x_3 \to x_4 \to x_5\right]$

$P_2 = a_{12}a_{24}a_{45}\left[x_1 \to x_2 \to x_4 \to x_5\right]$

$P_3 = a_{12}a_{25}\left[x_1 \to x_2 \to x_5\right]$

2. Individual loops and gain

$P_{11} = a_{23}a_{32}\left[x_2 \to x_3 \to x_2\right]$

$P_{21} = a_{23}a_{34}a_{42}\left[x_2 \to x_3 \to x_4 \to x_2\right]$

$P_{31} = a_{24}a_{42}\left[x_2 \to x_4 \to x_2\right]$

$P_{41} = a_{44}\left[x_4\right]$

3. Non touching loops

$P_{12} = P_{11,}P_{41} = a_{32}a_{23}a_{44}$

4. Δ and Δk

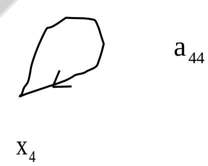
$\Delta = 1 - \left[P_{11} + P_{21} + P_{31} + P_{41}\right] + P_{12}$

$= 1 - \left[a_{32}a_{23} + a_{23}a_{34}a_{42} + a_{24}a_{42} + a_{44} + a_{23}a_{32}a_{44}\right]$

$\Delta_1 = 1$

$\Delta_2 = 1$

$\Delta_3 = 1 - a_{44}$



5. Transfer function by Mason's gain formula

$$T(s) = \sum_k \frac{P_k \Delta_k}{\Delta} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta}$$

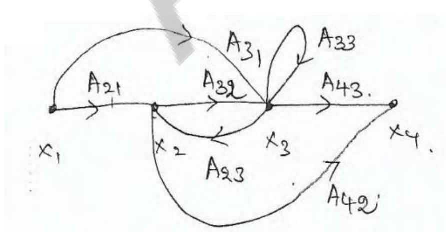
$$T(s) = \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{24}a_{45} + a_{12}a_{25}(1 - a_{44})}{1 - [a_{23}a_{32} + a_{23}a_{34}a_{42} + a_{24}a_{42} + a_{44}] + a_{23}a_{32}a_{44}}$$

2.Construct the signal flow graph for the following set of simultaneous equations

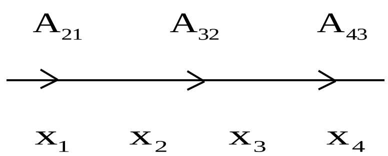
$X_2 = A_{21}X_1 + A_{23}X_3;\ \ X_3 = A_{31}X_1 + A_{32}X_2 + A_{33}X_3;$

$X_4 = A_{42}X_2 + A_{43}X_3$

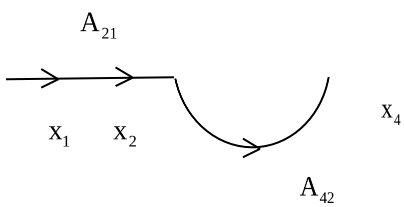
And obtain the overall transfer function using Mason's gain formula



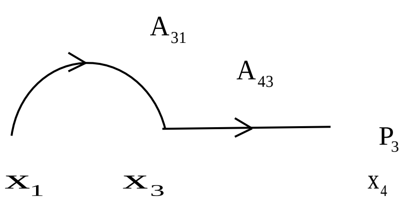
1. No of forward paths and gain



$P_1 = A_{21}A_{32}A_{43}$

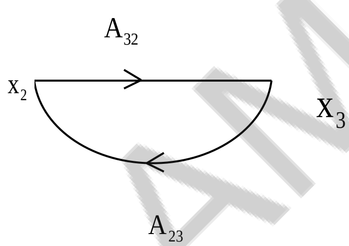


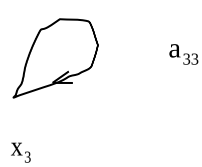
$P_2 = A_{21}A_{42}$



$P_3 = A_{31}A_{43}$

2. Individual loops and gain





$P_{11} = A_{23}A_{32}$

$P_{21} = A_{33}$

3. Non touching loops = Nil

4. Δ and Δk

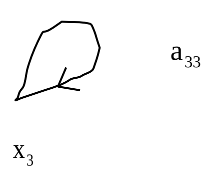
$\Delta = 1 - [P_{11} + P_{21}]$

$= 1 - [A_{32}A_{23} + A_{33}]$

$\Delta_1 = 1$

$\Delta_2 = 1 - A_{33}$

$\Delta_3 = 1$



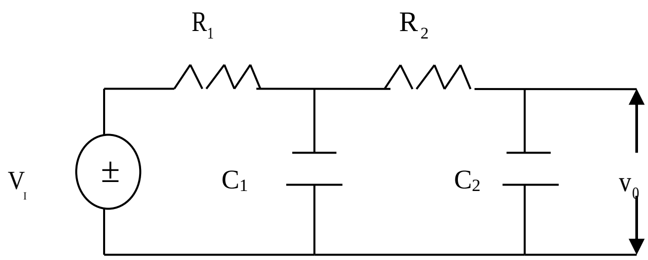
5. Transfer function By Mason's gain formula

$$T(s)=\sum_k \frac{P_k\Delta_k}{\Delta}=\frac{P_1\Delta_1+P_2\Delta_2+P_3\Delta_3}{\Delta}$$

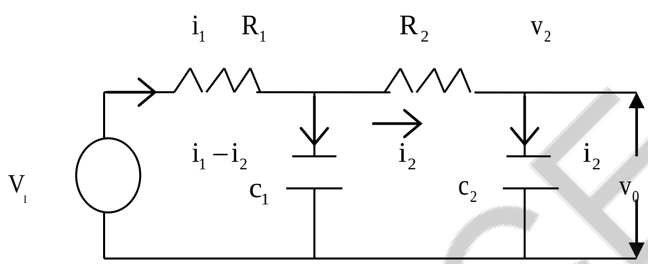
$$\frac{A_{21}A_{32}A_{43}+A_{21}A_{42}(1+A_{33})+A_{31}A_{43}}{1-[A_{32}A_{23}+A_{33}]}$$

V_0 / V_1

3.For the network shown below, draw the signal flow graph and find transfer function Mason;s gain formula.



Sol



Consider current through series element and voltage across the shunt element.

$V_i, i_1, V_1, i_2, V_2, V_o$

$i_1, v_1, i_2, v_2 \rightarrow$ mixed nodes

Nodes/Variables are

$V_i \rightarrow$ input node

$V_0 \rightarrow$ output node

Current through R_1

$$i=\frac{V_i-V_1}{R_1}$$

$$=\frac{V_i}{R_1}-\frac{V_1}{R_1}$$

$$I_1(s)=\frac{1}{R_1}V_i(s)-\frac{1}{R_1}V_1(s) \qquad (1)$$

Voltage across C_1

$$V_1 = \frac{1}{C_1}\int (i_1 - i_2)dt$$

$$V_1(s) = \frac{1}{C_1 s}\left[I_1(s) - I_2(s)\right]$$

$$V_1(s) = \frac{1}{C_1 s} I_1(s) - \frac{1}{C_1 s} I_2(s) \qquad (2)$$

current through R_2

$$i_2 = \frac{V_1 - V_2}{R_2}$$

$$I_2(s) = \frac{V_1(s) - V_2(s)}{R_2}$$

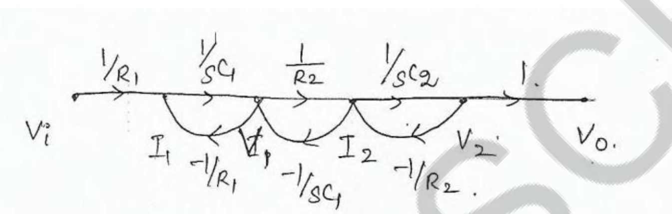
$$I_2(s) = \frac{1}{R_2} V_1(s) - \frac{1}{R_2} V_2(s) \qquad (3)$$

Voltage across C_2

$$V_2 = \frac{1}{C_2}\int i_2 dt$$

$$V_2 = \frac{1}{C_2 s} I_2(s) \qquad (4)$$

Construction of signal flow graph is as follows



$$-\frac{1}{R_1} \qquad -\frac{1}{SC_1} \qquad -\frac{1}{R_2}$$

$$-\frac{1}{R_1} \qquad -\frac{1}{SC_1} \qquad -\frac{1}{R_2}$$

1. No. of forward path and gains

$$V_i \rightarrow I_1 \rightarrow V_1 \rightarrow I_2 \rightarrow V_2 \rightarrow V_0$$

$$P_1 = \frac{1}{R_1}.\frac{1}{sC_1}.\frac{1}{R_2}.\frac{1}{sC_2} = \frac{1}{R_1R_2C_1C_2s^2}$$

K=1

2. Individual loops and gain

$$I_1 \rightarrow V_2 \rightarrow I_1 \qquad P_{11} = \frac{-1}{R_1C_1s}$$

$$V_2 \rightarrow I_2 \rightarrow V_2 \qquad P_{21} = \frac{-1}{R_2C_2s}$$

$$I_2 \rightarrow V_3 \rightarrow I_2 \qquad P_{31} = \frac{-1}{R_2C_2s}$$

3. Non touching loops

P_{11} and P_{31} are non touching loop pair

$$P_{11} = P_{11}P_{31} = \frac{1}{R_1R_2C_1C_2s^2}$$

Δ and Δk

$$\Delta = 1 - \left[P_{11} + P_{21} + P_{31}\right] + P_{12}$$

4.
$$= 1 + \frac{1}{R_1C_1s} + \frac{1}{R_2C_1s} + \frac{1}{R_2C_2s} + \frac{1}{R_1R_2C_1C_2s^2}$$

$$\Delta = \frac{\left[R_1R_2C_1C_2s^2 + \left(R_2C_2 + R_1C_1 + R_1C_2\right)s + 1\right]}{R_1R_2C_1C_2s^2}$$

$$\Delta_1 = 1$$

5. Transfer function by Mason's gain formula,

$$T(s) = \frac{1}{\Delta}\sum_k P_k\Delta_k$$

$$T(s) = \frac{V_0(s)}{V_1(s)} = \frac{\dfrac{1}{R_1R_2C_1C_2s^2}}{\dfrac{\left[R_1R_2C_1C_2s^2 + \left(R_2C_2 + R_1C_1 + R_1C_2\right)s + 1\right]}{R_1R_2C_1C_2s^2}}$$

$$\frac{V_0(s)}{V_1(s)} = \frac{1}{R_1R_2C_1C_2s^2 + \left(R_2C_2 + R_1C_1 + R_1C_2\right)s + 1}$$

*** INCLUDE THIS**

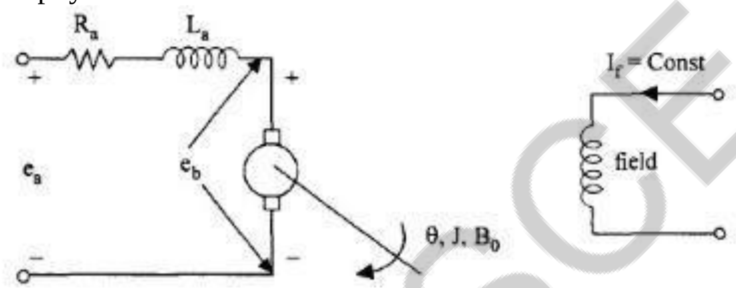
1. Explain about DC Servo Motor

A DC servo motor is used as an actuator to drive a load. It is usually a DC motor of low power rating. DC servo motors have a high ratio of starting torque to inertia and therefore they have a faster dynamic response.

- DC motors are constructed using rare earth permanent magnets which have high residual flux density and high coercively.
- As no field winding is used, the field copper losses is zero and hence, the overall efficiency of the motor is high.
- The speed torque characteristic of this motor is flat over a wide range, as the armature reaction is negligible.
- Moreover speed in directly proportional to the armature voltage for a given torque. Armature of a DC servo motor is specially designed to have low inertia.
- In some application DC servo motors are used with magnetic flux produced by field windings.
- The speed of PMDC motors can be controlled by applying variable armature voltage. These are called armature voltage controlled DC servo motors.
- Wound field DC motors can be controlled by either controlling the armature voltage or controlling rho field current. Let us now consider modelling of these two types or DC servo motors.

(a) Armature controlled DC servo motor

The physical model of an armature controlled DC servo motor is given in



The armature winding has a resistance R_a and inductance L_a.

The field is produced either by a permanent magnet or the field winding is separately excited and supplied with constant voltage so that the field current I_f is a constant. When the armature is supplied with a DC voltage of e_a volts, the armature rotates and produces a back e.m.f e_b

The armature current i_a depends on the difference of e_b and e_n. The armature has a moment of inertiaJ, frictional coefficient B_o

The angular displacement of the motor is θ. The torque produced by the motor is given by

$$T = K_T \, i_a$$

Where K_T is the motor torque constant.

The back emf is proportional to the speed of the motor and hence

$$e_b = K_b \dot{\theta}$$

The differential equation representing the electrical system is given by

$$R_a i_a + L_a \frac{di_a}{dt} + e_b = e_a$$

Taking Laplace transform of equation from above equation

$$T(s) = K_T I_a(s)$$
$$E_b(s) = K_b s \theta(s)$$
$$(R_a + s L_a) I_a(s) + E_b(s) = E_a(s)$$
$$I_a(s) = \frac{E_a(s) - K_b s \theta(s)}{R_a + sL_a}$$

The mathematical model of the mechanical system is given by

$$J \frac{d^2\theta}{dt^2} + B_0 \frac{d\theta}{dt} = T$$

Taking Laplace transform

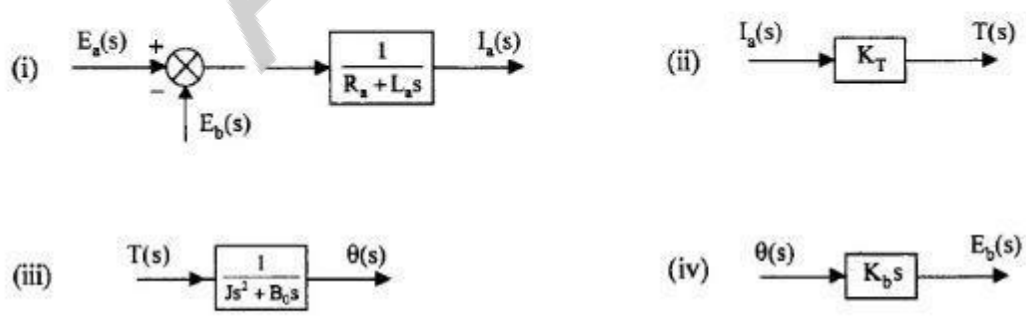
$$(Js^2 + B_0 s) \theta(s) = T(s)$$

$$\theta(s) = K_T \frac{E_a(s) - K_b s \theta(s)}{(R_a + sL_a)(Js^2 + B_0 s)}$$

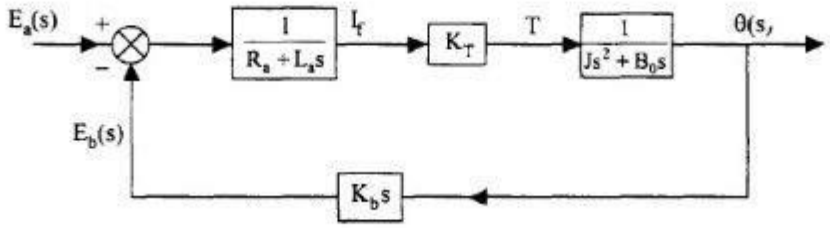
Solving for θ(s) ,we get

$$\theta(s) = \frac{K_T E_a(s)}{s[(R_a + sL_a)(Js + B_0) + K_T K_b]}$$

The block diagram representation of the armature controlled DC servo motor is developed in steps



Combining these blocks we have



Usually the inductance of the armature winding is small and hence neglected

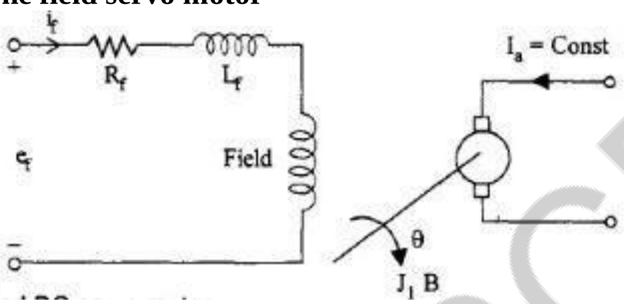
$$T(s) = \frac{\theta(s)}{E_a(s)} = \frac{K_T / R_a}{s\left[Js + B_0 + \frac{K_b K_T}{R_a}\right]}$$

$$= \frac{K_T / R_a}{s(Js + B)}$$

Where

$$B = B_0 + \frac{K_b K_T}{R_a}$$

Field Controlled Dc Servo Motor
The field servo motor



The electrical circuit is modeled as

$$I_f(s) = \frac{E_f(s)}{R_f + L_f s}$$

$$T(s) = K_T\, I_f(s)$$

$$(Js^2 + B_0)\, \theta(s) = T(s)$$

$$\frac{\theta(s)}{E_f(s)} = \frac{K_T}{s(Js + B_0)(R_f + L_f s)}$$

$$= \frac{K_T / R_f B_0}{s\left(\frac{J}{B_0} s + 1\right)\left(\frac{L_f}{R_f} s + 1\right)}$$

$$= \frac{K_m}{s(\tau_m s + 1)(\tau_f s + 1)}$$

Where Motor gain constant

$K_m = K_T / R_f B_0$

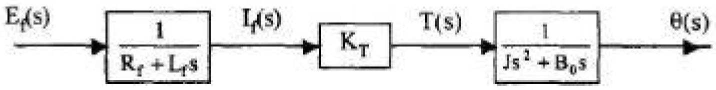
Motor time constant

$\tau_m = J/B_0$

Field time constant

$\tau_f = L_f/R_f$

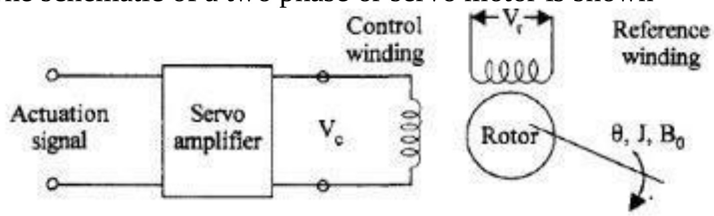
The block diagram is as shown as



2. Write a short note on AC Servo Motors

An AC servo motor is essentially a two phase induction motor with modified constructional features to suit servo applications.

The schematic of a two phase or servo motor is shown



It has two windings displaced by 90 degree on the stator One winding, called as reference winding, is supplied with a constant sinusoidal voltage.

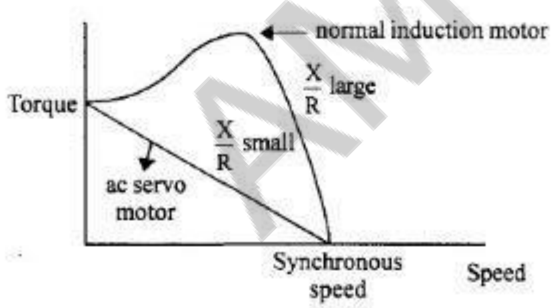
The second winding, called control winding, is supplied with a variable control voltage which is displaced by 90 ^{o} out of phase from the reference voltage.

The major differences between the normal induction motor and an AC servo motor are

The rotor winding of an ac servo motor has high resistance (R) compared to its inductive reactance (X) so that its ratio $\frac{X}{R}$ is very low.

For a normal induction motor, $\frac{X}{R}$ ratio is high so that the maximum torque is obtained in normal operating region which is around 5% of slip.

The torque speed characteristics of a normal induction motor and an ac servo motor are shown in fig



The Torque speed characteristic of a normal induction motor is highly nonlinear and has a positive slope for some portion of the curve.

This is not desirable for control applications as the positive slope makes the systems unstable. The torque speed characteristic of an ac servo motor is fairly linear and has 45 negative slope throughout.

The rotor construction is usually squirrel cage or drag cup type for an ac servo motor. The diameter is small compared to the length of the rotor which reduces inertia of the moving parts.

Thus it has good accelerating characteristic and good dynamic response.

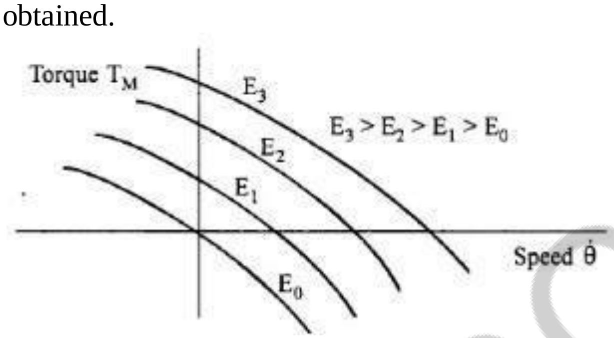
The supplies to the two windings of ac servo motor are not balanced as in the case of a normal induction motor.

The control voltage varies both in magnitude and phase with respect to the constant reference vulture applied to the reference winding.

The direction of rotation of the motor depends on the phase (± 90°) of the control voltage with respect to the reference voltage. For different rms values of control voltage the torque speed characteristics are shown in Fig.

The torque varies approximately linearly with respect to speed and also controls voltage.

The torque speed characteristics can be linearised at the operating point and the transfer function of the motor can be obtained.



3. Write a short note on Synchros
A commonly used error detector of mechanical positions of rotating shafts in AC control systems is the Synchro.
It consists of two electro mechanical devices.
Synchro transmitter

Synchro receiver or control transformer.

The principle of operation of these two devices is same but they differ slightly in their construction.
The construction of a Synchro transmitter is similar to a phase alternator.

The stator consists of a balanced three phase winding and is star connected.

The rotor is of dumbbell type construction and is wound with a coil to produce a magnetic field.

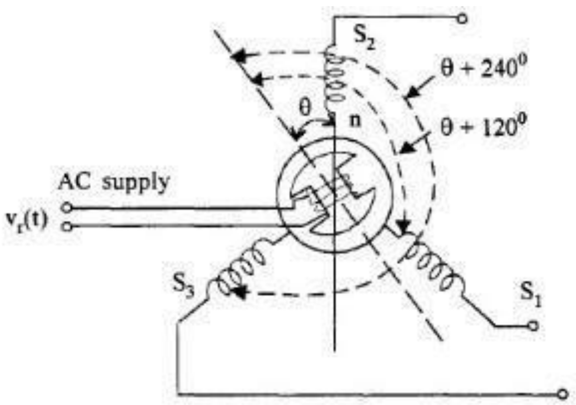
When a no voltage is applied to the winding of the rotor, a magnetic field is produced.

The coils in the stator link with this sinusoidal distributed magnetic flux and voltages are induced in the three coils due to transformer action.

Than the three voltages are in time phase with each other and the rotor voltage.

The magnitudes of the voltages are proportional to the cosine of the angle between the rotor position and the respective coil axis.

The position of the rotor and the coils are shown in Fig.



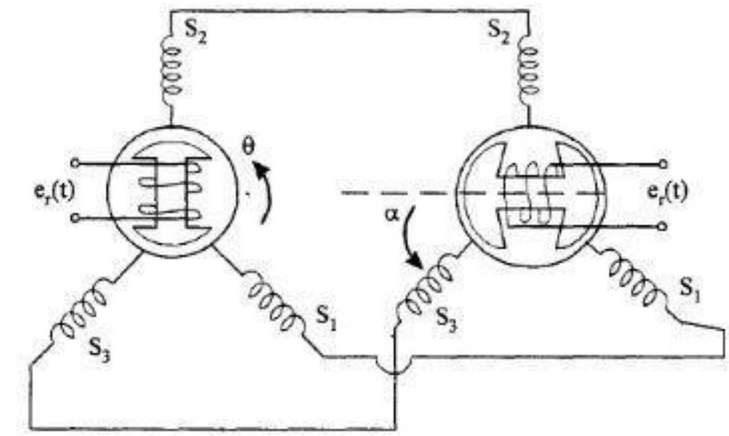
$$v_R(t) = v_r \text{ Sin } \omega_r t$$
$$v_{s_{1n}} = KV_r \text{ Sin } \omega_r t \cos(\theta + 120)$$
$$v_{s_{2n}} = KV_r \text{ Sin } \omega_r t \cos\theta$$
$$v_{s_{3n}} = KV_r \text{ Sin } \omega_r t \cos(\theta + 240)$$
$$v_{s_1 s_2} = v_{s_1 n} - v_{s_2 n} = \sqrt{3}\ KV_r \text{ Sin } (\theta + 240) \text{ Sin } \omega_r t$$
$$v_{s_2 s_3} = v_{s_2 n} - v_{s_3 n} = \sqrt{3}\ KV_r \text{ Sin } (\theta + 120) \text{ Sin } \omega_r t$$
$$v_{s_3 s_1} = v_{s_3 n} - v_{s_1 n} = \sqrt{3}\ KV_r \text{ Sin } \theta \text{ Sin } \omega_r t$$

When θ=90 the axis of the magnetic field coincides with the axis of coil S_2 and maximum voltage is induced in it as seen.

For this position of the rotor, the voltage c, is zero, this position of the rotor is known as the 'Electrical Zero' of die transmitter and is taken as reference for specifying the rotor position.

In summary, it can be seen that the input to the transmitter is the angular position of the rotor and the set of three single phase voltages is the output.

The magnitudes of these voltages depend on the angular position of the rotor as given



Hence

$$e_r(t) = K_1 V_r \cos\phi \sin\omega_r t$$

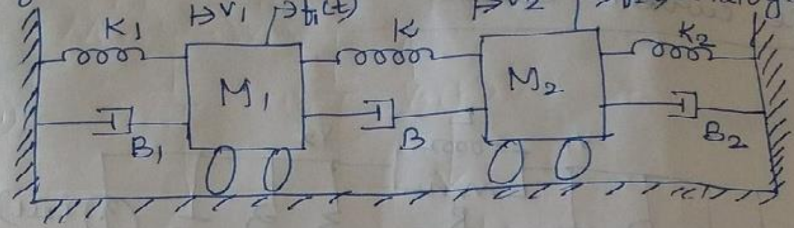
Now consider these three voltages to he applied to the stator of a similar device called control transformer or synchro receiver. The construction of a control transformer is similar to that of the transmitter except that the rotor is made cylindrical in shape whereas the rotor of transmitter is dumbell in shape. Since the rotor is cylindrical, the air gap is uniform and the reluctance of the magnetic path is constant. This makes the output impedance of rotor to be a constant.

Usually the rotor winding of control transformer is connected teas amplifier which requires signal with constant impedance for better performance. A synchro transmitter is usually required to supply several control transformers and hence the stator winding of control transformer is wound with higher impedance per phase. Since the some currents flow through the stators of the synchro transmitter and receiver, the same pattern of flux distribution will be produced in the air gap of the control transformer.

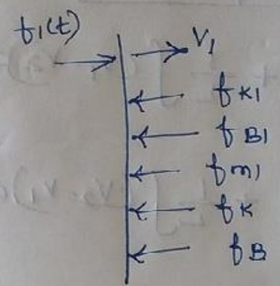
The control transformer flux axis is in the same position as that of the synchro transmitter. Thus the voltage induced in the rotor coil of control transformer is proportional to the cosine of the angle between the two rotors.

APRIL/MAY 2017

11a) Write the differential equations governing the system and draw force-current & force voltage analogous circuit.



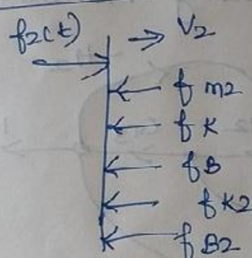
at node1



$$f_1(t) = f_{K1} + f_{B1} + f_{m1} + f_K + f_B$$

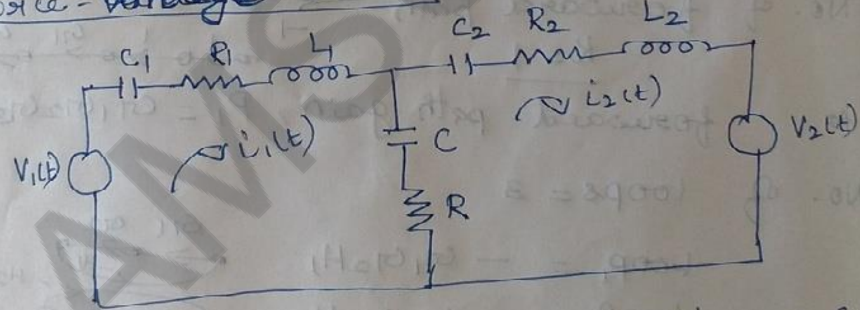
$$f_1(t) = K_1 \int V_1\,dt + B_1 V_1 + M_1 \frac{dV_1}{dt} + K \int (V_1 - V_2)\,dt + B[V_1 - V_2]$$

at node2



$$f_2(t) = K_2 \int V_2\,dt + B_2 V_2 + M_2 \frac{dV_2}{dt} + K \int (V_2 - V_1)\,dt + B[V_2 - V_1]$$

Force - ~~Voltage~~ circuit



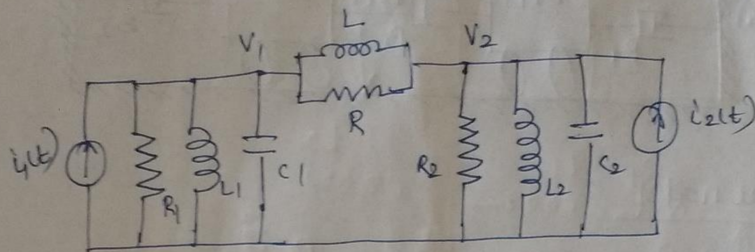
at loop1

$$V_1(t) = \frac{1}{C_1}\int i_1\,dt + R_1 i(t) + L_1 \frac{di_1}{dt} + \frac{1}{C}\int (i_1 - i_2)\,dt + R(i_1(t) - i_2(t))$$

at loop2

$$V_2(t) = \frac{1}{C_2}\int i_2\,dt + R_2 i_2(t) + L_2 \frac{di_2}{dt} + \frac{1}{C}\int (i_2 - i_1)\,dt + R(i_2(t) - i_1(t))$$

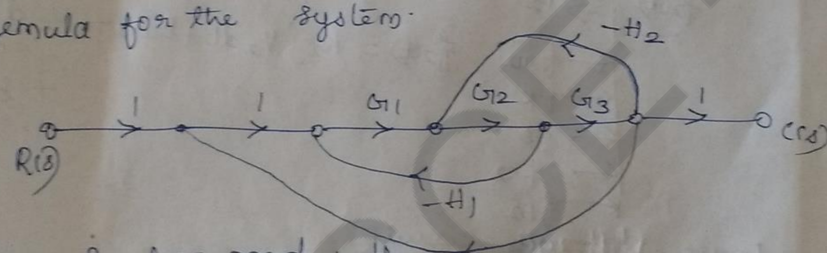
Force-current circuit



$$i_1(t) = \frac{V_1}{R_1} + \frac{1}{L_1}\int V_1\,dt + C_1\frac{dV_1}{dt} + \frac{1}{L}\int (V_1 - V_2)\,dt + \frac{V_1 - V_2}{R}$$

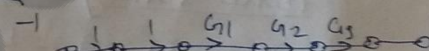
$$i_2(t) = \frac{V_2}{R_2} + \frac{1}{L_2}\int V_2\,dt + C_2\frac{dV_2}{dt} + \frac{1}{L}\int (V_2 - V_1)\,dt + \frac{V_2 - V_1}{R}$$

2. Obtain the transfer function using Mason's Gain formula for the system.



No. of forward path

$K = 1$



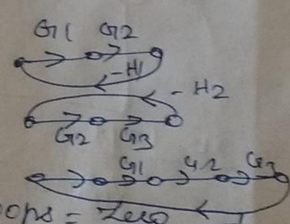
forward path gain, $P_1 = G_1G_2G_3$, $\Delta_k = 1$

No. of loops = 3

$Loop_1 = -G_1G_2H_1$

$Loop_2 = -G_2G_3H_2$

$Loop_3 = -G_1G_2G_3$



No. of two non-touching loops = Zero.

Transfer function, $T = \sum_{K=1} \frac{P_K\Delta_K}{\Delta}$

$$T = \frac{P_1 \Delta_1}{\Delta}$$

$$\Delta = 1 - [-G_1 G_2 H_1 - G_2 G_3 H_2 - G_1 G_2 G_3]$$

$$T = \frac{G_1 G_2 G_3}{1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3}.$$

Unit 2

TIME RESPONSE ANALYSIS

PART-A

1. What is an order of a system? APRIL/MAY 2011, Nov/Dec 2017
   The order of a system is the order of the differential equation governing the system. The order of the system can be obtained from the transfer function of the given system.
2. Define type number of the system Nov/Dec 2017
   The type number of the system is defined as number of poles which lies on the origin of the complex plane.
3. What is step signal?
   The step signal is a signal whose value changes from zero A at t=0 and remains constant at A for t>0.
4. What is ramp signal?
   The ramp signal is a signal whose value increases linearly with time from an initial value of zero at t=0. The ramp signal resembles a constant velocity.
5. State some standard signals used in time domain analysis Nov'15, APRIL /MAY'11&16, Nov/Dec 2018
   Step signal, Ramp signal, Parabolic signal and sinusoidal signal
6. What is transient response?
   The transient response is the response of the system when the system changes from one state to another.
7. What is steady state response?
   The steady state response is the response of the system when it approached infinity.
8. Define damping ratio. April/May 2019
   Damping ratio is defined as the ratio of actual damping to critical damping.
9. List the time domain specifications May/June 2016, NOV/DEC 2016
   The time domain specifications are
   i) Delay time
   ii) Rise time
   iii) Peak time
   iv) Peak overshoot
   v) Setting time
10. What is damped frequency of oscillation?
    In under damped system the response is damped oscillatory. The frequency of damped oscillation is given by $\omega_d = \omega_n \sqrt{1-\zeta^2}$
11. What will be the nature of response of second order system with different types of damping?
    - For undamped system the response is oscillatory.
    - For under damped system the response is damped oscillatory.
    - For critically damped system the response is exponentially rising.
    - For over damped system the response is exponentially rising but the rise time will be very large .
12. Define delay time.
    The time taken from for response to reach 50% of final value for the very first time is delay time.
13. Define rise time April / May 2010
    The time taken for response to raise from 0% to 100% for the very first time is rise time.

14. Define peak time.
The time taken for the response to reach the peak value for the first time is peak time.

15. Define peak overshoot. Nov/ Dec 2010, April/May 2017
Peak overshoot is defined as the ratio of maximum peak value measured from the Maximum value to final value.

16. Define setting time. Nov/Dec 2018
Setting time is defined as the time taken by the response to reach and stay within specified error.

17. What is the need for a controller?
The controller is provided to modify the error signal for better control action.

18. What are the different types of controllers?
The different types of the controller are
    - Proportional controller
    - PI controller
    - PD controller
    - PID controller

19. What is proportional controller?
It is device that produce a control signal which is proportional to the input error signal.

20. What is PI Controller?
It is device that produce a control signal consisting of two terms-one proportional to error signal and the other proportional to the integral of error signal.

21. What is PD Controller?
PD controller is a proportional plus derivative controller which produces an output signal consisting of two terms – one proportional to error signal and other proportional to the derivative of the signal.

22. What is the significance of integral controller and derivative controller in a PID controller?
The proportional controller stabilizes the gain but produces a steady state error. The integral control reduces or eliminated the steady state error.

23. Define Steady state error.
The steady state error is the value of error signal e(t) when t tends to infinity.

24. What is the drawback of static coefficients?
The main drawback of static coefficient is that it does not show the variation of error with time and input should be standard input.

25. What are the three constants associated with a steady state error?
The three steady state errors constant are
    - Positional error constant K_p
    - Velocity error constant K_v
    - Acceleration error constant K_a

26. What are the main advantages of generalized error co-efficients?
    i) Steady state is function of time
    ii) Steady state can be determined from any type of input

27. What are the effects of adding a zero to a system?
Adding a zero to a system results in pronounced early peak to system response thereby the peak overshoot increases appreciable.

28. Why derivative controller is not used in control system?
The derivative controller produces a control action based on rate of change of error signal and it does not produce corrective measures for any constant error. Hence derivative controller is not used in control system.

29. What is the effect of PI controller on the system performance? Nov/Dec 2019, April/May 2017, May/June 2016
The PI Controller increases the order of the system by one, which results in reducing the steady state error. But the system because less stable than the original system.

30. What is the effect of PD Controller of system performance? April/May 2017
The effect of PD controller is to increase the damping ratio of the system and so the peak overshoot is reduced.

31. What are the root loci?

The path taken by the root of the open loop transfer function when the loop gain is varied from 0 to infinity are called root loci.

32. What is the dominant pole? (NOV/DEC 2015, 2016)

The dominant pole is a pair of conjugate pole which decides the transient response of the system. In higher order system the dominant poles are very close to origin and all other poles of the system are widely separated and so they have less effect on transient response of the system.

33. What are the main significance of root locus?

i. The root locus technique is used for stability analysis.
ii. Using root locus techniques the range of value of K, for as stable system can be determined.

34. What are the breakaway point are break in points?

At break away point the root locus break from the real axis to enter into the complex plane. At break in point the root locus enters the real axis from then complex plane. To find the breakaway or break in points, from a equation for K from the characteristic equation and differentiate the equation of K with respect to s. Then find the roots of the equations dK/dS = 0. The root of dK/dS = 0 are breakaway or break in points provided for this value of root the gain K should be positive and real.

35. What are asymptotes? How will you find angle of asymptotes?

Asymptotes are the straight line which are parallel to root locus going to infinity and meet the root locus at infinity.

Angle of asymptotes = $\pm\frac{180°(2q+1)}{n-m}$ $\quad q = 0,1,2,3,....n-m$

N= number of poles

M = number of zeroes.

36. What is the centriod?

The meeting point of the asymptotes with the real axis is called centroid. The centroid is given by Centroid = (sum of the poles-sum of the zeros)/n-m

N= number of poles

M = number of zeroes.

37. What is magnitude criterion?

The magnitude criterion states that $s = s_a$ will be a point on root locus if for that value of s, magnitude of G(s)H(s) is equal to 1.

$$\left|G(s)H(s)\right| = K\frac{(\text{product length of vector from open loop zeros to the point } s = s_a)}{(\text{product length of vector from open loop poles to the point } s = s_a)} = 1$$

38. What is angle criterion?

The angle criterion states that $s = s_a$ will be a point on root locus if for that value of s, the argument or phase of G(s)H(s) is equal to an odd multiple 180 ° .

$$(\text{sum of the angle of vectors from zeros to the point } s = s_a) - (\text{sum of the angle of vectors from poles to the point } s = s_a) = \pm 180°(2q+1)$$

39. How will you find the root locus on real axis? (MAY/JUNE 2016)

To find the root locus on real axis choose the test point on real axis to the right of this test point is odd number then the test point lie on the root locus. If it is even the test point does not lie on the root locus.

Part – B & C QUESTIONS AND ANSWERS

1. Derive the time response analysis of a first order system for (i) Unit step input (ii) Unit ramp (iii) impulse input

(i) For Unit step input

The closed loop transfer function of first order system $\frac{C(s)}{R(s)} = \frac{1}{sT+1}$

If the input unit step, then r(t)=1, and R(s)= $\frac{1}{s}$

The response in s-domain, C(s)=R(s) $\frac{1}{(1+Ts)} = \frac{1}{s}.\frac{1}{(1+Ts)} = \frac{\frac{1}{T}}{s(s+\frac{1}{T})}$

By partial fraction expansion

$$C(s)=\frac{\frac{1}{T}}{s(s+\frac{1}{T})}=\frac{A}{s}+\frac{B}{(s+\frac{1}{T})}$$

$$A(s+\frac{1}{T})+B.s=\frac{1}{T}$$

put $s=\frac{-1}{T}, B=-1$

put s=0, A=1

$$\therefore C(s)=\frac{1}{s}+\frac{-1}{(s+\frac{1}{T})}$$

Response in time domain $c(t)=L^{-1}\left[C(s)\right]$

$$=L^{-1}\left[\frac{1}{s}+\frac{-1}{(s+\frac{1}{T})}\right]=1-e^{\frac{-t}{T}}$$

(ii) For Ramp input

The closed loop transfer function of first order system,

If the input is unit ramp then, r(t)=t and $R(s)=\frac{1}{s^2}$

The response in s- domain

$$C(s)=R(s)\frac{1}{(1+Ts)}=\frac{1}{s^2}.\frac{1}{(1+Ts)}=\frac{\frac{1}{T}}{s^2(s+\frac{1}{T})}$$

by partial fraction expansion

$$C(s)=\frac{\frac{1}{T}}{s(s+\frac{1}{T})}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{(s+\frac{1}{T})}$$

$$As\left(s+\frac{1}{T}\right)+B\left(s+\frac{1}{T}\right)+C.s^2=\frac{1}{T}$$

Put s=0, B=1

Put $s=\frac{-1}{T}, C=T$

Comparing the coefficients of s^2 terms, A+C=1 $\Rightarrow$ A=-T

$$\therefore C(s)=\frac{1}{s^2}+\frac{-T}{s}+\frac{T}{(s+\frac{1}{T})}$$

Response in time domain $c(t)=L^{-1}\left[C(s)\right]$

$$=L^{-1}\left[\frac{1}{s^2}+\frac{-T}{s}+\frac{T}{(s+\frac{1}{T})}\right]=t-T+Te^{\frac{-t}{T}}$$

(iii) For impulse input

The closed loop transfer function of first order system,

If the input impuse, then $r(t)=\delta(t)$ and R(s)=1

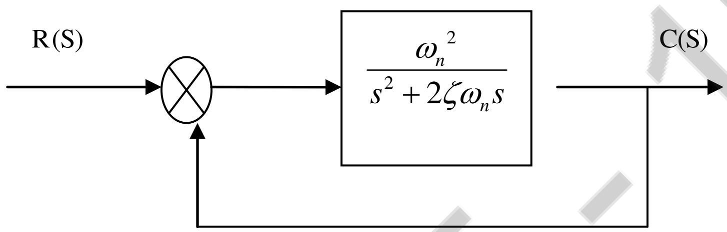
The response in s-domain, $C(s) = R(s)\frac{1}{(1+Ts)} = \frac{1}{(1+Ts)} = \frac{\frac{1}{T}}{(s+\frac{1}{T})}$

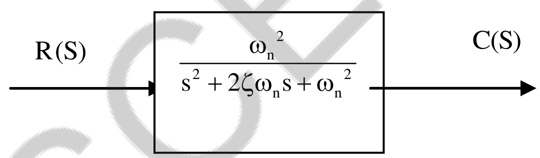
Response in time domain $c(t) = L^{-1}\left[C(s)\right]$

$$= L^{-1}\left[\frac{1}{(s+\frac{1}{T})}\right] = \frac{1}{T}e^{-\frac{t}{T}}$$

2. Discuss briefly about step response analysis second order system

The closed loop second order system is shown in fig.





The standard form of closed loop transfer function of second order system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega n^2}$$

where ω_n = undamped natural frequency rad/sec

ζ = Damping ratio

Depending on the value of ζ , the second order system is classified into 4 types.

1. Undamped system : ζ =0
2. Underdamped system: 0< ζ <1
3. Critically damped system: ζ =1
4. Overdamped system: ζ >1

3. **Response of undamped second order system for unit step input** **Nov/Dec 2019, April/May 2017**

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega^2{}_n}$$

for un damped system, ζ=0

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n{}^2}{s^2 + \omega^2{}_n}$$

When the input is unit step, r(t) = 1 and R(s)= $\frac{1}{s}$

$\therefore$ The response is s-domain, C(s)=R(s) $\frac{\omega_n{}^2}{s^2 + \omega_n{}^2}$

$$= \frac{1}{s} \cdot \frac{\omega_n{}^2}{s^2 + \omega_n^2}$$

By partial fraction expansion

$$C(s) = \frac{\omega_n{}^2}{s(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

$$A(s^2 + \omega_n^2) + Bs = \omega_n{}^2$$

put s=0, $\omega_n^2 A = \omega_n^2 \Rightarrow \boxed{A = 1}$

put s=jω_n, $j\omega_n B = \omega_n^2$

$B = -j\omega_n = -s$

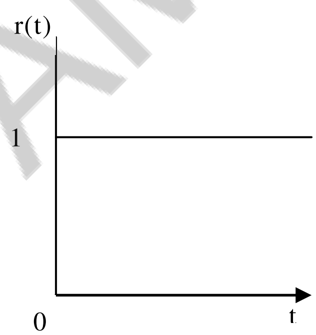
$\boxed{B = -s}$

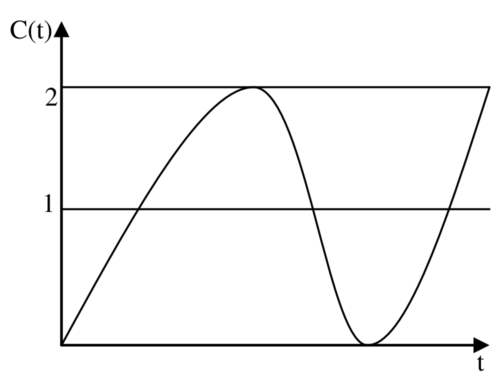
$$\therefore C(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

Time response $c(t) = L^{-1}\left[C(s)\right]$

$$= L^{-1}\left[\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right]$$

$$\boxed{c(t) = 1 - \cos\omega_n t}$$





The response of undamped second order system for unit step input is completely oscillatory.

4. **Response of under damped second order system for unit step input**. (Nov/Dec 2018)

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For under damped system, 0< ζ <1, and the roots of the characteristic equation are complex conjugate

The response is s-domain, $C(s) = R(s)\ \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

For unit step input, r(t) =1, $R(s)=\dfrac{1}{s}$

$$C(s) = R(s)\ \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s}.\ \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

By partial fraction expansion

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$A(s^2 + 2\zeta\omega_n s + \omega_n^2) + Bs^2 + Cs = \omega_n{}^2$$

Comparing constant terms,

$$A\,\omega_n{}^2 = \omega_n{}^2 \Rightarrow \boxed{A = 1}$$

Comparing the coefficient of s^2,

$$A+B=0 \Rightarrow \boxed{B = -1}$$

Comparing the coefficient of s

$$A(2\zeta\omega_n) + C = 0 \Rightarrow \boxed{C = -2\zeta\omega_n}$$

$$\therefore C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) + (\omega_n^2 - \zeta^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \quad \text{where } \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}.$$

The response in time domain, $c(t)=L^{-1}[C(s)]$

$$\therefore c(t) = L^{-1}\left[\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right]$$

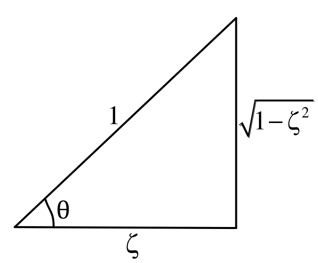
$$= 1 - e^{-\zeta\omega_n t}.\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}.e^{-\zeta\omega_n t}.\sin\omega_d t$$

$$= 1 - e^{-\zeta\omega_n^t}[\cos\omega dt + \frac{\zeta\omega_n}{\omega_n\sqrt{1 - \zeta^2}}\sin\omega dt]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}[\sqrt{1 - \zeta^2}.\cos\omega_d t + \zeta\sin\omega_d t]$$

On constructing right angle triangle withζ and $\sqrt{1 - \zeta^2}$, we get

$\sin\theta=\sqrt{1 - \zeta^2}$;$\cos\theta = \zeta$;$\tan\theta=\frac{\sqrt{1 - \zeta^2}}{\zeta}$

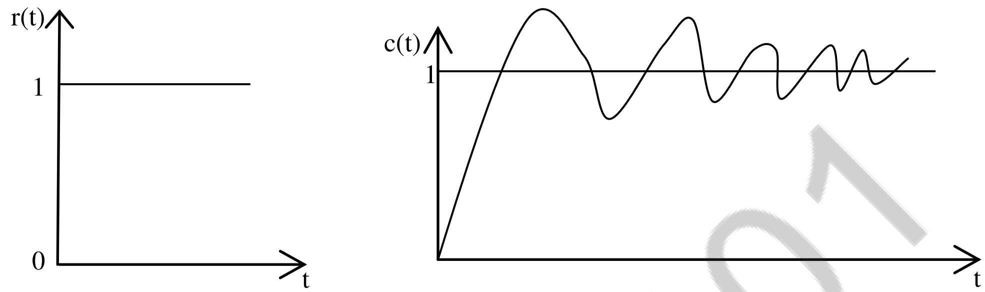


$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\left[\sin\theta.\ \cos\omega_d t + \cos\theta.\ \sin\omega_d t\right]$$

$$\boxed{c(t) = 1 - \frac{e^{-\zeta\omega_n^t}}{\sqrt{1 - \zeta^2}}\sin(\omega_d t + \theta)}$$

Where $\theta = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$

The response of under damped second order system for unit step input oscillator before setting to a final value.



3. Response of critically damped second order system for unit step input

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

for critical damping, $\zeta=1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

when the input is unit step r(t)=1, $R(s)=\frac{1}{s}$

$\therefore$ The response is s- domain,

$$C(s)=R(s).\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}.\frac{\omega_n^2}{(s+\omega_n)^2}$$

by partial fraction expansion,

$$C(s)=\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{(s+\omega_n)}$$

$$A(s+\omega_n)^2 + Bs + Cs(s+\omega_n) = \omega_n{}^2$$

put s=0, $\omega_n{}^2.A = \omega_n{}^2 \Rightarrow \boxed{A=1}$

put s=$-\omega_n$, $-\omega_n B = \omega_n{}^2 \Rightarrow \boxed{B=-\omega_n}$

Comparing the coefficient of s^2,

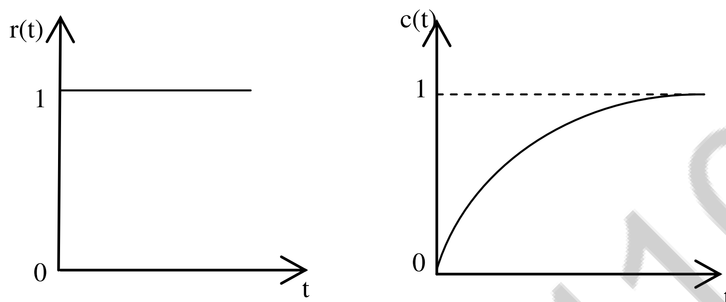
$$A + C = 0 \Rightarrow \boxed{C=-1}$$

$$\therefore C(s) = \frac{1}{s} + \frac{-\omega_n}{(s+\omega_n)^2} + \frac{-1}{s+\omega_n}$$

The response in time domain $c(t) = L^{-1}\left[C(s)\right] = L^{-1}\left[\frac{1}{s} + \frac{-\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right]$

$$c(t) = 1 - \omega_n t.e^{-\omega_n t} - e^{-\omega_n t}$$

$$\boxed{c(t) = 1 - e^{-\omega_n t}(1+\omega_n t)}$$



The response of critically damped closed loop second order system for unit step input, has no oscillations.

4.Response of overdamped second order system for unit step input.

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For over damped system $\zeta > 1,$ the roots of the denominator of transfer function are real and distinct. Let the roots of the denominator be s_a, s_b

$$s_{a,} s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\left[\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}\right]$$

Let $s_1 = -s_a$, and $s_2 = -s_b$

$$\therefore s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

The closed loop transfer function can be written in terms of s_1 and s_2 as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)}$$

For unit step input r(t)=1 and R(s)=1/s

$$\therefore C(s) = R(s)\frac{\omega_n^2}{(s+s_1)(s+s_2)} = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

by partial fraction expansion

$$C(s)=\frac{{\omega_n}^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$A(s+s_1)(s+s_2) + Bs(s+s_2) + Cs(s+s_1) = {\omega_n}^2$$

put s=0, $s_1 s_2 A = {\omega_n}^2$

$$A = \frac{{\omega_n}^2}{s_1 s_2} = \frac{{\omega_n}^2}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]}$$

$$= \frac{{\omega_n}^2}{\zeta^2\omega_n^2 + \omega_n^2\sqrt{\zeta^2-1}} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$\boxed{\therefore A = 1}$$

Put s=-s_1

B. $s_1(-s_1+s_2) = \omega_n^2$

$$B=\frac{-\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1\left[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]}$$

$$= \frac{-{\omega_n}^2}{s_1\left[2\omega_n\sqrt{\zeta^2-1}\right]} = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_1}$$

$$\boxed{B = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_1}}$$

put s=-s_2

$$C = (-s_2)(-s_2+s_1) = {\omega_n}^2$$

$$C = \frac{{\omega_n}^2}{-s_2(s_1-s_2)}$$

$$= \frac{{\omega_n}^2}{-s_2\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]}$$

$$= \frac{{\omega_n}^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_2} = \frac{{\omega_n}^2}{2\sqrt{\zeta^2-1}}.\frac{1}{s_2}$$

$$\boxed{\therefore C = \frac{\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_2}}$$

$$\therefore C(s) = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_1}.\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_2}.\frac{1}{(s+s_2)}$$

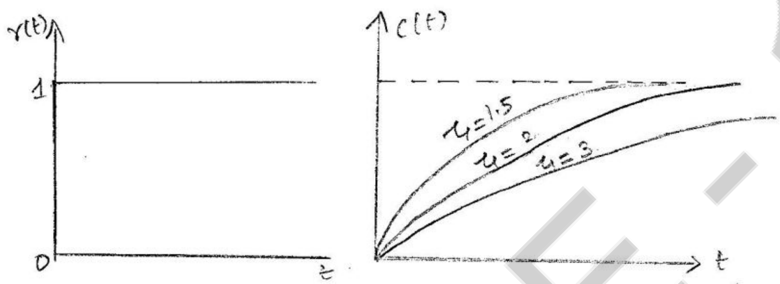
The response in time domain, c(t)

$$c(t)=L^{-1}\left[\frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_1}.\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_2}.\frac{1}{(s+s_2)}\right]$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_1}e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_2}e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)$$

Where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$ and $s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$



The response of over damped closed loop system or unit step input has no oscillations, but it takes longer time for the response to reach the final steady value.

5. What are the time domain specifications? Define them

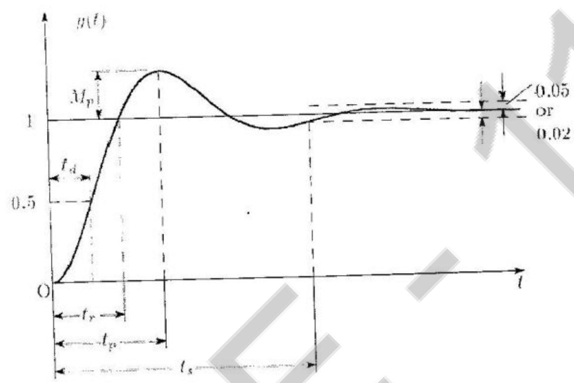
Time domain specifications

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications

1. Delay time(t_d)
2. Rise time (t_r)
3. Peak time (t_p)
4. Maximum overshoot (M_p)
5. Settling time (t_s)
6. Steady state error (e_{ss})

- Delay time(t_d) is the time required to reach at 50% of its final value by a time response signal during its first cycle of oscillation.

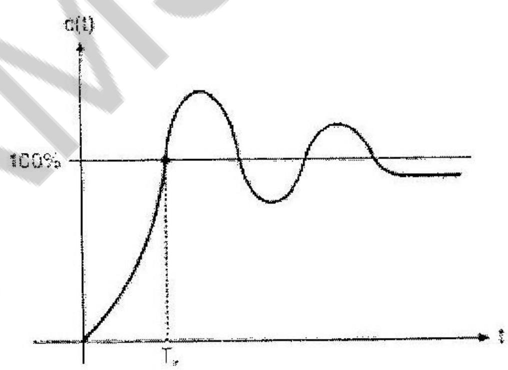
- Rise time (t_r) is the time required to reach at final value by a under damped time response signal during its first cycle of oscillation. If the signal is over damped then rise time is counted as the time required by the response to rise form 10% to 90% of its final value.
- Peak time (t_p) is simply the time required by response to reach its first peak i.e the peak of first cycle of oscillation, or first overshoot.
- Maximum overshoot (M_p) is straight way difference between the magnitude of the highest peak of time response and magnitude of its steady state. Maximum overshoot is expressed in terms of percentage of steady-state value of the response. As the first peak of response is normally maximum in magnitude, maximum overshoot is simply normalized difference between first peak and steady- state value of a response.
- Settling time (t_s): Time required for a response to become steady. It is defined as the time required by the response to reach and steady within specified range of 2% to 5% of its final value.
- Steady state error (e_{ss}) is the difference between actual output and desired output at the infinite range of time



6. Derive the expressions for time domain specifications of a second order system subjected to a step input

(April/May 2019)

Expression for Rise time t_r



Transient response of second order system is given by

$$c(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At rise time c(t)=1

$$\Rightarrow 1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} . \sin(\omega_d t_r + \theta)$$

$$-\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} . \sin(\omega_d t_r + \theta) = 0$$

equation will get satisfied if

$$\sin(\omega_d t_r + \theta) = 0;$$

$$\Rightarrow (\omega_d t_r + \theta) = n\pi \quad where \quad n = 1, 2...$$

Let $n = 1$

$$\omega_d t_r + \theta = \pi$$

$$\therefore t_r = \frac{\pi - \theta}{\omega_d}$$

Expression for Peak time t_p:

Transient response of second order system is given by

$$c(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

Where $\theta = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$

As at t=t_p, c(t) will achieve its maxima, according to Maxima theorem.

$$\frac{dc(t)}{dt} \Big|_{t=t_p} = 0$$

So differentiating c(t) w.r.t t, we can write

$$\frac{d}{dt} c(t) = 0 \Rightarrow \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} .(-\zeta\omega_n) \sin(\omega_d t + \theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \right) \cos(\omega_d t + \theta)\omega_d = 0$$

substituting $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\frac{\zeta\omega_n e^{-\zeta\omega n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta) - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}\cos(\omega_d t+\theta) = 0$$

$$\zeta\sin(\omega_d t+\theta) - \sqrt{1-\zeta^2}\cos(\omega_d t+\theta) = 0$$

$$\therefore \tan(\omega_d t+\theta) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\text{Now, } \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\therefore \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\theta$$

$$\tan(\omega_d t+\theta) = \tan\theta$$

from trignometric formula,

$$\tan(n\pi + \theta) = \tan\theta$$

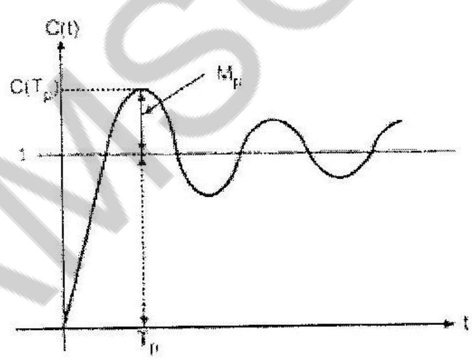
$$\omega_d t = n\pi \qquad \text{where } n = 1,2,3$$

But t_p and required for first peak overshoot n=1

$$\omega_d t_p = \pi$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Expression for maximum peak overshoot(%Mp)



$$M_p = c(t_p) - 1$$

$$M_p = 1 - \frac{e^{\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta) - 1$$

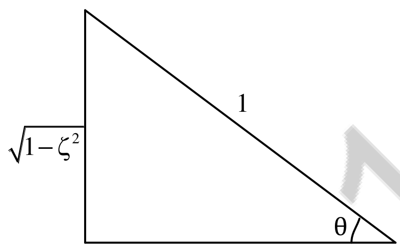
$$M_p = -\frac{e^{\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta)$$

but $t_p = \frac{\pi}{\omega_d}$, substituting

$$M_p = \frac{-e^{\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}} \sin(\pi + \theta)$$

Now, $\sin(\pi+\theta) = -\sin(\theta)$

$$M_p = \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\theta$$



$$\theta = \tan^{-1} X, \quad X = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\frac{X}{1} = \tan\theta$$

$\sin\theta = \frac{X}{\sqrt{1+X^2}}$ and substitute value of X

$$M_p = \frac{e^{-\zeta\omega_n t_P}}{\sqrt{1-\zeta^2}}$$

substitute $t_p = \frac{\pi}{\omega_d}$

$$\therefore M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Expression for setting time t_s

The setting time t_s is the required by the output to settle down within 2% of tolerance band. So, t_s is the time when output becomes 98% of its final value and remains within the range of ± 2 %

$$c(t) \text{ at } (t=t_s) = 0.98$$

Now at t= t_s, the transient oscillatory term completely vanishes. The only term which controls the amplitude of the output within $\pm 2\%$. Hence value of t_s is obtained considering only exponentially decaying envelope, neglecting all other terms.

$$c(t) \text{ at } (t=t_s) = 1 - e^{\zeta\omega_n t_s}$$

$$0.98 = 1 - e^{\zeta\omega_n t_s}$$

$e^{\zeta\omega_n t_s} = 0.02$

$$t_s = \frac{3.912}{\zeta\omega_n}$$

In practice the settling time is assumed to be

$$t_s = \frac{4}{\zeta\omega_n} = 4T \quad \text{for } \pm 2\% \text{ tolerance}$$

where $T = \frac{1}{\xi\omega_n}$ is called constant of system

similarly for $\pm 5\%$ of tolerance band.

$$c(t) \text{ at } (t = t_s) = 0.95$$

$$0.95 = 1 - e^{-\zeta\omega_n t_s}$$

$$t_s = \frac{2.995}{\zeta\omega_n} \approx \frac{3}{\zeta\omega_n} = 3T$$

7. Discuss the effects of P, PI, PD and PID Controllers Nov/Dec 2015, May/June 2016, Nov/Dec 2016, Nov/Dec 2019

Controllers: A Controller is a device introduced in the system to modify the error signal and to produce a control signal.

The controller modifies/improves the transient response of the system

The different types of controllers are

- Proportional controller(P controller)
- Integral controller (I controller)
- PI controller
- PD controller
- PID Controller

Proportional controller (P controller)

- The proportional controller is a device that produces a control signal, u(t) proportional to the input error signal e(t).

  In P-controller , u(t) α e(t)

  $$U(t) = K_p e(t) \ldots\ldots\ldots\ldots\ldots(1)$$

  Where K_p is the proportional gain or proportional constant On taking Laplace transform to (i)

  $$U(s) = K_p E(s)$$

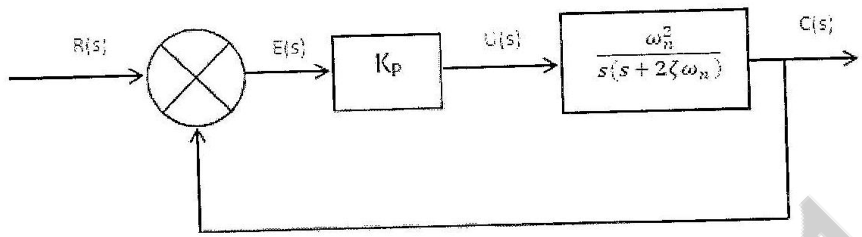
  $$\frac{U(s)}{E(s)} = K_p \ldots\ldots\ldots\ldots\ldots(2)$$

  Equation (2) is the transfer function of P controller

  → The proportional controller amplifies the error signal by amount K_p

  → The introduction of controller on the system increases the loop gain by an amount K_p

→ The increase in loop gain improves steady state tracking accuracy, disturbance signal rejection and relative stability and also makes system less sensitive to parameter variation.



$$G(s)H(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n S + \omega_n^2}$$

Where ζ is damping ratio and ω_n is undamped natural frequency.

For steady state response,

$$K_p = \lim_{s\to 0} G(s)H(s) = \infty;\ e_{ss} = 0$$

$$K_p = \lim_{s\to 0} sG(s)H(s) = \frac{\omega_n}{2\zeta}; e_{ss} = \frac{2\zeta}{\omega_n} = \text{cons tan t}$$

If transient response is to be improved, damping ratio must be changed.

In general good time response demands,

- Less settling time
- Less overshoot
- Less rise time
- Smallest steady state error

→ Increasing the gain K_v to very large values, steady state error may be reduced but due to high gain, settling time and peak overshoot increases and this may lead to instability of the system

→ Drawback : it leads to constant steady state error

Integral controller (I controller)

The integral controller is a device that produces a control signal u(t) which is proportional to integral of the input error signal [e(t)]

In I controller, $u(t) \propto \int e(t)dt$

$u(t) = K_i e(t)dt$...................(1)

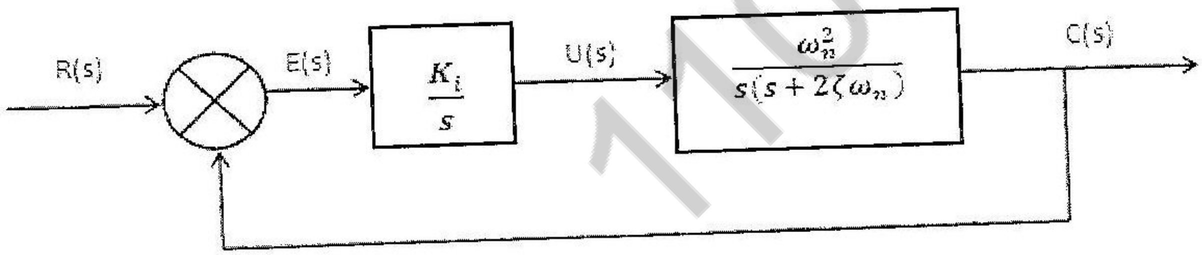
Where K_i is the integral constant

On taking Laplace transform to (i) $U(s) = K_i \frac{E(s)}{s}$

$$\frac{U(s)}{E(s)} = \frac{K_i}{s} \text{..................(2)}$$

Eqn(2) is the transfer function of I controller

- → The integral controller removes or reduces the steady state error without need for manual reset. Hence I controller is called automatic reset.
- → Drawback: it may lead to oscillatory response of increasing or decreasing amplitude, which is undesirable and the system may become unstable.



PI Controller

The proportional plus integral controller produces an output signal consisting of two terms, one proportional to error signal and the other proportional to the integral of the error signal

In PI Controller, $u(t) \ \alpha \left[e(t) + \int e(t)dt \right]$

$$u(t) = K_p e(t) + K_I \int e(t)dt$$

$$u(t) = K_p e(t) + \frac{K_p}{T_I} \int e(t)dt \text{.................(1)}$$

where $K_i = \frac{K_p}{T_i}$; K_p is the proportional gain and T_i is the integral time.

On taking Laplace Transform to (1),

$$U(s) = K_p E(s) + \frac{K_p}{T_i}\frac{E(s)}{s}$$

$$= E(s)\left[K_p + \frac{K_p}{T_i s} \right]$$

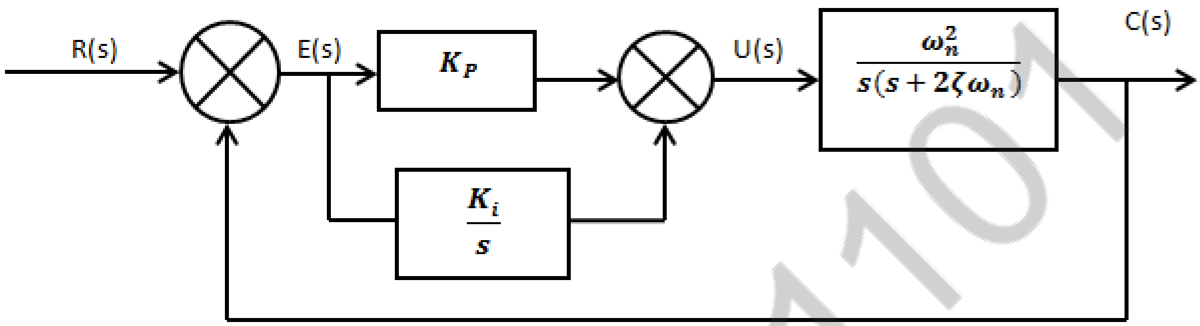
$$= E(s) K_p \left[1 + \frac{1}{T_i s} \right]$$

$$\frac{E(s)}{U(s)} = K_p \left[1 + \frac{1}{T_i s} \right] \text{...........................(2)}$$

Equation (2) is the transfer function of PI Controller

The advantages of both P controller and I Controller are combined in PI controller. The proportional control action increases the loop gain and makes the system less sensitive to variations of system parameters.

The integral control action is adjusted by varying the integral time. The change in value of K_p affects both the proporational and integral parts of control action. The inverse of the integral time T_i is called the reset rate.



Effects of PI Controller:

$$G(s)=\frac{(K_p+\frac{K_i}{s})\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$\text{Assuming } K_p=1,$$

$$G(s)=\frac{\left(1+\frac{K_i}{s}\right)\omega_n^2}{s(s+2\zeta\omega_n)}=\frac{(K_i+s)\omega_n^2}{s^2(s+2\zeta\omega_n)}$$

i.e system becomes TYPE2 in nature

$$\frac{C(s)}{R(s)}=\frac{(K_i+s)\omega_n^2}{s^3+2\zeta\omega_n s^2+\omega_n^2 s+K_i\omega_n^2}$$

i.e it becomes third order.

As order increases by one, system relatively becomes less stable as K_i must be designed in such a way that system will remain in stable condition. Second order system is always stable.

Hence transient response gets affected if controller is not designed properly. While,

For steady state response,

$$K_p = \lim_{s\to 0} G(s)H(s) = \infty;\ e_{ss} = 0$$
$$K_p = \lim_{s\to 0} sG(s)H(s) = \infty; e_{ss} = 0$$

Hence as type is increased by one, error becomes zero for ramp type of inputs, i.e., steady state of system gets improved and becomes more accurate in nature.

Hence PI controller ha following effects:

- → It increases order of the system
- → It increases the TYPE of the system
- → Design of K_i must be proper to maintain stability of system. So it makes system relatively less stable.
- → Steady state error reduces tremendously for same type of inputs.

In general PI controller improves steady state part affecting the transient part.

PD Controller

The proportional plus derivative controller produces an output signal consisting of two terms: one proportional to error signal and the other proportional to the derivative of error signal.

In PD Controller, $u(t)\alpha\left[e(t)+\frac{d}{dt}e(t)\right]$

$u(t) = K_p e(t) + K_p T_d e(t)$........................(1)

Where K_p is the proportional gain and T_d is the derivative time

On taking Laplace transform to (i),

$$U(s) = K_p E(s) + K_p T_d sE(s)$$
$$\frac{U(s)}{E(s)} = K_p(1 + T_d s)\text{...............(2)}$$

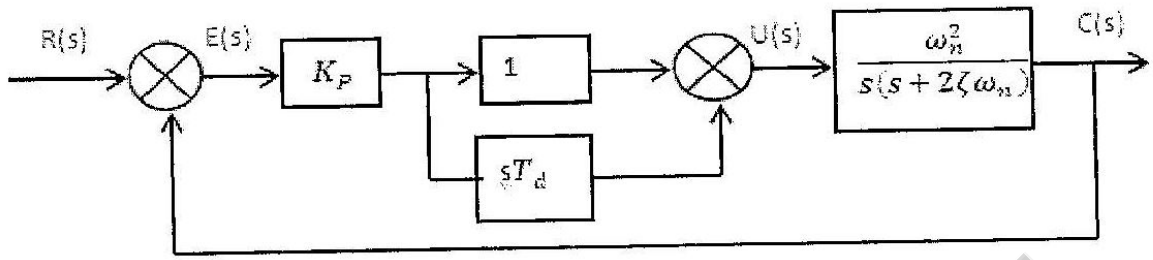
Equation (2) is the transfer function of PD Controller.

The derivative control acts on rate of change of error and not on the actual error signal. The derivative control is effective only during transient periods and so it does not produce corrective measures for any constant error. Hence the derivative controller is never used alone, but it is employed in association with proportional and integral controllers.

The derivative controller does not affect the steady state error directly but anticipates the error, initiates an early corrective action and tends to increase the stability of the system.

It amplifies noise signal and may cause a saturation effect in the actuator.

The derivative control action is adjusted by varying the derivative time. The change in the value of K_p affects both P and D parts of control action. The derivative control action is called as rate control.



Effects of PD Controller:

$$G(s) = \frac{K_p(1+sT_d)\omega_n^2}{s(s+2\zeta\omega_n)}$$

Assuming $K_p = 1$,

$$G(s) = \frac{(1+sT_d)\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$\frac{C(s)}{R(s)} = \frac{(1+sT_d)\omega_n^2}{s^2 + s[2\zeta\omega_n + \omega_n^2 T_d] + \omega_n^2}$$

Comparing the denominator with standard form, ω_n is same as P type controller.

$$2\zeta'\omega_n = 2\zeta\omega_n + \omega_n^2 T_d$$

$$\zeta' = \zeta + \frac{\omega_n T_d}{2}$$

Because of this controller, damping ratio increases by factor $\frac{\omega_n T_d}{2}$

For steady state response,

$$K_p = \lim_{s\to 0} G(s)H(s) = \infty;\ e_{ss} = 0$$

$$K_p = \lim_{s\to 0} sG(s)H(s) = \frac{\omega_n}{2\zeta}; e_{ss} = \frac{2\zeta}{\omega_n}$$

As there is no change in coefficients, error also will remain same. Hence PI controller has following effecs:

- → It increases the damping ratio
- → ω_n for system remains unchanged.
- → TYPE number of the system remains unchanged.
- → It reduces peak overshoot
- → It reduces settling time
- → Steady state error remains unchanges

In general PD controller improves transient part without affecting steady state

PID controller

The PID controller produces an output signal consisting of three terms: one proportional to error signal, another one proportional to integral of error signal and that one proportional to derivative of error signal

In PID controller, $u(t) \propto [e(t) + \int e(t)dt + \frac{d}{dt}e(t)]$

$$U(t) = K_p e(t) + \frac{K_p}{T_i}\int e(t)dt + K_p T_d \frac{d}{dt}e(t) \text{..................(1)}$$

Where K_p is the proportional gain, T_i integral time and T_d is the derivative time.

On taking Laplace transform to (1),

$$U(s) = K_p E(s) + \frac{K_p}{T_i}\frac{E(s)}{s} + K_p T_d E(s)$$

$$U(s) = E(s) K_p \left[1 + \frac{1}{T_i s} + T_d s\right]$$

$$\frac{U(s)}{E(s)} = K_p[1 + \frac{1}{T_i s} + T_d s] \text{.......................(2)}$$

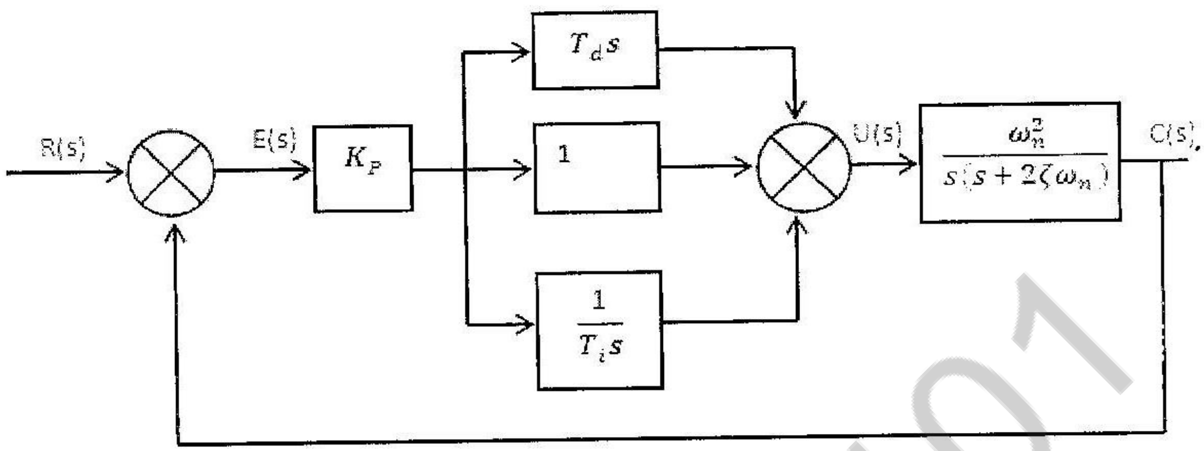
Equation (2) is the transfer function of PID controller.

The combination of proportional control action. Integral action and derivative control action is called PID control.

The proportional controller stabilizes the gain but produces a steady state error

The integral controller reduces (or) eliminates the steady error.

The derivative controller reduces the rate of change of error.



Problems

1. A system has the following transfer function

$$\frac{C(s)}{R(s)} = \frac{20}{s+10}$$

Determine its unit impulse and unit step response with zero initial conditions.

Sol:

a) Unit impulse input

For unit impulse input R(s)=1

$$\frac{C(s)}{R(s)} = \frac{20}{s+10}$$

$$C(s) = R(s)\frac{20}{s+10}$$

$$=1.\ \frac{20}{s+10}$$

Time Response $c(t)=L^{-1}[C(s)]$

$$c(t) = L^{-1}\left[\frac{20}{s+10}\right]$$

$$\boxed{c(t) = 20e^{-10t}}$$

b) Unit step input

For unit step input, R(s) =1/S

$$\frac{C(s)}{R(s)} = \frac{20}{s+10}$$

Response in 's' domain $C(s)=R(s)\ \frac{20}{s+10}$

$$C(s) = \frac{1}{s}\frac{20}{(s+10)}$$

$$= \frac{A}{s} + \frac{B}{s+10}\left[\text{by partial fraction expansion}\right]$$

$A(s+10) + Bs = 20$

comparing coefficients of s,

A+B=0 $\rightarrow$ (1)

comparing constant terms

10A=20 $\Rightarrow \boxed{A = 2}$

$\therefore \boxed{B = -2}$

substituting A and B

$$C(s) = \frac{2}{s} - \frac{2}{s+10}$$

Response in time domain c(t) = L^{-1}[C(s)]

$$c(t) = L^{-1}\left[\frac{2}{s}\right] - L^{-1}\left[\frac{2}{s+10}\right]$$

$$\boxed{c(t) = 2 - 2e^{-10t}}$$

2. Obtain the unit step response and unit impulse response of the unity feedback system having open loop transfer function

$$G(s) = \frac{10}{s(s+2)}$$

Sol Given $G(s) = \dfrac{10}{s(s+2)}$ $H(s) = 1$

The closed loop transfer function $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} = \frac{10}{s^2+2s+10}$$

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+2s+10}$$

(a) Unit step input

For unit step input, r(t) =1, R(s)=1/s

Response in s domain C(s) = R(s) $\dfrac{10}{s^2+2s+10}$

$$C(s) = \frac{1}{s}\frac{10}{s^2+2s+10}$$

$$C(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 10} \left[\text{by partial fraction expansion}\right]$$

$$A(s^2 + 2s + 10) + Bs^2 + Cs = 10$$

comparing constant terms

$$10A=10 \Rightarrow \boxed{A = 1}$$

comparing the coefficients of s terms

$$2A+C=0 \Rightarrow \boxed{C = -2}$$

Comparing the coefficients of s^2,

$$A+B=0 \rightarrow \boxed{B = -1}$$

$$\therefore C(s) = \frac{1}{s} - \frac{s+2}{s^2 + 2s + 10}$$

$$= \frac{1}{s} - \frac{s+2}{s^2 + 2s + 1 - 1 + 10}$$

$$= \frac{1}{s} - \frac{s+2}{(s+1)^2 + 9}$$

$$= \frac{1}{s} - \frac{s+1}{(s+1)^2 + 9} - \frac{1}{(s+1)^2 + 9}$$

$$= \frac{1}{s} - \frac{s+1}{(s+1)^2 + 9} - \frac{3}{3((s+1)^2 + 9)}$$

$$\boxed{\begin{array}{l} L^{-1}\left[\dfrac{\omega}{(s+a)^2 + \omega^2}\right] = e^{-at}\sin \omega t \\ L^{-1}\left[\dfrac{s+a}{(s+a)^2 + \omega^2}\right] = e^{-at}\cos \omega t \end{array}}$$

Response in time domain c(t) = $L^{-1}\left[C(s)\right]$

$$c(t) = 1 - e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t$$

$$c(t) = 1 - e^{t}[\cos 3t + 0.33\sin 3t]$$

b)Impulse response

for impulse input, R(s)=1

$$\therefore C(s) = R(s)\frac{10}{s^2 + 2s + 10}$$

$$C(s) = \frac{10}{s^2 + 2s + 10}$$

$$C(s) = \frac{10}{(s+1)^2 + 3^2}$$

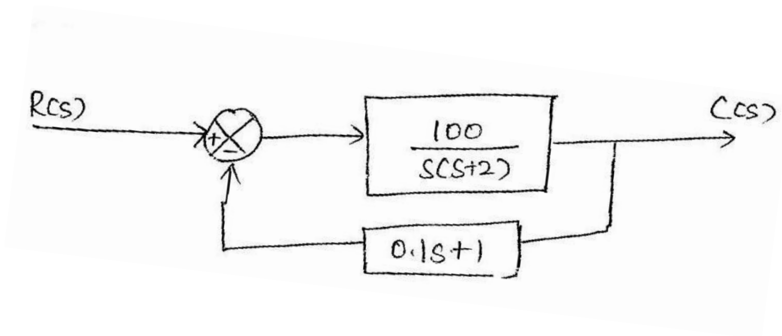
$$c(t) = L^{-1}\left[C(s)\right] \qquad \boxed{L^{-1}\left[\frac{\omega}{(s+a)^2 + \omega^2}\right] = e^{-at}\sin \omega t}$$

$$= \frac{10}{3}\frac{3}{(s+1)^2 + 3^2}$$

$$=3.33\ e^{-t}\sin 3t$$

$$\boxed{c(t) = 3.33\ e^{-t}\sin 3t}$$

3. A positional control system with velocity feedback is shown in fig. What is the response of the system for unit step input?



Sol:

The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s).H(s)}$$

$$\text{given } G(s) = \frac{100}{s(s+2)} \quad H(s) = 0.1s+1$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{100}{s(s+2)}}{1+\frac{100}{s(s+2)}(0.1s+1)}$$

$$= \frac{\frac{100}{s(s+2)}}{\frac{s(s+2)+100(0.1s+1)}{s(s+2)}}$$

$$= \frac{100}{s^2+2s+10s+100} = \frac{100}{s^2+12s+100}$$

The characteristic polynomial is $s^2+12s+100$

$$\text{roots are } s_{1,}s_2 = \frac{-12 \pm \sqrt{144-4x100}}{2}$$

$$= \frac{-12 \pm j16}{2}$$

$$= -6 \pm j8$$

The roots are complex conjugate. The system is under damped. So the response of the system will have damped oscillations.

The response in s-domain C(s) = R(s) $\frac{100}{s^2+12s+100}$

Since input is unit step, R(s)=1/s

$\therefore C(s) = \frac{1}{s} . \frac{100}{s^2 + 12s + 100}$

$= \frac{A}{s} + \frac{Bs + C}{s^2 + 12s + 100}$ [By partial fraction expansion]

$A(s^2 + 12s + 100) + Bs^2 + Cs = 100$

comparing the constant terms,

$100A=100 \Rightarrow \boxed{A = 1}$

comparing the coefficients of s,

$12A+C \Rightarrow \boxed{C = -12}$

comparing the coefficients of s^2,

$A + B = 0 \Rightarrow \boxed{B = -1}$

$$\therefore C(s) = \frac{1}{s} + \frac{s + 12}{s^2 + 12s + 100}$$

$$= \frac{1}{s} - \frac{s + 12}{s^2 + 12s + 36 + 64}$$

$$= \frac{1}{s} - \frac{s + 6 + 6}{(s \ + 6)^2 + 8^2}$$

$$= \frac{1}{s} - \frac{s + 6}{(s + 6)^2 + 8^2} - \frac{6}{(s + 6)^2 + 8^2}$$

$$= \frac{1}{s} - \frac{s + 6}{(s + 6)^2 + 8^2} - \frac{6}{8} \frac{8}{(s + 6)^2 + 8^2}$$

The time domain response is obtained by taking inverse Laplace transform of C(s)

$\therefore$ Time response, $c(t)=L^{-1}\{C(s)\}$

$$c(t) = L^{-1}\left\{\frac{1}{s} - \frac{s + 6}{(s + 6)^2 + 8^2} - \frac{6}{8} \frac{s}{(s + 6)^2 + 8^2}\right\}$$

$$= 1\text{-}e^{-6t} \cos 8t - \frac{6}{8} e^{-6t} \sin 8t$$

$$\boxed{c(t) = 1 - e^{-6t}\left[\frac{6}{8} \sin 8t + \cos 8t\right]}$$

4. Find all the time domain specifications for a unity feedback control system whose open loop transfer function is given as $G(s)= \frac{25}{s(s + 6)}$

The open loop transfer function $G(s)= \frac{25}{s(s + 6)}$ $H(s) = 1$

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s).H(s)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{25}{s(s+6)}}{1+\frac{25}{s(s+6)}.1} = \frac{25}{s(s+6)+25} = \frac{25}{s^2+6s+25}$$

$$\frac{C(s)}{R(s)} = \frac{25}{s^2+6s+25}$$

The characteristic equation is $s^2+6s+25=0$

By comparing the equation with standard form $s^2+2\zeta\omega_n s+\omega_n^2=0$, we get

$$\omega_n^2 = 2.5$$
$$\omega_n = 5$$

$$2\zeta\omega_n = 6$$
$$\zeta = \frac{6}{2x\omega_n} = \frac{6}{2x5} = 0.6$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 5\sqrt{1-0.36} = 4\text{rad / sec}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \tan^{-1}\left(\frac{\sqrt{1-0.36}}{0.6}\right) = 53.12^\circ = 0.92\text{rad.}$$

1. Rise time $t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-0.92}{4} = 0.55\sec$

2. Peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = 0.785s$

3. Delay time $t_d = \frac{1+0.7\zeta}{\omega_n} = \frac{1+0.7x0.6}{5} = 0.284s$

4. Setting time $t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.6x5} = 1.33s$

5. % Peak overshoot $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100\%$

$= e^{-0.6x\pi/\sqrt{1-0.6^2}} x100\%$

$\%M_p = 9.5\%$

Results

$t_r = 0.55\sec$
$t_p = 0.785\sec$
$t_d = 0.284\sec$
$t_s = 1.33\sec$
$\%M_p = 9.5\%$

5. The differential equation of the system is given by $\frac{d^2y}{dt^2}+5\frac{dy}{dx}+16y=16x$. Find the time domain specifications and output response expression.

Sol:

The given differential equation $\frac{d^2y}{dt^2}+5\frac{dy}{dx}+16y=16x$

Taking Laplace transform, we get

$$s^2Y(s)+5sY(s)+16Y(s)=16X(s)$$
$$\frac{Y(s)}{X(s)}=\frac{16}{s^2+5s+16}$$

Comparing with standard form of second order system,

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n}$$
$$\omega_n^2=16$$
$$\omega_n=4\text{rad}/\text{sec}$$

$$2\zeta\omega_n=5$$
$$\zeta=\frac{5}{2x4}=0.625$$

Damping ratio ζ =0.625

Natural frequency of oscillation = $\omega_n=4\text{rad}/\text{sec}$

Damping frequency $\omega_d=\omega_n\sqrt{1-\zeta^2}$

$$=4\sqrt{1-(0.625)^2}$$
$$=3.1225 \text{ rad/sec}$$

$$\theta=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}=\tan^{-1}\frac{\sqrt{1-0.625^2}}{0.625}=51.3^\circ=0.8949\text{rad}/\text{sec}$$

Delay time $t_d=\frac{1+0.7\zeta}{\omega_n}=\frac{1+.07(0.625)}{4}=0.3593\text{sec}$

Rise time $t_r=\frac{\pi-\theta}{\omega_d}=\frac{3.14-0.8949}{3.1225}=0.719\text{sec}$

Peak time $t_p=\frac{\pi}{\omega_d}=\frac{3.14}{3.1225}=1.006\text{sec}$

$\%\text{Peak overshoot } (M_p) = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \text{x}100$

$= e^{-(3.14\text{x}0.625)/\sqrt{1-0.625^2}} \text{x}100$

$= 8.09\%$

setting time t_s =

for 2% tolerance, $t_s = 4/\zeta\omega_n = \frac{4}{0.625\text{x}4} = 1.6\,\text{sec}$

for 5% tolerance, $t_s = 3/\zeta\omega_n = \frac{3}{0.625\text{x}4} = 1.2\,\text{sec}$

Output response of the system

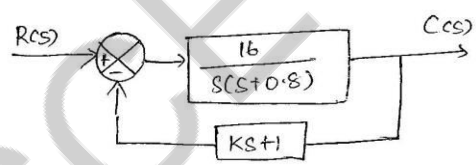
Since ζ =0.625, it is under damped system. The response of the second order under damped system is given by

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$$=1- \frac{e^{-0.625\text{x}4\text{x}t}}{\sqrt{1-0.625^2}} \sin(3.1225t + 0.8949)$$

$$c(t) = 1 - 1.2810e^{-2.5t} \sin(3.1225t + 0.8949)$$

6. The unity feedback system is characterized as shown in fig. What is the response c(t) to the unit step input. Given that $\zeta = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.



Sol

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s).H(s)}$

$$G(s) = \frac{16}{s(s+0.8)}; H(s) = Ks+1$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{16}{s(s+0.8)}}{1+\frac{16}{s(s+0.8)}(Ks+1)} = \frac{16}{s^2+0.8s+16Ks+16}$$

$$= \frac{16}{s^2+(0.8+16K)s+16}$$

By comparing with standard form of second order transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{16}{s^2 + (0.8 + 16k)s + 16}$$

$$\omega_n^2 = 16$$
$$\omega_n = 4$$

$$2\zeta\omega_n = 0.8 + 16K$$
$$K = \frac{2\zeta\omega_n - 0.8}{16}$$
$$= \frac{2x0.5x4 - 0.8}{16}$$
$$\boxed{K = 0.2}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{16}{s^2 + (0.8 + 16x0.2)s + 16} = \frac{16}{s^2 + 4s + 16}$$

Output response

The response in S domain, $C(s)=R(s).\frac{16}{s^2 + 4s + 16}$

For unit step input, R(s)=1/s

$$C(s) = \frac{1}{s}.\frac{16}{s^2 + 4s + 16}$$
$$= \frac{A}{s}\frac{Bs + C}{s^2 + 4s + 16}\left[\text{by partial fraction expansion}\right]$$
$$A(s^2 + 4s + 16) + Bs^2 + Cs = 16$$

comparing the constant term,

$$16A=16 \Rightarrow \boxed{A = 1}$$

Comparing the coefficients of s^2 term

$$A + B = 0 \Rightarrow \boxed{B = -1}$$

Comparing the coefficients of s term

$$4A + C = 0 \Rightarrow C = -4A - -4 \ \boxed{C = -4}$$

$$\therefore C(s) = \frac{1}{s} - \frac{s + 4}{s^2 + 4s + 16}$$
$$= \frac{1}{s} - \frac{s + 4}{s^2 + 4s + 4 + 12}$$
$$= \frac{1}{s} - \frac{s + 4}{(s + 2)^2 + 12}$$
$$= \frac{1}{s} - \frac{s + 2}{(s + 2)^2 + 12} - \frac{2}{(s + 2)^2 + 12}$$
$$= \frac{1}{s} - \frac{s + 2}{(s + 2)^2 + 12} - \frac{2}{\sqrt{12}}\frac{\sqrt{12}}{(s + 2)^2 + 12}$$

Time domain response is obtained by taking inverse Laplace transform, of C(s)

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\frac{\sqrt{12}}{(s+2)^2+12}\right]$$

$$=1\text{-}e^{-2t}\cos\sqrt{12}t - \frac{2}{\sqrt{12}}e^{-2t}\sin\sqrt{12}t$$

$$=1\text{-}e^{-2t}\cos\sqrt{12}t - \frac{2}{2\sqrt{3}}e^{-2t}\sin\sqrt{12}t$$

$$\boxed{c(t) = 1 - e^{-2t}[\cos\sqrt{12}t + \frac{1}{\sqrt{3}}e^{-2t}\sin\sqrt{12}t}$$

Damped frequency of oscillation

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464 \text{ rad/sec}$$

$$\text{Rise time } t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi - 1.047}{3.464} = 0.6046\,\text{sec}$$

$$\text{where } \theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \tan^{-1}\left(\frac{\sqrt{1-0.5^2}}{0.5}\right) = 60^\circ = 1.047\text{radian}$$

$$\text{Peak time } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907\,\text{sec}$$

$$\%\text{ Maximum overshoot,} \quad \%\,M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}x100\%$$

$$= e^{-0.5x\pi/\sqrt{1-.05^2}}x100\%$$

$$= 16.3\%$$

Setting time t_s=

$$\text{for 5\% error, } t_s = 3T = \frac{3}{\zeta\omega_n} = \frac{3}{0.5x4} = 1.5\,\text{sec}$$

$$\text{for 2\% error, } t_s = 4T = \frac{4}{\zeta\omega_n} = \frac{4}{0.5x4} = 2\,\text{sec}$$

7. The unity feedback control system is characteristic by an open loop transfer function G(s)=K/[s(s+10)]. Determine the gain K, so that the system will have damping ratio of 0.5 for this value of K, determine peak overshoot and peak time for a unit step input.

Sol

The closed loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s).H(s)}$$

$$G(s) = \frac{K}{s(s+10)}, H(s) = 1$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s^2+10s+K}$$

The standard form of second order equation of a closed loop system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

comparing these two equations,

$$\omega_n^2 = K \Rightarrow \omega_n = \sqrt{K}$$

$$2\zeta\omega_n = 10 \Rightarrow \zeta = \frac{5}{\sqrt{k}}$$

$$\text{for } \zeta\text{=0.5, K=}\frac{25}{\zeta^2} = \frac{25}{0.25} = 100$$

$$\boxed{K = 100}$$

$$\therefore \omega_n = \sqrt{K} = \sqrt{100} = 10$$

(b) Peak time(t_p)

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-(0.5)^2}} = 0.363\sec$$

$$\boxed{\begin{aligned}\%M_p &= 16.3\% \\ t_p &= 0.363\sec\end{aligned}}$$

8. The open loop transfer function of a unity feedback control system is given by G(s)= $\frac{K}{s(sT+1)}$ where K and T are positive constants. By what factor should the amplifier gain be reduced so that the peak overshoot of unit step response of the system is reduced form 75% to 25% APRIL/MAY 2017

Sol

The closed loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$G(s) = \frac{K}{s(sT+1)}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(sT+1)}}{1+\frac{K}{s(sT+1)}} = \frac{K}{Ts^2+s+K} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

Comparing this with standard second order system equation, the

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = \frac{K}{T} \Rightarrow \quad \omega_n = \sqrt{\frac{K}{T}}; \quad 2\zeta\omega_n = \frac{1}{T}$$

$$\therefore \zeta = \frac{1}{2\omega_n T} = \frac{1}{2\sqrt{KT}}$$

Let the peak overshoot M_{p1}correspond to $\zeta = \zeta_1$ and M_{p2} be the peak overshoot for $\zeta = \zeta_2$ and corresponding gains be K_1 and K_2 respectively

$$M_{p1} = e^{-\zeta_1\pi/\sqrt{1-\zeta_1^2}} = 0.75$$

taking natural logarthims on both sides,

$$\frac{-\zeta_1\pi}{\sqrt{1-\zeta_1^2}} = \ln 0.75 = -0.2877$$

from which $\zeta_1 = 0.091$

IIIly,

$$M_{p2} = e^{-\zeta_2\pi/\sqrt{1-\zeta_2^2}} = 0.25$$

taking ln on both sides,

$$\frac{-\zeta_2\pi}{\sqrt{1-\zeta_2^2}} = \ln 0.25 = -1.3863$$

$$\sqrt{1-\zeta_2^2} = 2.266\zeta_2$$

$$\zeta_2 = 0.4$$

$\zeta_1 \alpha \frac{1}{\sqrt{K_1}}$ and $\zeta_2 \alpha \frac{1}{\sqrt{K_2}}$ since T is same in both the cases

$$\frac{\zeta_1^2}{\zeta_2^2} = \frac{K_2}{K_1} = \frac{(0.091)^2}{(0.4)^2} = \frac{1}{19.4}$$

$$\text{(or)} \boxed{K_2 = \frac{1}{19.4} K_1}$$

Hence the original gain has to be reduced by factor 19.4 to reduce the overshoot from 75% to 25%

9. For a unity feedback control system, the open loop transfer function $G(s)=[10(s+2)]/s^2(s+1)$ find

1. The position, velocity , acceleration error constants
2. The steady state error, when $R(s)= \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

Sol

$$G(s) = \frac{10(s+2)}{s^2(s+1)}. \ H(s) = 1$$

1. Position, velocity and acceleration error constant

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s)$

$$= \underset{s\to 0}{Lt} \frac{10(s+2)}{s^2(s+1)} = \infty$$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, sG(s)$

$$= \underset{s\to 0}{Lt}\, s \frac{10(s+2)}{s^2(s+1)} = \infty$$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2 G(s)$

$$= \underset{s\to 0}{Lt}\, s^2 \frac{10(s+2)}{s^2(s+1)}$$

$$= \underset{s\to 0}{Lt} \frac{10(s+2)}{(s+1)} = \frac{10x2}{1} = 20$$

(2) To find steady state error

The error signal in s domain $E(s) = \frac{R(s)}{1+G(s).H(s)}$

$R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$, $\quad G(s)=\frac{10(s+2)}{s^2(s+1)}; H(s) = 1$

$$E(s) = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{1+\frac{10(s+2)}{s^2(s+1)}} = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{\frac{s^2(s+1)+10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

$$= \underset{s\to 0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2(s+1)}{s^2(s+1)+10(s+2)} - \frac{(s+1)}{3s(s^2(s+1)+10(s+2))}\right\}$$

$$=0-0+\frac{1}{60} = \frac{1}{60}$$

Steady state error $\boxed{e_{ss} = \frac{1}{60}}$

10. Consider a unity feedback system with closed loop transfer function $\frac{C(s)}{R(s)} = \frac{Ks+b}{s^2+as+b}$. Determine the transfer function G(s). show that the steady state error with unit ramp is given by $\frac{(a-K)}{b}$

Sol

For unity feedback system, H(s)=1

The closed loop transfer function, $M(s)=\frac{C(s)}{R(s)}$

$$M(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s).H(s)}=\frac{G(s)}{1+G(s)}$$

$$\therefore M(s)=\frac{G(s)}{1+G(s)}$$

$$G(s)=M(s)(1+G(s))$$
$$G(s)=M(s)+M(s).G(s)$$
$$G(s)-M(s)G(s)=M(s)$$
$$G(s)(1-M(s))=M(s)$$
$$\boxed{G(s)=\frac{M(s)}{1-M(s)}}\ M(s)=\frac{Ks+b}{s^2+as+b}\ \text{(given)}$$

$\therefore$ open loop transfer function

$$G(s)=\frac{M(s)}{1-M(s)}=\frac{\frac{Ks+b}{s^2+as+b}}{1-\frac{Ks+b}{s^2+as+b}}=\frac{Ks+b}{(s^2+as+b)-Ks+b}$$
$$=\frac{Ks+b}{s^2+(a-K)s}=\frac{Ks+b}{s(s+(a-K)}$$

$$\text{Velocity error constant, } K_v=\underset{s\to 0}{Lt}\, sG(s)H(s)$$
$$=\underset{s\to 0}{Lt}\, sG(s)$$
$$=\underset{s\to 0}{Lt}\, s\frac{Ks+b}{s(s+(a-K))}=\frac{b}{a-K}$$

With velocity input, steady state error,

$$e_{ss}=\frac{1}{K_v}=\frac{a-K}{b}$$

Hence proved

11. For a unity feedback control system having open loop transfer function $\frac{K(s+2)}{s(s+5)(4s+1)}$

The input applied is r(t)=1-3t. Find the minimum value of K. so that the steady state error is less than 1.

Sol

$$G(s) = \frac{K(s+2)}{s(s+10)(s+1)};\ H(s)=1$$

Error constants

$$K_p = \underset{s\to 0}{Lt}\, G(s)H(s) = \underset{s\to 0}{Lt} \frac{K(s+2)}{s(s+5)(4s+1)} = \infty$$

$$K_v = \underset{s\to 0}{Lt}\, sG(s)H(s) = \underset{s\to 0}{Lt}\, s\frac{K(s+2)}{s(s+5)(4s+1)}$$

$$= \frac{K(s+2)}{(s+5)(4s+1)} = \frac{2K}{5}$$

Total steady state error due to r(t)=1+3t

$$e_{ss} = \frac{1}{1+K_p} + \frac{3}{K_v}$$

$$= \frac{1}{1+\infty} + \frac{3}{\frac{2}{5}k} = 0 + \frac{3}{0.4K} = \frac{3}{0.4K}$$

$$e_{ss} < 1(\text{given}) \quad \Rightarrow \frac{3}{0.4k} < 1$$

$\Rightarrow \boxed{K > 7.5}$ For steady state error to be less than 1

12. Determine the type and order of the system with the following transfer function

(1) $\frac{s+4}{(s-2)(s+3)}$

Sol: order is 2
Type number 0

(2) $\frac{10}{s^3(s^2+2s+1)}$

Sol: order is 5
Type number 3

INCLUDE THIS

ROOT LOCUS

1. Sketch the root locus of the system whose open loop transfer function is $G(S) = \frac{K}{s(s+2)(s+4)}$.. Find the value of K, So that the damping ratio of the closed loop system is 0.5

Solution:

Step 1: To locate poles and zeros

The poles of open loop transfer function are the roots of the equation $s(s+2)(s+4) = 0$

Poles are lying at s = 0, -2, -4.

Let us denoted poles $p_1 = 0, p_2 = -2, p_3 = -4$

Step 2: To find the root locus on the real axis

The root locus starts from pole $p_1 = 0$& terminal at $p_2 = -2$ and it form the part of root locus and the root locus starts from p_3 & the terminates at open loop zero at infinity.

Step 3: to find asymptotes and centroid

$$\text{angle of asymptotes} = \frac{\pm 180°(2q+1)}{n-m} \quad q = 0,1,2,3....n-m$$

Here n = 3, m = 0. $\therefore q = 0,1,2,3.$

$$\text{when } q = 0, \quad \phi_A = \frac{\pm 180°}{3} = \pm 60°$$

$$\text{when } q = 1, \quad \phi_A = \frac{\pm 180° \times 3}{3} = \pm 180°$$

$$\text{when } q = 2, \quad \phi_A = \frac{\pm 180° \times 5}{3} = \pm 300° = \pm 60°$$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{n-m}$$

$$\sigma_A = \frac{0-2-4-0}{3} = -2$$

Step 4: To find the break away and break in points

$$\text{The closed loop transfer function} = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+2)(s+4)}}{1+\frac{K}{s(s+2)(s+4)}} = \frac{K}{s(s+2)(s+4)+K}$$

The characteristic equation is given by

$$s(s+2)(s+4)+K=0$$
$$s(s^2+6s+8)+K=0$$
$$s^3+6s^2+8s+K=0$$
$$K=-\left[s^3+6s^2+8s\right]$$
$$\frac{dK}{ds}=\left[s^2+12s+8\right]$$
$$\text{put } \frac{dK}{ds}=0 \Rightarrow 3s^2+12s+8=0$$
$$s=\frac{-12\pm\sqrt{12^2-4\times3\times8}}{2\times3}=-0.845 \text{ or } -3.154$$

Check for K;

When s=-0.845, the K is given by $K=-\left[(-0.845)^3+6(-0.845)^2+8(-0.845)\right]=3.08$ Since K is +ve and real for s = -0.845, this point is actual break away point.

When s = -3.154, the value is given by $K=-\left[(-3.154)^3+6(-3.154)^2+8(-3.154)\right]=-3.08$

Since K, is negative for s = -3.154, this is not a actual breakaway point.

Step 5: To find angle of departure

since there are no complex pole (or) zero, there is no need to find angle of departure

Step 6: To find the crossing point imaginary axis.

The characteristics equation is given by:

$$s^3+6s^2+8s+K=0$$

Put $s=j\omega$

$$(j\omega)^3+6(j\omega)^2+8(j\omega)+K=0$$
$$-j\omega^3-6\omega^2+8j\omega+K=0$$

Equating imaginary part to zero,

$$-j\omega^3+j8\omega=0$$
$$-j\omega^3=-j8\omega$$
$$\omega^2=8 \Rightarrow \omega=\pm\sqrt{8}$$
$$\omega=\pm2.8$$

Equating real parts to zero

$$-6\omega^2 + K = 0$$

$$K = 6\omega^2 = 6 \times 8 = 48$$

The crossing point of root locus is $\pm j2.8$

The value of K corresponding to this point is K = 48. Thus is the limiting value of K for stability.

The complete root locus sketch is shown in fig. The root locus has three branches. One branch starts at the pole at s = -4, travel the '-ve' real axis to meet the zero at infinity, the other two root locus branches starts at s = 0 and s = -2 & travel the –ve real axis breakaway from real axis at s= -0.845, then cross imaginary axis $s = \pm j2.8$ & travel parallel to asymptotes to meet zero at infinity.

To find the value of K corresponding to G = 0.5

Given that G = 0.5

$$\cos\theta = 0.5 \Rightarrow \theta = \cos^{-1} 0.5 = 60°$$

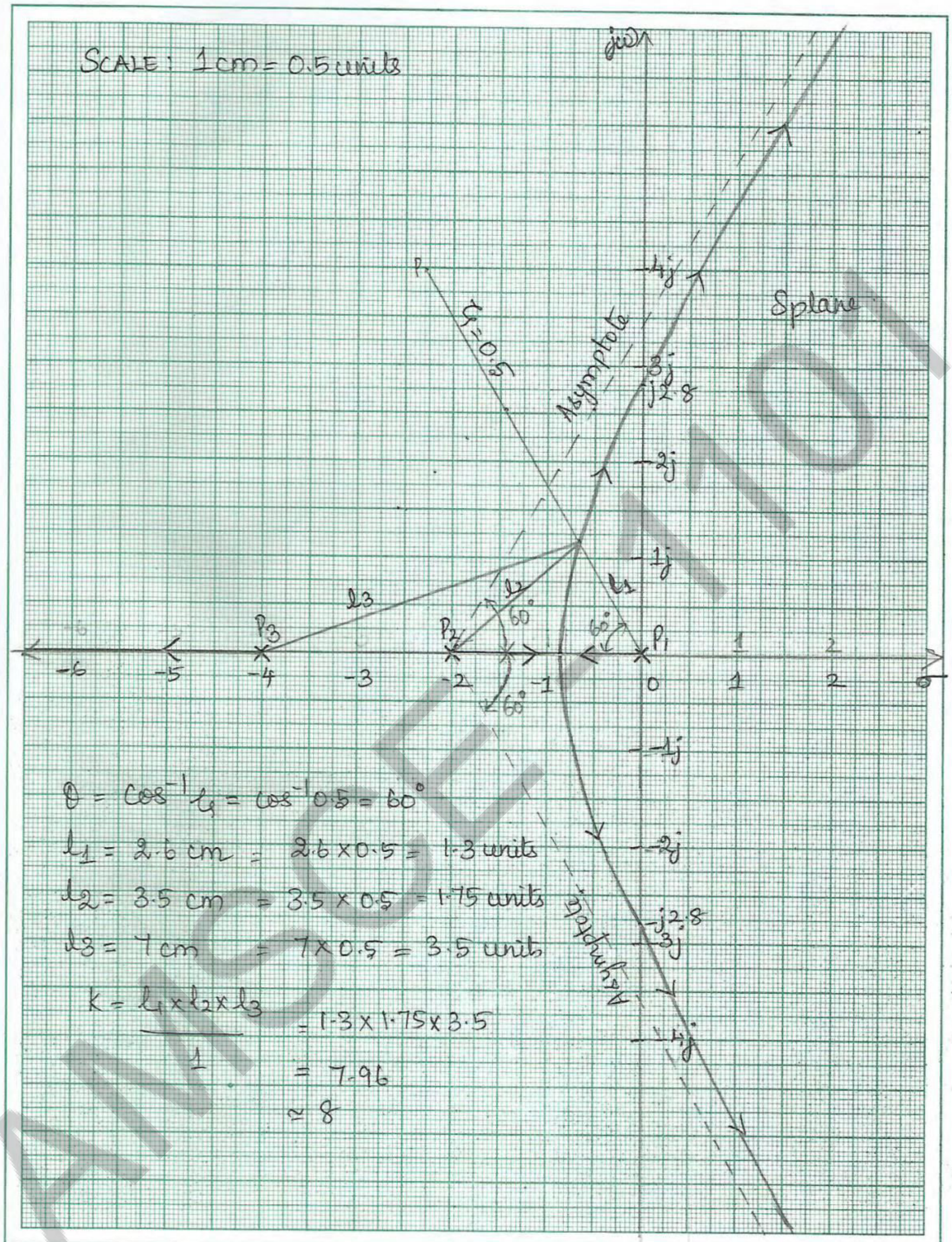
Draw a line OP, such that the angle between line OP & -ve real axis is 60^0 $(\theta = 60°)$

The meeting point of OP and root locus is s_d

k at s = s_d

$$= \frac{\text{Product of length of vector from all pole to the point } s = s_d}{\text{Product of length of vector from all zeros to the point } s = s_d}$$

$$= \frac{\ell_1 \times \ell_2 \times \ell_3}{1} = \frac{1.3 \times 1.75 \times 3.5}{1} = 7.96 \approx 8$$



$\theta = \cos^{-1}\zeta = \cos^{-1}0.5 = 60^\circ$

$l_1 = 2.6\ cm = 2.6 \times 0.5 = 1.3$ units

$l_2 = 3.5\ cm = 3.5 \times 0.5 = 1.75$ units

$l_3 = 7\ cm = 7 \times 0.5 = 3.5$ units

$$k = \frac{l_1 \times l_2 \times l_3}{1} = 1.3 \times 1.75 \times 3.5 = 7.96 \simeq 8$$

2. The open loop transfer of a unity feedback control system is given by, $G(s) = \frac{K}{s^2\left(s^2+4s+13\right)}$ **Sketch the root locus.**

Solution:-

Step 1: To locate poles and zeros

Poles

$$s = 0,\ \frac{-4 \pm \sqrt{4^2 - 4\times 13}}{2}$$
$$= 0, -2+j3, -2-j3$$

Let $P_1 = 0, P_2 = -2+j3, P_3 = -2-j3$

Zeros: Nil

Step 2: To find root locus on the real axis there is only one pole at origin. Hence the entire –ve real axis will be a part of root locus.

Step 3: To find angles of asymptotes and centroid

Angle of asymptote $\phi_A = \frac{\pm 180°\left(2q+1\right)}{n-m}$

q = 0, 1, 2, ……n-m

Here n = 3, q = 0, 1, 2, 3.

When q =0, $\phi_A = \frac{\pm 180°}{3} = \pm 60°$

When q =1, $\phi_A = \frac{\pm 180°}{3} \times 3 = \pm 180°$

When q = 2, $\phi_A = \frac{\pm 180°}{3} \times 5 = \pm 300° = \mp 60°$

When q = 3, $\phi_A = \frac{\pm 180°}{3} \times 7 = \pm 420° = \pm 60°$

Centroid $\sigma_A = \frac{\text{Sum of poles - Sum of zeros}}{n-m}$

$$= \frac{0-2+j3-2-j3-0}{3} = \frac{-4}{3} = -1.33$$

Step 4: To find the breakaway and break in points

The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s^2+4s+13)}}{\frac{K}{s(s^2+4s+13)}} = \frac{K}{s(s^2+4s+13)+K}$$

The characteristics equation is $s(s^2+4s+13)+K=0$

$$s(s^2+4s+13)+K=0$$

$$K=-(s^2+4s+13)$$

$$\frac{dK}{ds} = -\left[3s^2+8s+13\right]$$

$$\frac{dK}{ds} = 0 \Rightarrow 3s^2+8s+13=0$$

$$s = \frac{-8 \pm \sqrt{8^2-4\times 13\times 3}}{2\times 3}$$

$$s = -1.33 \pm j\,1.6$$

Check for K:

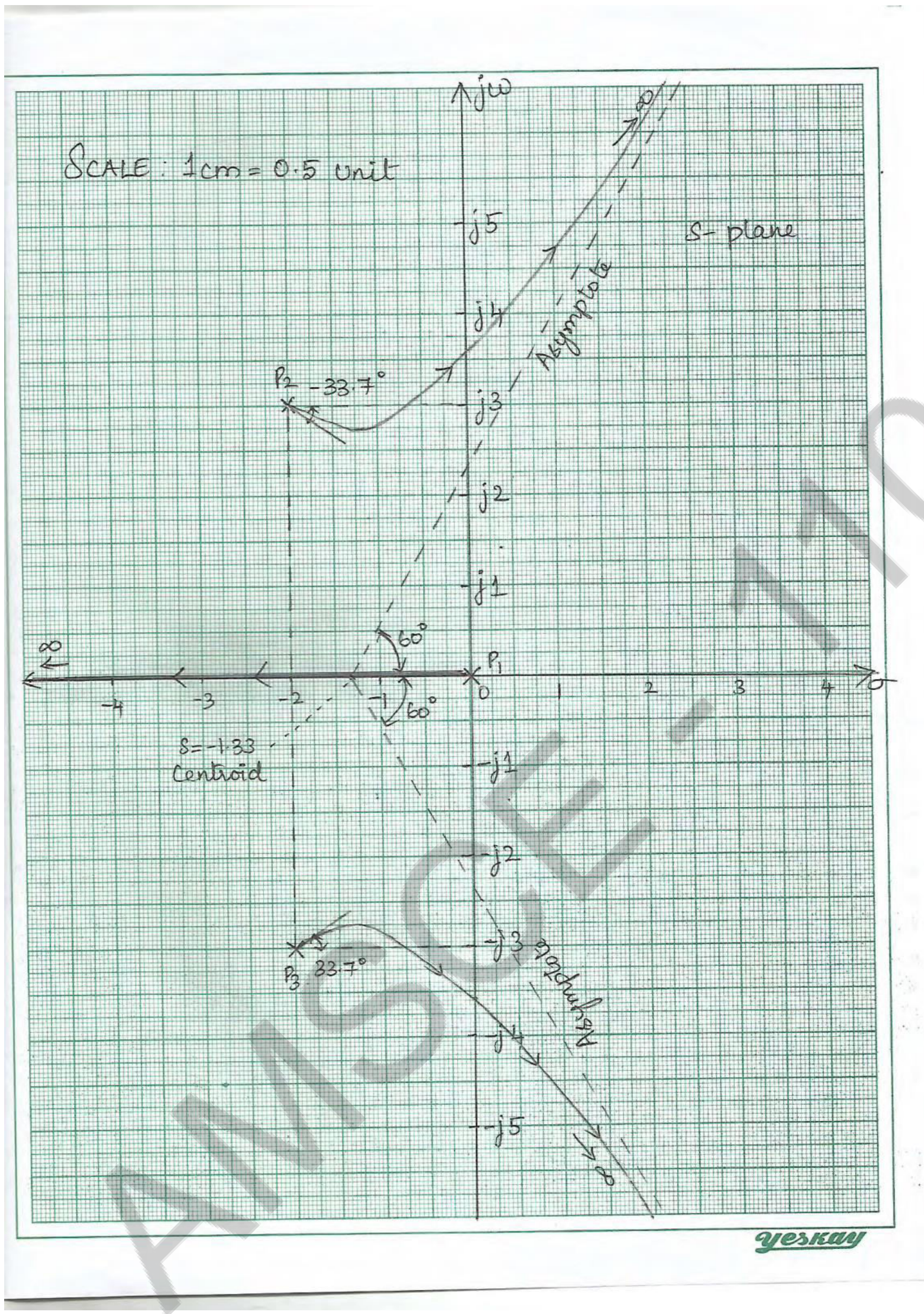
When s= -1.33 + j1.6 the value of K is given by

$$K = -(s^2+4s^2+13s)$$

$$= -\left[(-1.33+j1.6)^3 + 4(-1.33+j1.6)^2 + 13(-1.33+j1.6)\right]$$

$$\neq \text{real} + \text{ve}$$

Similarly when s = -1.33 – j 1.6, the value of K is not positive & real. Therefore, the root locus has reither breakaway nor breakin points,



Step 5: To find the angle of departure consider complex pole P_2. Draw velocities from all other poles to the pole P_2. Let the angles of these vectors be $\theta_1 \& \theta_2$

Here $\theta_1 = 180° - \tan^{-1} 3/2 = 123.7°, \quad \theta_2 = 90°$

Angle of departure from the complex pole P_2

$$
\begin{aligned}
&= 180° - (\theta_1 + \theta_2) \\
&= 180° - (123.7° + 90°) \\
&= -33.7°
\end{aligned}
$$

The angle of departure at complex pole P_3 is negative of the angle of departure at complex pole A.

Angle of departure at pole $P_3 = +33.7^0$

Step 6: To find the crossing point an imaginary axis

The characteristics equation is given by

$$s^3 + 4s^2 + 13s + K = 0$$

Put s = jω

$$
\begin{aligned}
&(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0 \\
&\Rightarrow -j\omega^3 - 4\omega^2 + j13\omega + K = 0
\end{aligned}
$$

Equating imaginary parts to zeros,

$$
\begin{aligned}
&-\omega^2 + 13\omega = 0 \\
&-\omega^3 = -13\omega \\
&\omega^2 = 13 \Rightarrow \omega = \pm\sqrt{13} = \pm 3.6
\end{aligned}
$$

Equating real part to zero

$$-4\omega^2 + K = 0 \Rightarrow K = 4\omega^2 = 4 \times 3 = 52$$

The crossing point of root locus is ± 3.6

The value of K at this crossing point is 52.

The complete root sketch is shown in fig.

3. The open loop transfer function of 0 unity feedback system is given by, $G(S) = \frac{K(s+9)}{s(s^2+4s+11)}$. Sketch the root locus of the system.

Solution:-

Step 1: To locate poles & zeros

Poles

$$s\left(s^2+4s+11\right)=0$$
$$s=0,-2+j2.64,-2-j2.64$$

Let $P_1 = 0$, $P_2 = -2+j2.64$, $P_3 = -2-j2.64$

Zeros: s +9 = 0 & s = -9

Let Z = -9

Step 2: To find root locus on real axis the position of real axis from s = 0 to s= -9 will be a part of root locus & from s = -9 to s = ∞ will not be part of root locus.

Step 3: To find angle of asymptotes & centroid angle of asymptotes

$$\phi_A = \pm\frac{180°(2q+1)}{n-m}$$

Where q = 0, 1, ….n - m

Here n = 3, m = 1, q = 0, 1, 2

When q = 0, $\phi_A = \pm\frac{180°}{2} = \pm 90°$

When q = 1, $\phi_A = \pm\frac{180°}{2}\times 3 = \pm 270 = \mp 90°$

When q = 2, $\phi_A = \pm\frac{180°}{2}\times 5 = \pm 450° = \mp 90°$

$$\sigma_A = \frac{\sum \text{Poles} - \sum \text{zeors}}{n-m}$$

Centroid $= \frac{0-2+j2.64-2-j2.64-(-9)}{2}$

$= 2.5$

Step 4: To find break away and break is points

The characteristics equation of the system is

$$1+\frac{K(s+9)}{s(s^2+4s+11)}=0$$

$$K=\frac{-s(s^2+4s+11)}{s+9}$$

$$\frac{dK}{dS}=0 \Rightarrow 2s^3+31s^2+61s=0$$

$$\Rightarrow s(s^2+15.5s+30.5)=0$$

$$\Rightarrow s=0, s=-13.157,\ s=-2.313$$

There are no valid breakaway (or) break in points.

Step 5: To find the angle of departure

Consider complex pole P_2. Draw vectors from all poles and zeros to pole P_2.

$$\theta_1 = 180° - \tan^{-1}\frac{2.64}{2} = 127.1°$$

$$\theta_2 = 90°$$

$$\theta_3 = \tan^{-1}\frac{2.64}{7} = 20.7°$$

The Angle of departure from complex pole P_2 = 180°-(127.1°+90°)+20.7° = -16.4°

The angle of departure from complex pole P_3 is negative of the angle of departure of from complex pole P_2.

Angle of departure from complex pole $P_3 = 16.4^0$

Step 6: To find the crossing point of imaginary axis.

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

The characteristics equation is 1+G(s)

$$\Rightarrow 1+\frac{K(s+9)}{s(s^2+4s+11)}=$$

$$\Rightarrow s(s^2+4s+11)$$

$$\Rightarrow s^3+4s^2+11s+Ks+9K=0$$

$$\Rightarrow s^3+4s^2+(11+K)s+9K=0$$

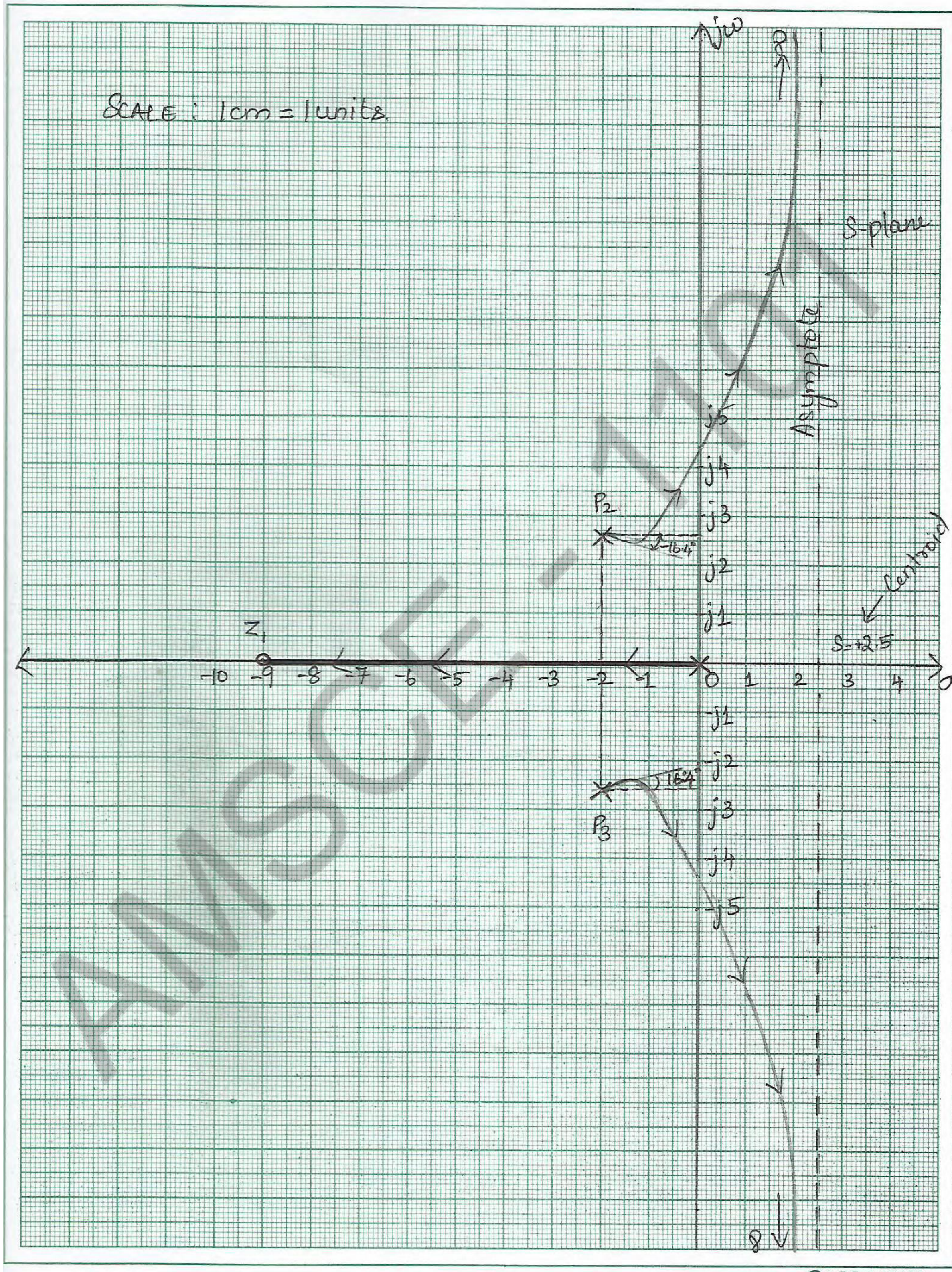
Put s = jω

$$(j\omega)^3+4(j\omega)^2+11(j\omega)+Kj\omega+9K=0$$

$$-j\omega^3-4\omega^2+j11\omega+jK\omega+9K=0$$

Equating imaginary part to zero

$$-\omega^3 + 11\omega + K\omega = 0$$
$$\omega^3 = (11 + K)\omega$$
$$\omega^2 = 11 + K \qquad \rightarrow (1)$$


SCALE : 1cm = 1units.
jω
S-plane
Asymptote
Centroid
S=+2.5
P2
P3
Z1
-16.4°
16.4°
j5
j4
j3
j2
j1
-j1
-j2
-j3
-j4
-j5
σ
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4

Equating real part to zero,

–

$$-4\omega^2 + 9K = 0 \Rightarrow 9K = 4\omega^2$$

But, $\omega^2 = 11 + K$

$$\therefore 9K = 4(1+K) = 44 + 4K$$
$$\therefore 9K - 4K = 44$$
$$5K = 44$$
$$K = \frac{44}{5} = 8.8$$

Put K = 8.8 in eqn (1)

$$\omega^2 = 11 + 8.8 = 19.8$$
$$\omega = \pm\sqrt{19.8} = \pm 4.4$$

The crossing point of root locus = ± j 4.4. the value of K corresponding to this point is 8.8.

The complete root locus sketch is shown in fig.

4. Sketch the root locus plot of the system whose OLTF is given as APRIL/MAY 2017

$$G(s).H(s) = \frac{K}{s(s+4)(s^2+4s+13)}$$

Solution:-

Step 1: To locate poles and zeros

Poles:

$$s(s+4)(s^2+4s+13) = 0$$
$$s = 0, -4, -2+j3, -2-j3$$

Let $P_1 = 0$, $P_2 = -4$, $P_3 = -2+j3$, $P_4 = -2-j3$

Step 2: To locate root locus on real axis

The portion between s =0 & s = -4 is a part of the root locus.

Step 3: To find angle of asymptotes & centroid

Angle of asymptotes $\phi_A = \frac{\pm 180(2q+1)}{n-m}$

$q = 0, 1, ...n-m$

Here n = 4, m = 0, q = 0, 1, 2, 4,

When q = 0, $\phi_A = \pm\frac{180}{4}\times 1 = \pm 45°$

When q = 1, $\phi_A = \pm\frac{180°}{4}\times 3 = \pm 135°$

When q = 2, $\phi_A = \pm\frac{180°}{4}\times 5 = \pm 225°$

When q = 3, $\phi_A = \pm\frac{180°}{4}\times 7 = \pm 315°$

Centroid

$$\sigma_A = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$$
$$= \frac{(0-4-2+j3-2-j3)-(0)}{4-0}$$
$$\sigma_A = -2$$

Step 4: To find breakaway and break in points

The characteristics equation is

$$1 + G(s)H(s) = 0$$
$$1 + \frac{K}{s(s+4)(s^2+4s+13)} = 0$$
$$K = -s(s+4)(s^2+4s+13)$$
$$K = -(s^4+18s^3+29s^2+52s)$$
$$\frac{dK}{dS} = 0 \Rightarrow 4s^3+24s^2+58s+52 = 0$$
$$\Rightarrow s = -2, s = -2+j1.58, s = -2-j1.58$$

The valid break away point are

B_1 = -2, B_2 = -2+j 1.58, B_3 = -2-j 1.58

Step 5: To find angle of departure

Consider complex pole P_3. Draw vectors from all other poles to pole P_3.

Now

$$\theta_1 = 125°$$
$$\theta_2 = 90°$$
$$\theta_3 = 55°$$

Angle of departure from

$$P_3 = 180° - (\theta_1 + \theta_2 + \theta_3)$$
$$= 180° - (125° + 90° + 55°)$$
$$= -90°$$

Angle of departure from $p_4 = -(-90°) = +90°$

Step 6: To find crossing point of imaginary axis

The characteristics equation is

$$s^4 + 8s^3 + 29s^2 + 52s + K = 0$$

Using Routh Hurwitz criterion

	Column 1			
s^4:	1	29	K	ROW 1
s3:	8	52		ROW 2
s^2:	22.5	K		ROW 3
s^1:	52-0.3K			ROW 4
s^0:	K			ROW 5

For stability K > 0, (from S^0 row)

And 52 – 0.35 K > 0 (from S^1 row)

K > 0, K < 148.6

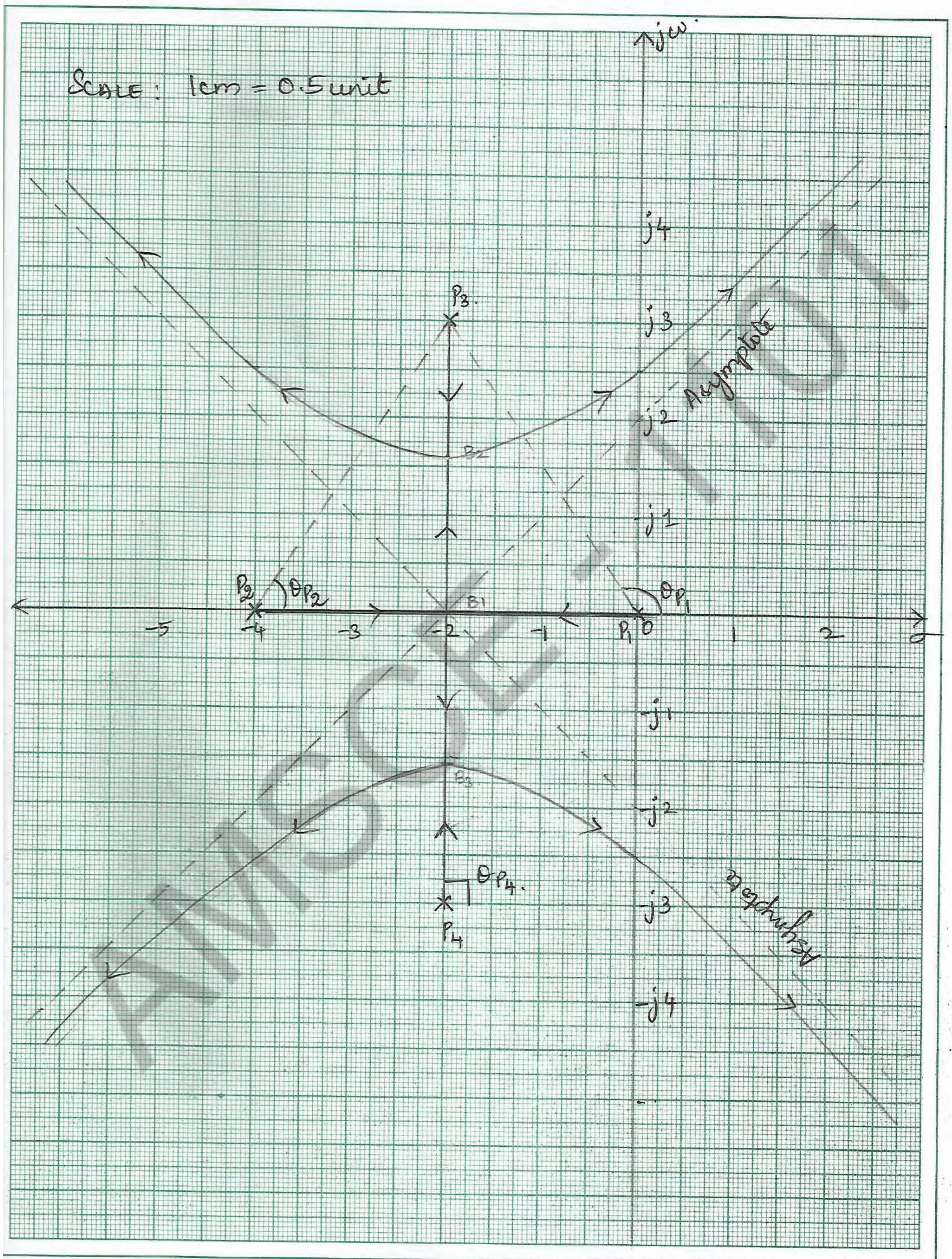
For system to be stable, the maximum value of K is 148.6

The auxiliary equation is $22.5\, s^2 + K = 0$

$$22.5s^2 + 148.6 = 0$$
$$s = \pm j2.56$$

The crossing point of imaginary axis is 2.56 & corresponding value of K is 148.6

The complex root locus sketch is shown in fig.


SCALE: 1cm = 0.5 unit
jω
j4
j3
j2
j1
-j1
-j2
-j3
-j4
P3
P2
P1
P4
θp2
θp1
θp4
Asymptote
Asymptote
-5
-4
-3
-2
-1
0
1
2
σ

2`

Time response analysis

5. With neat steps, write down the procedure for the construction of root locus.

<u>Procedure for constructing the root locus of the loop transfer function when k is varied from 0 to ∞.</u>

1. Symmetry: The root locus plot is always symmetrical with respect to the real axis is s – plane

2. Starting and Ending points: the root locus originates from an open loop pole ie, K = 0 and terminates at open loop zero is K = ∞

3. Number of Loci: The number of separate root locus (N) depends upon the number of pole (n) and number of zeros (m) of the loop transfer function.

N = n for n> m

N = m for m > n

Where n is the number of finite poles of G(s) H(s)

M is the number of finite zeros of G(s) H(s)

Thus, the number of separate root locus is equal to the number of poles (or) zeros which ever is greater.

4. Existence on real axis: Some of the loci will lie on the real axis. A point on the real axis if the sum of open loop transfer function poles and zeros to the point is odd.

5. The number of asymptotic lines: Asymptotes is defined as a line on which the root locus touches at infinity.

For the function, G(s) H(s) having n finite poles and m finite zeros , the no. of asymptotes q = n-m

6. Angle of asymptotes: If the number of poles is greater than the number of zeros n > m; then n – m branches will move to infinity and these cbranches move along the asymptotes. For root locus, the angle of asymptotes, $\phi_A = 0 \pm \frac{180^\circ(2q+1)}{n-m}$

Where q is a positive integer having values 0, 1, 2…. (n – m)

7. Centre of Asymptote or centroid : The point at which asymptotes intersect on real axis in s – plane is called centroid & is given by

$$\sigma_A = \frac{\sum \text{poles of } G(s)H(s) - \sum \text{zeros of} G(s)H(s)}{n-m}$$

8. Breakaway (or) break in points: Breakaway point is defined as the point at which root locus comes out of the real axis and breakin point is defined as the point at which root locus enters the real axis.

The breakaway (or) break in points are the points on the root locus at which multiple roots of the characteristic equation occur.

The following are the steps to determine the breakaway (or) break in points

(a) Find the characteristics equation, 1+ G(s) H(s) = 0

(b) Write K in terms of s

(c) Derive $\frac{dK}{ds}$ & put $\frac{dK}{ds} = 0$

(d) The roots of equation $\frac{dK}{ds} = 0$ may be breakaway (or) break in points

If the value of K is positive & real for any root of $\frac{dK}{dS} = 0$, then the corresponding root is avalid break away (or) break in points

9. Intersection of root locus with imaginary axis

The point of intersection of root locus with the imaginary axis in the s – plane can be determined by use of the Routh criterion. Alternatively by letting s = jω in the characteristic equation and separate real part and imaginary part. Two equations are obtained: one by equating real parts to zero and the other by equating imaginary part to zero. Solve the two equations for ω and K.

The value of ω gives the point where the root locus crosses the imaginary axis & the value of K gives value of gain K at crossing point. Also this value of K is the limiting value of K for stability of the system.

10. Angle of departure (or) arrival: The root locus leaves from a complex pole & arrives at a complex zero. These two angles are known as angle of departure and angle of arrival, respectively.

$$\left.\begin{array}{l}\text{Angle of departure}\\ \text{(from a complex}\\ \text{pole A)}\end{array}\right\} = 180^\circ - \begin{pmatrix}\text{sum of angles to the complex}\\ \text{pole A from other poles}\end{pmatrix} + \begin{pmatrix}\text{Sum of angles of vectors}\\ \text{to the complex pole A from zeros.}\end{pmatrix}$$

$$\left.\begin{array}{l}\text{Angle of arrival at}\\ \text{a complex zero A}\end{array}\right\} = 180^\circ - \begin{pmatrix}\text{Sum of angles of vectors}\\ \text{to the complex zero A from}\\ \text{all other zeros}\end{pmatrix} + \begin{pmatrix}\text{Sum of angles of vectors to}\\ \text{the complex zero A from}\\ \text{poles}\end{pmatrix}$$

11. Value of K at a point on the root locus

The value of K at a point S1 on the root locus is determine by measuring the vectors from the poles and zeros of loop transfer function to point S1 on the root of is given as

$$K = \frac{1}{|G(s)\,H(s)|} = \frac{\prod_{j-1}^{n+m} |s_1 + P_j|}{\prod_{i-1}^{n} |s_1 + Z_i|}$$

$$= \frac{\text{Product of all vectors lengths from poles of G(s) H(s) to } s_1}{\text{Product of all vectors lengths from zeros of G(s) H(s) to } s_1}$$

UNIT – 3

FREQUENCY RESPONSE AND SYSTEM ANALYSIS

PART-A

1. What is meant by frequency response ? April/May 2017

A frequency response is the steady state response of s system when the input to the system is a sinusoidal signal.

2. List out the different frequency domain specification? (NOV/DEC 2015, MAY/JUNE 2016, Nov/Dec 2017)

The frequency domain specifications are

- Resonant peak
- Resonant frequency
- Bandwidth
- Cut- off rate
- Gain margin
- Phase margin

3. Define – Resonant Peak

The maximum value of the magnitude of closed loop transfer function called resonant peak.

4. What is bandwidth?

The bandwidth is the range of frequencies for which the system gain is more than - 3 dB. The bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time.

5. Define Cut – off rate ?

The slope of the log – magnitude curve near the cut – off is called cut – off rate. The cut –off rate indicates the ability to distinguish the signal from noise.

6. Define – Gain margin? (MAY/ JUNE 2013)

The gain margin, K_g is defined as the reciprocal of the magnitude of the open loop transfer function at phase cross over frequency. Gain margin $kg = \frac{1}{|G(j\omega)|_{\alpha=\omega pc}}$

7. Define Phase Cross over frequency? April/May 2019

The frequency at which, the phase of open loop transfer function is -180° is called phase cross over frequency ω_{pc}.

8. What is Phase margin? (MAY/JUNE 2013 & NOV/DEC 2011)

It is the amount of phase lag at the gain cross over frequency required to bring system to the verge of instability. The phase margin, $\gamma=180^o+\phi_{gc}$.

9. Define Gain cross over frequency? (APRIL/MAY 2011, May 2016, April/May 2019 & Nov/Dec 2019)

The gain cross over frequency ω_{gc}, is the frequency at which the magnitude of the open loop transfer function is unity.

10. What is Bode plot?

The Bode plot is the frequency response plot of the transfer function of a system. A bode plot consist of two graphs. One is the plot of magnitude of sinusoidal transfer versus log ω. The other is a plot of the phase angle of a sinusoidal function versus log ω.

11. What are the main advantages of Bode plot?

The main advantages are:

(i) Multiplication of magnitude can be to addition.
(ii) A simple method for sketching an approximate log curve is available
(iii) It is based on asymptotic approximation. Such approximation is sufficient if rough information on the frequency response characteristics is needed.
(iv) The phase angle curves can be easily draw if a template for the phase angle curve of 1+jω is available.

12. Define Corner frequency? April/May 2018

The frequency at which the two asymptotes meet in a magnitude plot is called corner frequency.

13. Define Phase lag and phase lead?

A negative phase angle is called phase lag. A positive phase angle is called phase lead.

14. What are M circles? (NOV/DEC 2015, MAY/JUNE 2016)

The magnitude M of closed loop transfer function with unity feedback will be in the form of circle on complex plane for each constant value of M. The family these circles are called M circles.

15. What is Nichols chart?

The chart consisting of M & N loci in the log magnitude versus phase diagram is called Nichol's chart.

16. What are two contours Nichol's chart?

Nichols chart of M and N contours superimposed on ordinary graph. The M contours are the magnitude of closed loop system in decibel and the N contours are the phase angle locus of closed loop s system.

17. What is non – minimum phase transfer function?

A transfer function which has one or more zeros in the right half S – plane is known as non – minimum phase transfer function.

18. What are the advantages of Nichols chart? (APRIL/MAY 2015)

The advantage are:

(i) It is used to find the close loop frequency response from open loop frequency response.
(ii) Frequency domain specification can be determined from Nichols chart.
(iii) The gain of the system can be adjust to satisfy the given specification.

19. What are N circles? **(NOV/DEC 2015, MAY/JUNE 2016)**

If the phase of closed loop transfer function with unity feedback is $\propto$, then N = tan $\propto$. For each constant value of N, a circle can be drawn in the complex plane. The family of these circles are called N circles.

20. What are the two types of compensation?

The two types of compensation are

(i) Cascade or series compensation
(ii) Feedback compensation or parallel compensation

21. What are the three types of compensator? **(MAY/JUNE 2013)**

The three types of compensators are

1. Lag compensator
2. Lead compensator
3. Lag – lead compensator

22. What are the uses of lead compensator? **(NOV/DEC 2011)**

The uses of lead compensator are

- Speeds up the transient response
- Increases the margin of stability of a system
- Increases the system error constant to a limited extent.

23. What is the use of lag compensator? **(APRIL/MAY 2011)**

The lag compensator improves the steady state behaviour of a system, while nearly preserving its transient response.

24. When lag – lead compensator is required?

The lag – lead compensator is required when both the transient and steady state response of a system has to be improved.

25. What is a compensator? **(APRIL/MAY 2011)**

A device inserted into the system for the purpose of satisfying the specification is called as a compensator.

26. When lag/ lead/ lag – lead compensation is employed? **(APRIL/MAY 2011, May/June 2016, April/May 2017, Nov/Dec 2017)**

Lag compensation is employed for stable system for improvement in steady state performance. Lead compensation is employed for stable/ unstable system for improvement in transient state performance. Lag – lead compensation is employed for stable/unstable system for improvement in both steady state and transient state performance.

27. What are the effects of adding a zero to a system?

Adding a zero to a system results in pronounced early peak to system response thereby the peak overshoot increase appreciably.

28. What are the characteristics of phase lead network? (APRIL/MAY 2015)

- In lead compensation, if the bandwidth increases, the speed off response will also get increased
- The lead compensator having the phase lead frequency response characteristics which will improve the transient response and will also extent to steady state response.

29. What is the significant of Nichol's plot? (NOV/DEC 2016)

The complete closed loop frequency response can be obtained by using Nichol's chart. All the frequency domain specification can be obtained by sketching open loop magnitude – phase plot on the Nichol's chart.

30. What is series compensation? (NOV/DEC 2016)

If the compensator is placed in the forward path of the plant then, the compensation is termed as series compensation.

PART – B
BODE PLOT

1. For the following transfer function, sketch the Bode plot. Also determine gain margin & phase margin.

$$G(s)H(s) = \frac{5}{s(10+s)(20+s)}$$

Solution:-
The sinusoidal transfer function G(jω) is obtained by replacing s by jω in the s – domain transfer function.

$$G(s)H(s) = \frac{5}{s\times 10(1+0.1s)20(1+0.05s)}$$

$$= \frac{5}{200s(1+0.1s)(1+0.05s)}$$

$$= \frac{0.025}{s(1+0.1s)(1+0.05s)}$$

Put s = jω

$$G(j\omega)H(j\omega) = \frac{0.025}{j\omega(1+j0.1\omega)(1+j0.05\omega)}$$

Magnitude plot

The corner frequencies are $\omega_{c1} = \frac{1}{0.1} = 10\,\text{rad/sec}$

$$\omega_{c2} = \frac{1}{0.05} = 20\ \text{rad/sec}$$

Team	Corner frequency rad /sec	Slope db/dec	Change in slope db/dec

$\frac{0.025}{j\omega}$	-	-20	-
$\frac{1}{1+j0.1\omega}$	$\omega_{c1}=\frac{1}{0.1}=10$	-20	-20 – 20 = -40
$\frac{1}{1+j0.05\omega}$	$\omega_{c2}=\frac{1}{0.05}=20$	-20	-40-20 = -60

Choose a low frequency ω_ℓ such that $\omega_\ell < \omega_{c1}$ & choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let ω_ℓ = 0.1 rad/ sec, ω_h = 50 rad/sec

Let $A = |G(j\omega)|$ in dB

Calculating of gain A at $\omega = \omega_\ell, \omega_{c1}, \omega_{c2} + \omega_h$

At $\omega = \omega_\ell$, $\quad A = 20\log\left|\frac{0.025}{j\omega}\right| \omega = \omega_\ell = 0.1$

$$= 20\log\left|\frac{0.02}{0.1}\right|$$

$$= -12.04\,\text{dB}$$

At $\omega = \omega_{c1}$ $\quad A = 20\log\left|\frac{0.025}{j\omega}\right| \omega = \omega_{c1} = 10$

$$= 20\log\left|\frac{0.025}{10}\right|$$

$$= -52.04 \text{ db}$$

At $\omega = \omega_{c2}$ $\quad A = \left[\text{slope from } \omega_{c1} \text{to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + \text{A at } \omega = \omega_{c1}$

$$= 40 \times \log\left|\frac{20}{10}\right| + (-52.04)$$

$$-12.04 + (-52.04) = -64.08\,\text{dB}$$

At $\omega = \omega_h$ $\quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + \text{A at } \omega = \omega_{c2}$

$$= -60 \times \log\left(\frac{50}{20}\right) + (-64.08) = -88 \text{ db}$$

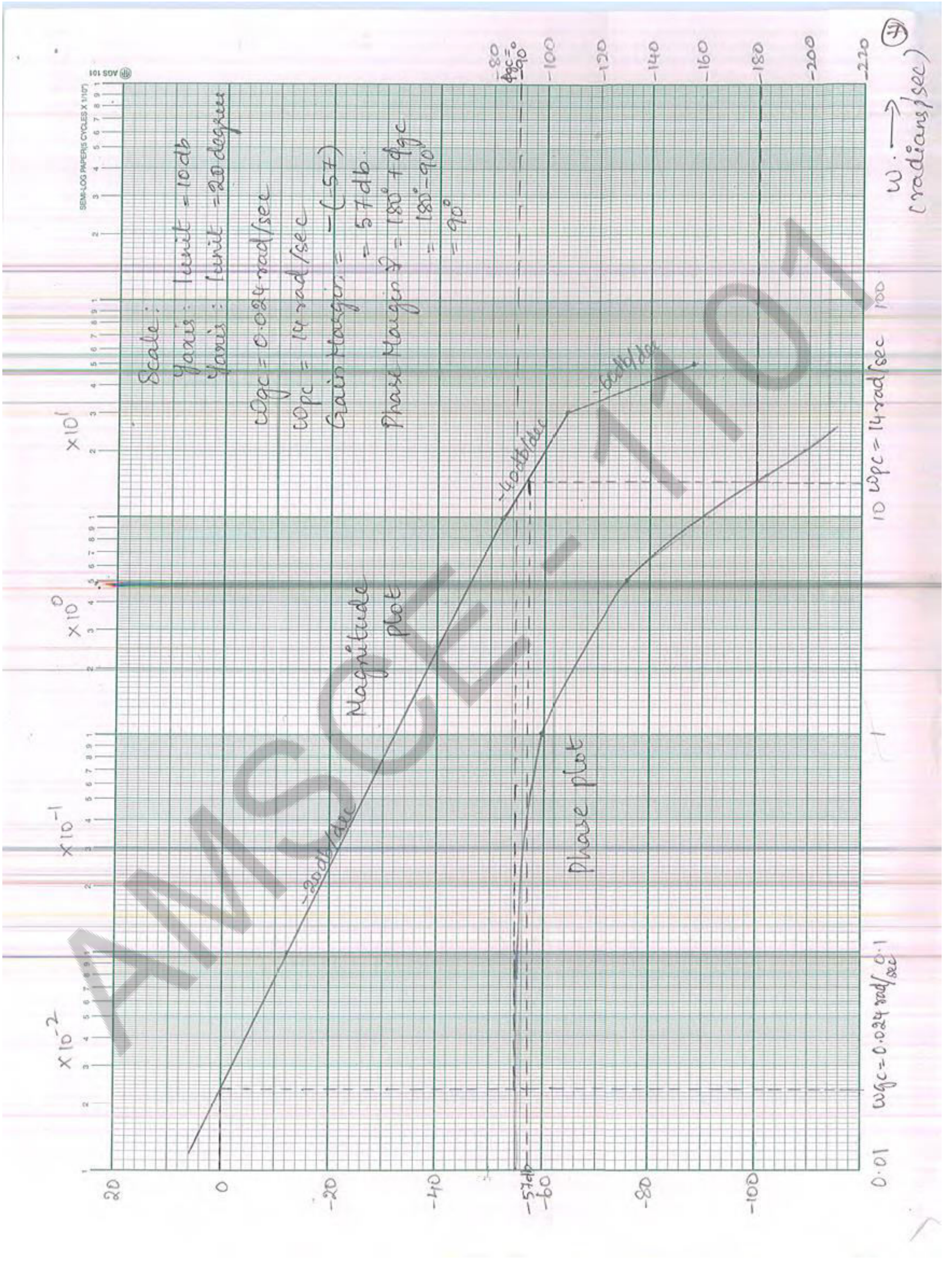
ω rad/sec	A dB
0.1	-12.04
10	-52.04
20	-64.08
50	-88

Phase angle plot

$\phi = \underline{|G}(j\omega) = -90 - \tan^{-1} 0.1\omega - \tan^{-1} 0.05\omega$

ω rad/sec	$\phi = \underline{|G}(j\omega)$ deg
0.01	-90.08
0.1	-90.85
1	-98.57
5	-130.6
10	-161.6

14	-179.45
15	-183.17
20	-198.43



From graph, gain crossover frequency ω_{gc} = 0.024rad/ sec

Phase crossover frequency ω_{pc} = 14rad/ sec
Gain margin = 57 db
Phase margin $\gamma = 90°$

2. Sketch the Bode plot for the following transfer function and determine the phase margin and gain margin $G(s) = \dfrac{20}{s(1+3s)(1+4s)}$

Solution:-

The sinusoidal transfer function of G(jω) is obtained by replacing s by jω in the given transfer function.

$$G(j\omega) = \frac{20}{j\omega(1+j3\omega)(1+j4\omega)}$$

Magnitude plot

The corner frequencies, $\omega_{c1} = \dfrac{1}{4} = 0.25\ \text{rad/sec}$

$$\omega_{c2} = \frac{1}{3} = 0.33\ \text{rad/sec}$$

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/sec
$\dfrac{20}{j\omega}$	-	-20	-
$\dfrac{1}{1+j3\omega}$	$\omega_{c1} = \dfrac{1}{4} = 0.25$	-20	-20 – 20 = - 40
$\dfrac{1}{1+j4\omega}$	$\omega_{c2} = \dfrac{1}{3} = 0.33$	-20	-40 - 20 = -60

Choose a frequency ω_ℓ such that $\omega_\ell < \omega_{c1}$ and choose a frequency ω_h such that $\omega_h > \omega_{c2}$

Let $\omega_\ell = 0.15\text{rad/sec}$ and $\omega_h = 2\text{rad/sec}$

Calculation of gain A at $\omega = \omega_\ell, \omega_{c1}, \omega_{c2} + \omega_h$

At $\omega = \omega_\ell$, $\quad A = |G(j\omega)| = 20\log\left|\dfrac{20}{j\omega}\right| \omega = \omega_\ell = 0.15$

$$= 20\log\left|\frac{20}{0.15}\right| = 42.5\ \text{dB}$$

At $\omega = \omega_{c1}$, $\quad A = |G(j\omega)| = 20\log\left|\dfrac{20}{0.15}\right| \omega = \omega_{c1} = 0.25$

$$= 20\log\left|\frac{20}{0.25}\right| = 38\text{dB}$$

At $\omega = \omega_{c2}$ $\quad A = \left[\text{Slope from } \omega_{c1} \text{to} \omega_{c2} \times \log\dfrac{\omega_{c2}}{\omega_{c1}}\right] + \text{A at } \omega = \omega_{c1}$

$$= -40 \times \log\frac{0.33}{0.25} + 38 = 33\text{dB}$$

At $\omega = \omega_h$ $\quad A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_h \times \log\dfrac{\omega_h}{\omega_{c2}}\right] + \text{A at } \omega = \omega_{c2}$

$$= -60 \times \log\frac{2}{0.33} + 33 = -13.95 \approx 14\text{dB}$$

ω rad/sec	A dB

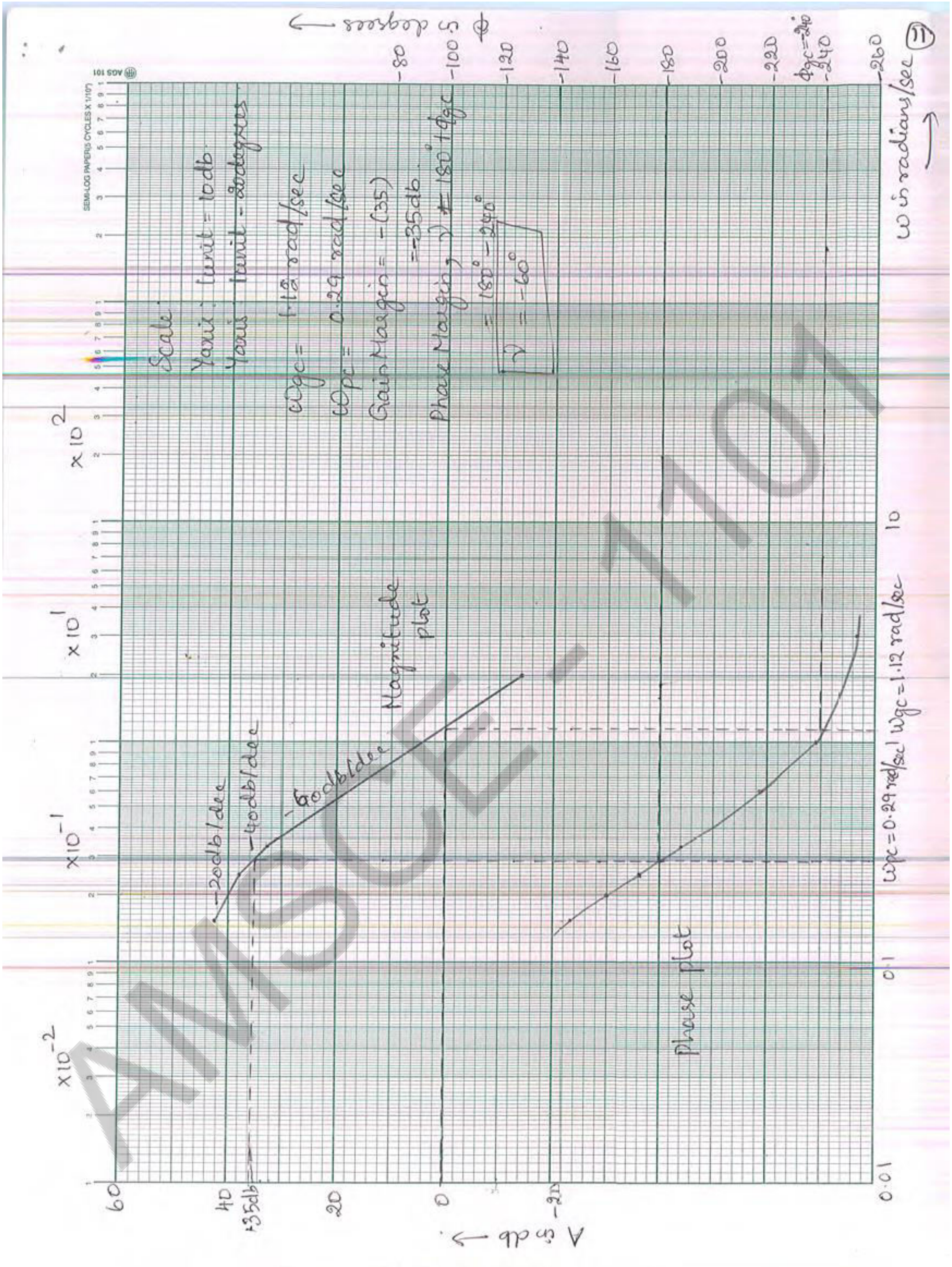
0.15	42.5
0.25	38
0.33	33
2	-14

Phase angle plot

$\phi = \angle G(j\omega) = -90^\circ - \tan^{-1} 3\omega - \tan^{-1} 4\omega$

ω rad/sec	$\phi = \angle G(j\omega)$ deg
0.15	-146
0.2	-160
0.25	-172
0.33	-188
0.6	-218
1	-238
2	-253

From graph,
Gain cross over frequency, ω_{gc} = 1.12 rad/sec
Phase cross over frequency, ω_{pc} = 0.29 rad/sec
Gain margin = -35db
Phase margin $\gamma = 180^\circ + \phi gc$
$= 180^\circ - 240^\circ$
$= -60^\circ$



3. Sketch the Bode plot for the following transfer function and determine the system gain K for the gain cross over frequency to be 5rad/ sec. $G(s)=\frac{ks^2}{(1+0.2s)(1+0.02s)}$ **APRIL/MAY 2017**

Solution:-

The sinusoidal transfer function G(jω) is obtained by replacing s by (jω) in the s-domain transfer function

Put s = jω

$$\therefore G(j\omega) = \frac{k(j\omega)^2}{(1+j0.2\omega)(1+j0.02\omega)}$$

Let k = 1

$$\therefore G(j\omega) = \frac{(j\omega)^2}{(1+j0.2\omega)(1+0.02\omega)}$$

Magnitude plot

The corner frequency are $\omega_{c1} = \frac{1}{0.2} = 5\text{rad/sec}$

$$\omega_{c2} = \frac{1}{0.02} = 50\text{rad/sec}$$

Term	Corner frequency rad/sec	Slope db/ dec	Change in slope db/deg
$(j\omega)^2$	-	+40	-
$\frac{1}{1+j0.2\omega}$	$\omega_{c1} = \frac{1}{0.2} = 5$	-20	40 – 20 = 20
$\frac{1}{1+j0.02\omega}$	$\omega_{c2} = \frac{1}{0.02} = 50$	-20	20 – 20 = 0

Choose a low frequency ω_ℓ such that $\omega_\ell < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let $\omega_\ell = 0.5\text{rad/sec}$ and $\omega_h = 100\text{rad/sec}$

Let $A = |G(j\omega)| \text{in db}$

Calculating of gain A at $\omega = \omega_\ell, \omega_{c1}, \omega_{c2} + \omega_h$

At $\omega = \omega_\ell$, $A = 20\log|(j\omega)^2|$

$$= 20\log(\omega)^2$$

$$= 20\log(0.5)^2 = -12\text{db}$$

At $\omega = \omega_{c1}$, $A = 20\log|(j\omega)^2|$

$$= 20\log(\omega)^2$$

$$= 20\log(0.5)^2 = 28\text{db}$$

At $\omega = \omega_{c2}$, $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A \text{ at } \omega = \omega_{c1}$

$$= 20 \times \log\frac{50}{5} + 28 = 48\text{db}$$

At $\omega = \omega_h$ $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A \text{ at } \omega = \omega_{c2}$

$$= 0 \times \log\frac{100}{50} + 48 = 48\text{db}$$

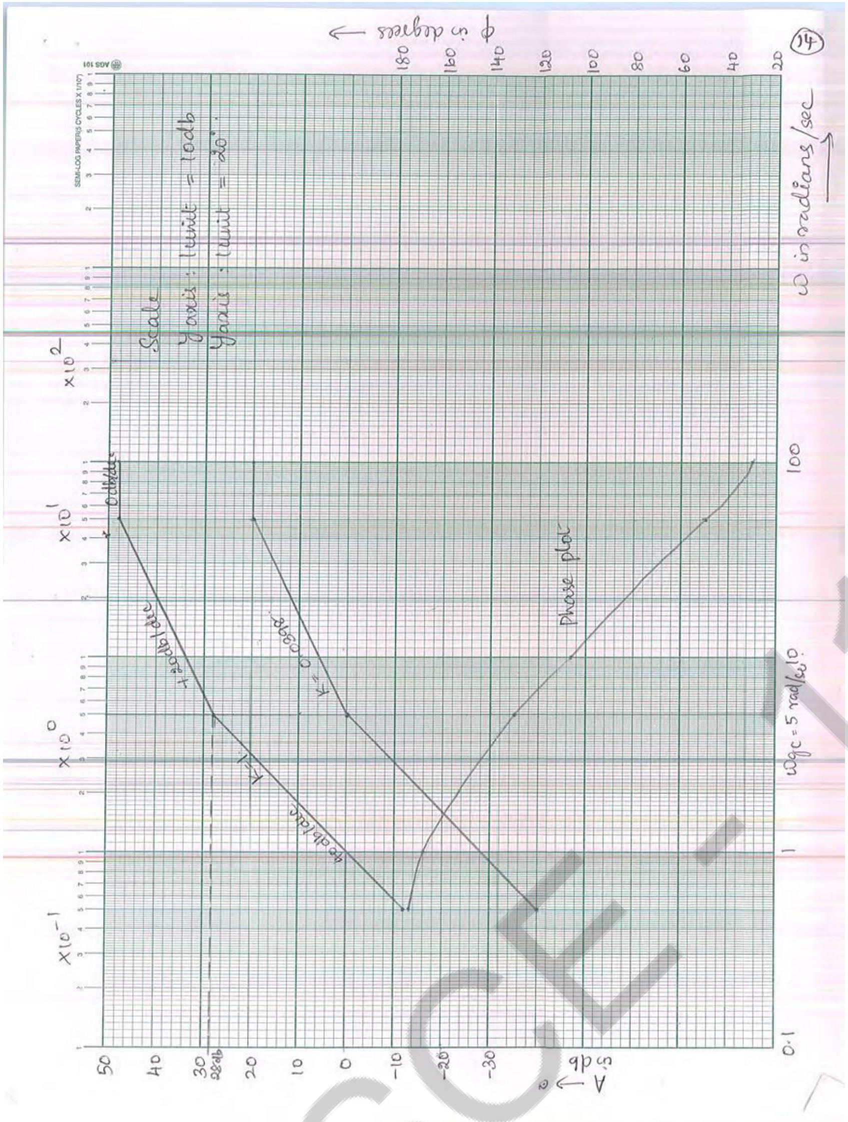


Table-1

ω rad / sec	A dB
0.5	-12
5	28
50	48
100	48

Phase plot

$$\phi = \angle G(j\omega) = 180° - \tan^{-1} 0.2\omega - \tan^{-1} 0.02\omega$$

Table 2

ω = rad/sec	$\phi = \angle G(j\omega)$ deg
0.5	174
1	168
5	130
10	106
50	50

100	30

Bode plot for the above table 1 & 2 is shown in fig.

To find K

Gain cross over frequency ω_{gc} = 5 rad/sec (given) At $\omega = \omega_{gc}$ = 5 rad/sec, the gain is 28 dB.
If gain cross over frequency is 5rad/sec, then at that frequency, the dB gain should be zero.

Hence to every point of magnitude plot a dB gain of -28dB should be added.
The value of k is calculated by equating
20 log k to -28 dB

20 log k = -28 dB

$$20\log k = -28\text{dB}; k = 10^{-28/20}; k = 0.0398$$

4. Given $G(s) = \dfrac{ke^{-0.2s}}{s(s+2)(s+8)}$. Find K so that the system is stable with

(a) gain margin equal to 6db (b) Phase margin equal to 45°

Solution:-

Put k =1 and convert the given transfer function to time constant form (or) bode form

$$\therefore G(s) = \frac{e^{-0.2s}}{s(s+2)(s+8)} = \frac{e^{-0.2s}}{s \times 2(1+0.5s) \times 8(1+0.125s)}$$

$$= \frac{0.0625e^{-0.2s}}{s(1+0.5s)(1+0.125s)}$$

The sinusoidal transfer function G(jω) is obtained by replacing s by jω.

$$\therefore G(j\omega) = \frac{0.0625e^{-j0.2\omega}}{j\omega(1+j0.5\omega)(1+j0.125\omega)}$$

Magnitude plot

The corner frequency are, $\omega_{c1} = \dfrac{1}{0.5} = 2\text{rad/sec}$

$$\omega_{c2} = \frac{1}{0.125} = 8\text{rad/sec}$$

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{0.0625}{j\omega}$	-	-20	-
$\dfrac{1}{1+j0.5\omega}$	$\omega_{c1} = \dfrac{1}{0.5} = 2$	-20	-20 – 20 = -40
$\dfrac{1}{1+j0.125\omega}$	$\omega_{c2} = \dfrac{1}{0.125} = 8$	-20	-40 – 20 = -60

Choose a low frequency ω_ℓ such that $\omega_\ell < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let $\omega_\ell = 0.5\text{rad/sec}$ and $\omega_h = 50\text{rad/sec}$

Calculation of gain A at $\omega = \omega_\ell, \omega_{c1}, \omega_{c2} + \omega_h$

At $\omega = \omega_\ell$, $\quad A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\left|\frac{0.0625}{0.5}\right| = -18\,db$

At $\omega = \omega_{c1}$, $\quad A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\left|\frac{0.0625}{2}\right| = -30\,db$

At $\omega = \omega_{c2}$, $\quad A = \left[\text{slope form } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A \text{ at } \omega = \omega_{c1}$

$$= -40 \times \log\frac{8}{2} + (-30) = -54\,db$$

At $\omega = \omega_h$, $\quad A = \left[\text{slope form } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A \text{ at } \omega = \omega_{c2}$

$$= -60 \times \log\frac{50}{8} + (-54) = -102\,db$$

ω rad/sec	A db
0.5	-18
2	-30
8	-54
50	-102

Phase angle plot

$$\phi = \angle G(j\omega) = -0.2 \times \omega \times \frac{180}{\pi} - 90° - \tan^{-1} 0.5\omega - \tan^{-1} 0.125\omega$$

ω rad/sec	$\phi = \angle G(j\omega)$ deg
0.01	-90
0.1	-94
0.5	-114
1	-134
2	-172
3	-202
4	-226

The above Bode plot for the above transfer function is shown in fig.

To find K

With k = 1, gain margin = 32 db
But required gain margin is 6db. Hence to every point of magnitude plot, a db gain of 26 db is added.

$$20\log k = 26$$

$$k = 10^{26/20} = 19.95$$

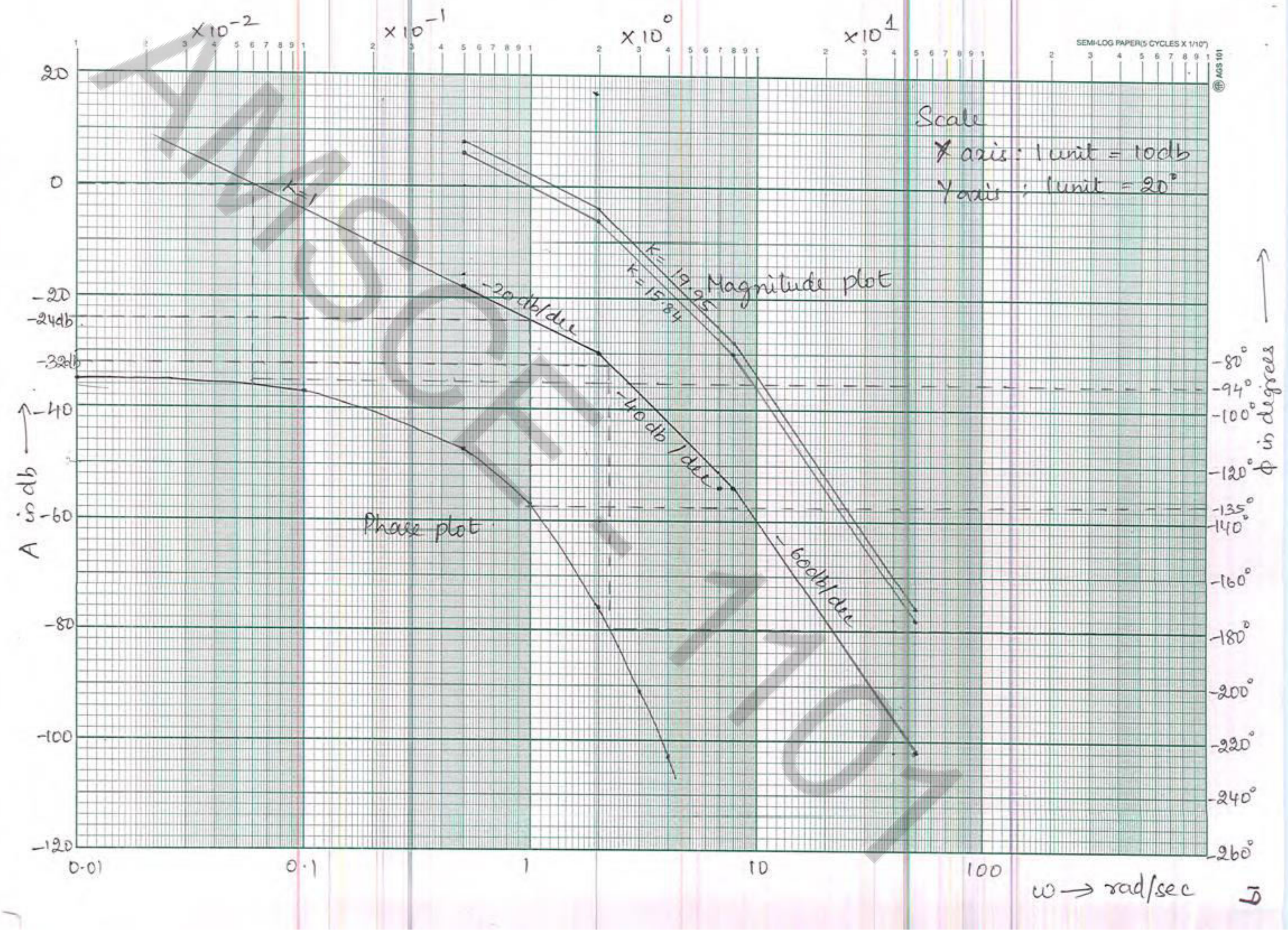
Phase margin $\gamma = 180° + \phi_{gc}$
When $\quad \gamma_{new} = 45°, \phi_{gc\,new} = \gamma_{new} - 180° = 45° - 180° = -135°$

When K = 1, the db gain at $\phi_{gc} = -135°$ is -24 db.

The gain must be made zero, to have PM = 45°. Hence to every point of magnitude plot a db gain of 24 db should be added.

The value of k is calculated by

$$20\log k = 24; k = 10^{24/20}; k = 15.84$$



Scale
X axis : 1 unit = 10db
Y axis : 1 unit = 20°
Magnitude plot
Phase plot
K = 1
K = 19.95
K = 15.84
-20db/dec
-40db/dec
-60db/dec
A in db
φ in degrees
ω → rad/sec
-24db
-32db
-94°
-135°
SEMI-LOG PAPER(5 CYCLES X 1/10")

5. Sketch the bode plot for the following transfer function & determine phase margin

$$G(s)=\frac{75(1+0.2s)}{s(s^2+16s+100)}$$

$$G(s)=\frac{75(1+0.2s)}{s(s^2+16s+100)}=\frac{75(1+0.2s)}{s\times 100\left(\frac{s^2}{100}+\frac{16s}{100}+1\right)}$$

$$=\frac{0.75(1+0.2s)}{s(1+0.01s^2+0.16s)}$$

Put $s = j\omega$

$$G(j\omega)=\frac{0.75(1+j0.2\omega)}{j\omega(1-0.01\omega^2+j0.16\omega)}$$

$$=\frac{0.75(1+j0.2\omega)}{j\omega(1-0.01\omega^2+j0.16\omega)}$$

Magnitude plot

The corner frequencies are, $\omega_{c1}=\frac{1}{0.2}=5\,\text{rad/sec}$

$\omega_{c2}=\omega_n=10\,\text{rad/sec}$

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{0.75}{j\omega}$	-	-20	-
$1+j0.2\omega$	$\omega_{c1}=\frac{1}{0.2}=5$	+20	-20 +20 = 0
$\frac{1}{1+0.01\omega^2+j0.16\omega}$	$\omega_{c2}=\omega_n=10$	-40	0 – 40 = -40

Choose a low frequency ω_ℓ such that $\omega_\ell<\omega_{c1}$ and choose a high frequency ω_h such that $\omega_h>\omega_{c2}$

Let $\omega_\ell=0.5\,\text{rad/sec}$ and $\omega_h=20\,\text{rad/sec}$

Calculation of gain A at $\omega=\omega_\ell,\omega_{c1},\omega_{c2}+\omega_h$

At $\omega=\omega_\ell$, $\quad A=20\log\left|\frac{0.75}{j\omega}\right|=20\log\frac{0.75}{0.5}=3.5\,\text{db}$

At $\omega=\omega_{c1}$, $\quad A=20\log\left|\frac{0.75}{j\omega}\right|=20\log\frac{0.75}{5}=-16.5\,\text{db}$

At $\omega=\omega_{c2}$, $\quad A=\left[\text{slope from }\omega_{c1}\text{ to }\omega_{c2}\times\log\frac{\omega_{c2}}{\omega_{c1}}\right]+A\text{ at }\omega=\omega_{c1}$

$$=0\times\log\frac{10}{5}+(-16.5)=-16.5\,\text{db}$$

At $\omega=\omega_h$, $\quad A=\left[\text{slope from }\omega_{c2}\text{ to }\omega_h\times\log\frac{\omega_h}{\omega_{c1}}\right]+A\text{ at }\omega=\omega_{c2}$

$$=-40+\log\frac{20}{10}+(-16.5)=-28.5\,\text{db}$$

ω rad/sec	A dB
0.5	3.5
5	-16.5
10	-16.5
20	-28.5

Phase angle plot.

$$\phi = \underline{|G}(j\omega) = \tan^{-1} 0.2\omega - 90° - \tan^{-1}\frac{0.16\omega}{1-0.01\omega^2}, \text{for} \omega \le \omega_n$$

$$\phi = \underline{|G}(j\omega) = \tan^{-1} 0.2\omega - 90°\left(\tan^{-1}\frac{0.16}{1-0.01\omega^2} + 180°\right) \text{for } \omega > \omega_n$$

ω rad/sec	$\phi = \angle G(j\omega)\text{deg}$
0.5	-88
1	-88
5	-92
10	-116
20	-148
50	-168
100	-174

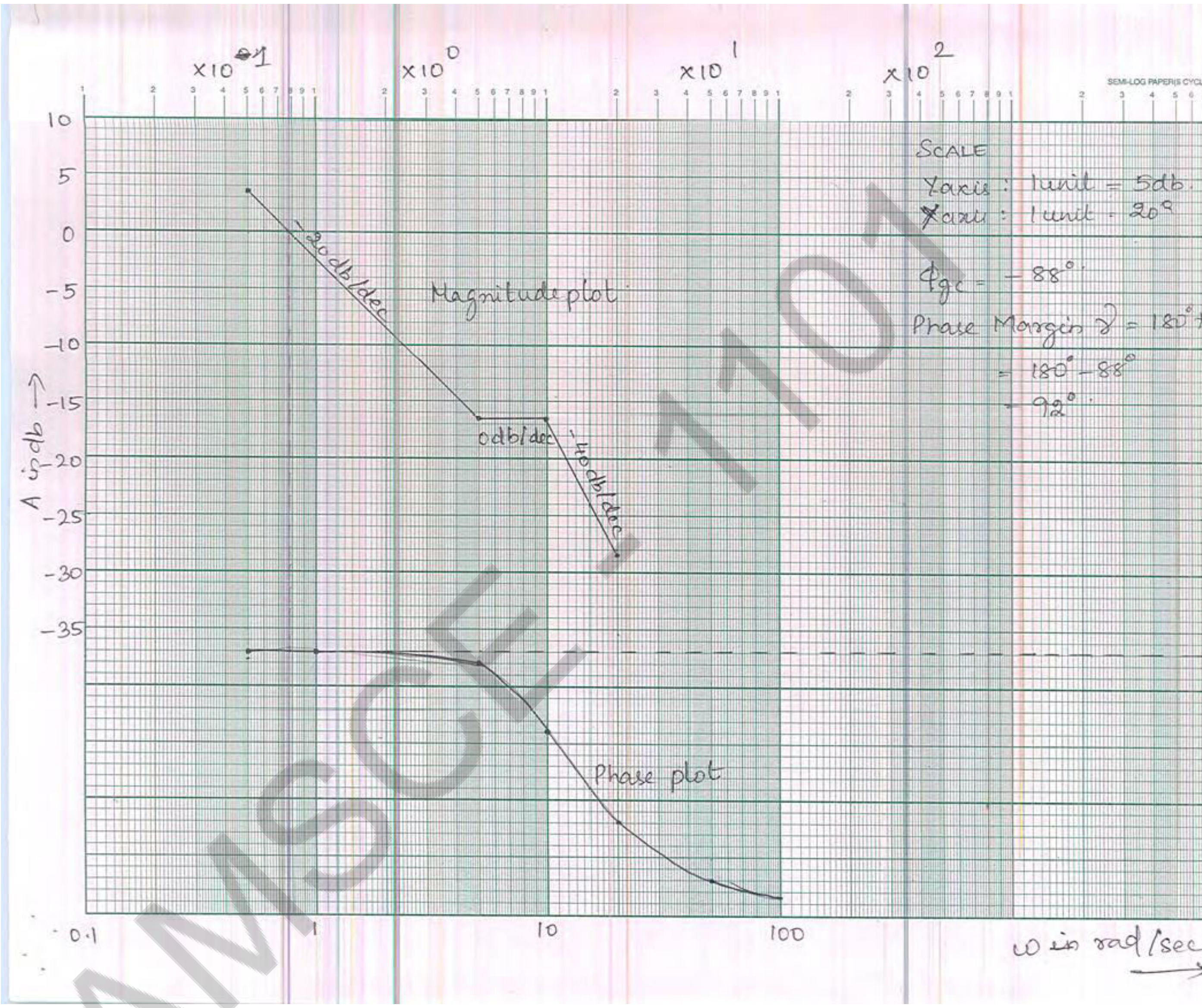
From graph

ϕ_{gc} = -88°

Phase Margin γ = 180° + ϕ_{gc}

= 180 ° - 88° = 92°

Gain Margin = + ∞ [As phase plot crosses the -180° at infinity. |G(jω)| at infinity = -∞ db]



POLAR PLOT

1. The open loop transfer function of a unity feedback system is given by $G(s)=\frac{1}{s^2(1+s)(1+2s)}$**. Sketch the polar plot and determine the gain margin and phase margin**

Solution

Given that $G(s) = \frac{1}{s^2(1+s)(1+2s)}$

Put s = jω, $G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega)(1+j2\omega)}$

$= \frac{1\angle 0°}{\omega^2\angle 180°\sqrt{1+\omega^2}\angle \tan^{-1}\omega\sqrt{1+4\omega^2}\angle \tan^{-1}2\omega}$

$G(j\omega) = \frac{1}{\omega^2\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}\angle\left(-180° - \tan^{-1}\omega - \tan^{-1}2\omega\right)$

$|G(j\omega)| = \frac{1}{\omega^2\sqrt{1+\omega^2}\sqrt{1+4\omega^2}} = \frac{1}{\omega^2\sqrt{(1+\omega^2)(1+4\omega^2)}}$

$= \frac{1}{\omega^2\sqrt{1+5\omega^2+4\omega^4}}$

$\angle G(j\omega) = -180° - \tan^{-1}\omega - \tan^{-1}2\omega$

Corner frequencies

$\omega_{c1} = \frac{1}{1} = 1$ rad / sec

$\omega_{c2} = \frac{1}{2} = 0.5$ rad / sec

Magnitude and phase plot of G(jω)

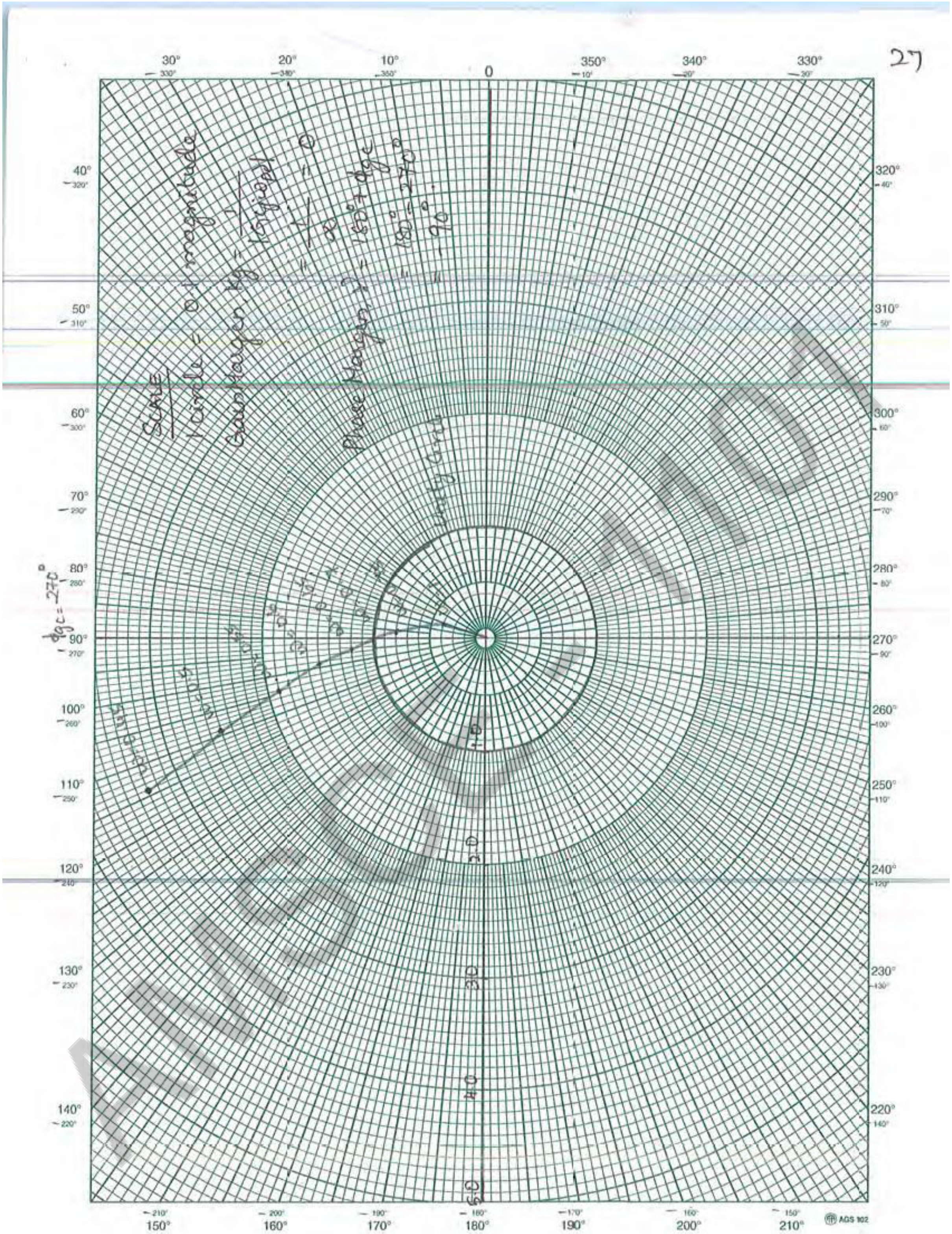
ω rad/sec	|G(jω)|	∠G(jω) Deg
0.45	3.33	-246
0.5	2.5	-251
0.55	1.9	-256
0.6	1.5	-261
0.65	1.2	-265
0.7	0.97 ≅ 1	-269
0.75	0.8	-273
1.0	0.3	-288

From Polar graph

Gain Margin, $K_g = \frac{1}{\infty} = 0$

Phase Margin, γ = 180° - 270°

= - 90°



2. The open loop transfer function of a unity feedback system is given by $G(s)=\dfrac{1}{s(1+s)^2}$. Sketch the polar plot and determine the gain margin and phase margin.

Solution

Given that $G(s) = \frac{1}{s(1+s)^2}$

Put s = jω

$\therefore G(j\omega) = \frac{1}{j\omega(1+j\omega)^2} = \frac{1}{j\omega(1+j\omega)(1+j\omega)}$

$= \frac{1}{\omega\angle 90°\sqrt{1+\omega^2}\tan^{-1}\omega\sqrt{1+\omega^2}\tan^{-1}\omega}$

$= \frac{1}{\omega\left(\sqrt{1+\omega^2}\right)^2}\angle\left(-90° - 2\tan^{-1}\omega\right)$

$|G(j\omega)| = \frac{1}{\omega\left(1+\omega^2\right)} = \frac{1}{\omega+\omega^3}$

$\angle G(j\omega) = -90° - 2\tan^{-1}\omega$

Corner frequencies.

$\omega_{c1} = \frac{1}{1} = 1$ rad / sec

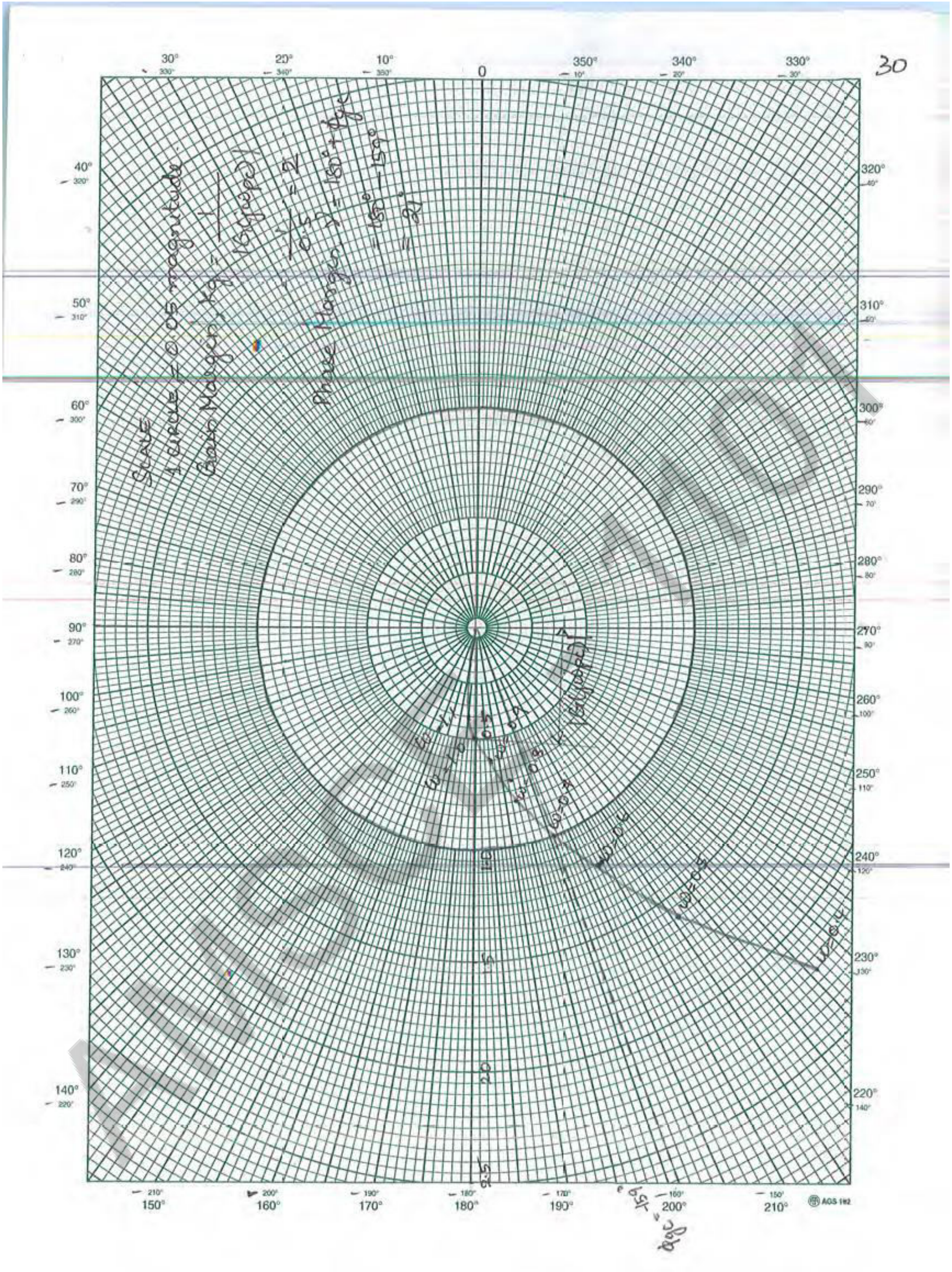
Table Magnitude and phase plot of G(jω)

ω rad/sec	|G(jω)|	∠G(jω) Deg
0.4	2.2	-134
0.5	1.6	-143
0.6	1.2	-151
0.7	1	-159
0.8	0.8	-167
0.9	0.6	-174
1.0	0.5	-180
1.1	0.4	-185

From Polar graph,

Gain Margin kg = 2

Phase Margin γ = 21°



3. Consider a unity feedback system having an open loop transfer function, $G(s)=\dfrac{k}{s(1+0.5s)(1+4s)}$

Sketch the polar plot and determine the value of k so that (i) Gain Margin is 20db
(ii) Phase Margin is 30°
Solution

Given that $G(s) = \frac{k}{s(1+0.5s)(1+4s)}$

Put k = 1 and s = jω in G(s)

$\therefore\ G(j\omega) = \frac{1}{j\omega(1+j0.5\omega)(1+j4\omega)}$

$= \frac{1}{\omega\angle 90^\circ\sqrt{1+(0.5\omega)^2}\ \tan^{-1}0.5\omega\sqrt{1+(4\omega)^2}\angle\tan^{-1}4\omega}$

$= \frac{1}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+16\omega^2}\left(+90^\circ+\tan^{-1}0.5\omega+\tan^{-1}4\omega\right)}$

$= \frac{1}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+16\omega^2}}\angle\left(-90^\circ-\tan^{-1}0.5\omega-\tan^{-1}4\omega\right)$

$|G(j\omega)| = \frac{1}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+16\omega^2}}$

$\angle G(j\omega) = -90^\circ - \tan^{-1}0.5\omega - \tan^{-1}4\omega$

Corner frequencies

$\omega_{c1} = \frac{1}{4} = 0.25$ rad / sec

$\omega_{c2} = \frac{1}{0.5} = 2$ rad / sec

32 Table Magnitude and Phase plot of G(jω)

ω rad/sec	|G(jω)|	∠G(jω) Deg
0.3	2.11	-149
0.4	1.3	-159
0.5	0.87	-167
0.6	0.61	-174
0.8	0.35	-184
1.0	0.22	-193
1.2	0.15	-199

From polar plot, k = 1

Gain Margin kg = $\frac{1}{0.44} = 2.27$

Gain Margin in db = 20 log 2.27 = 7.12 db

Phase Margin, $\gamma = 180^\circ + \phi_{gc} - 180^\circ - 165^\circ = 15^\circ$

To find k

Case (i)

Let G_B be the magnitude of open loop transfer fn G(jω) at -180° with k = 1

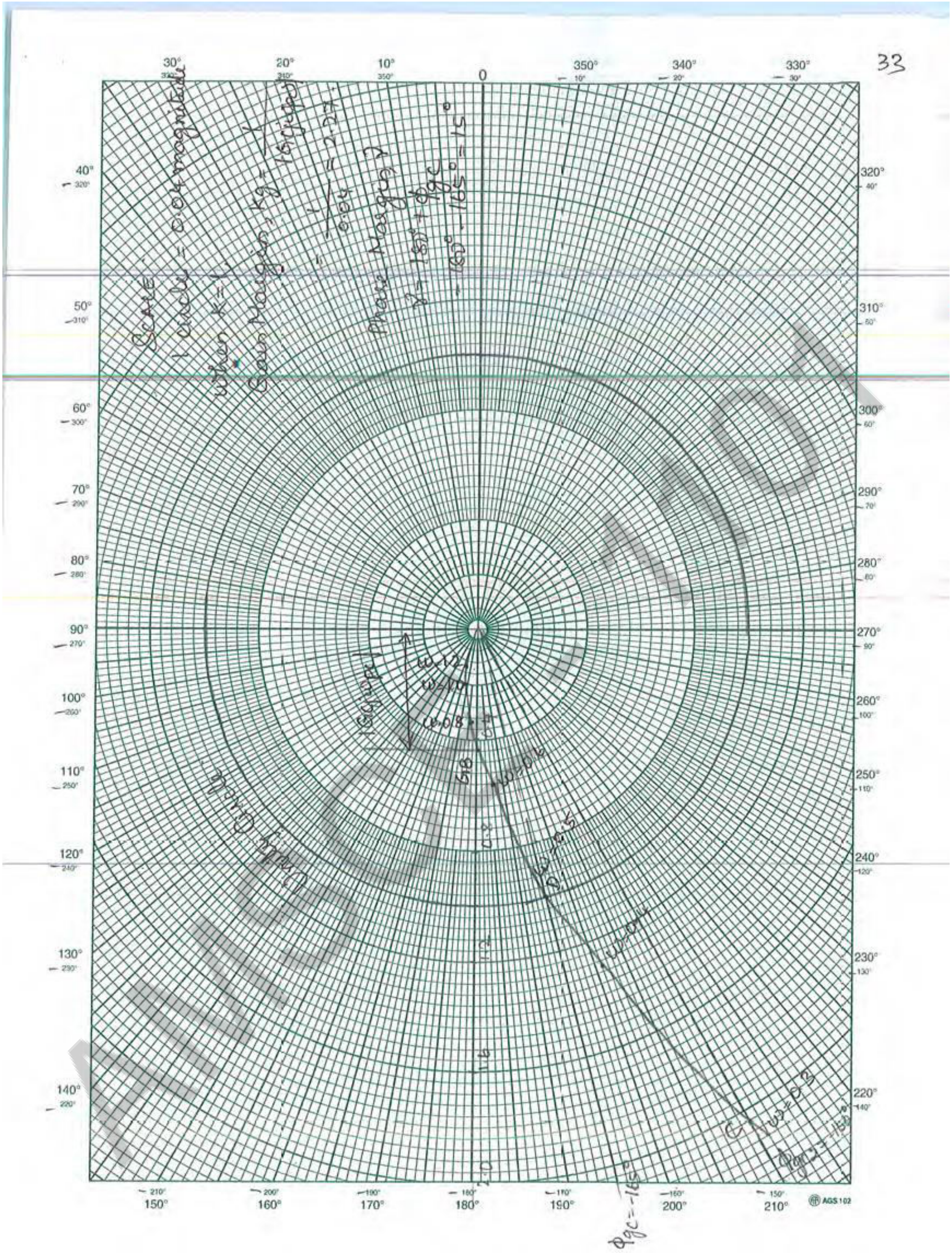
Let G_A be the magnitude of open loop transfer function G(jω) at -180° with k = ? & gain margin of 20 db.

Now $20 \log \frac{1}{G_A} = 20 \Rightarrow \log \frac{1}{G_A} = \frac{20}{20} = 1$

$\Rightarrow G_A = 0.1$

The value of $k = \frac{G_A}{G_B} = \frac{0.1}{0.44} = 0.227$

k = 0.227



Case (ii)

With k = 1, the phase margin is 15°. This has to be increased to 30°. Hence the gain has to be decreased.

Let ϕ_{gc2} be the phase of G(jω) for a phase margin of 30°.

$\therefore\ 30° = 180° + \phi_{gc2}$

$\phi_{gc2} = 30° - 180° = -150°$

In the polar plot the -150° line cuts the locus of G(jω) at point c and cut the unity circle at point D.

Let G_C be magnitude of G(jω) at point C

G_D be magnitude of G(jω) at point D

From polar plot, G_C = 2.04

G_D = 1

Now $k = \frac{G_D}{G_C} = \frac{1}{2.04} = 0.49$

k = 0.49

*** INCLUDE THIS ***

Compensator design using Bode plots

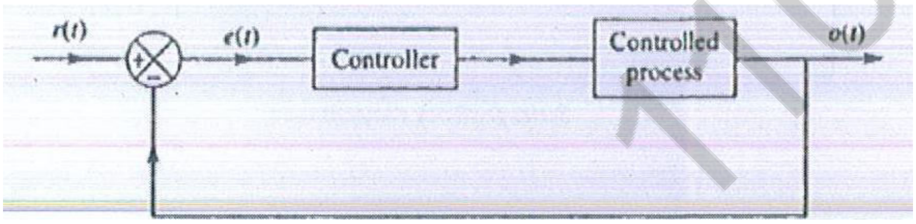
1. Write short notes on different types of compensation

Types of compensation

Series Compensation or Cascade Compensation

This is the most commonly used system where the controller is placed in series with the controlled process.

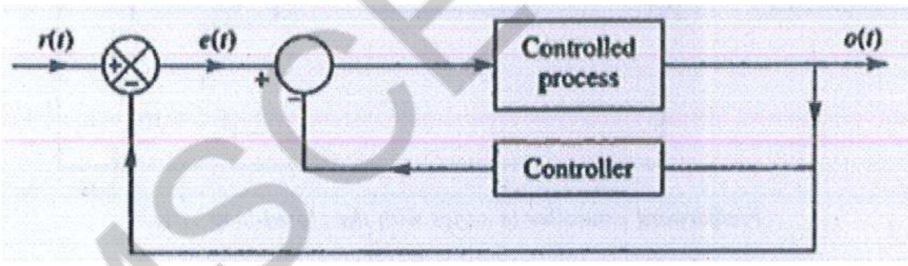
Figure shows the series compensation.



Series compensation

Feedback compensation or Parallel compensation

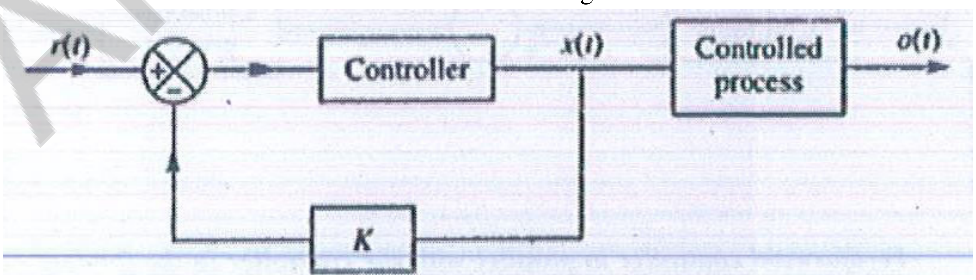
This is the system where the controller is placed in the sensor feedback path as shown in fig.



Feedback compensation or parallel compensation

State Feedback Compensation

This is a system which generates the control signal by feeding back the state variables through constant real gains. The scheme is termed state feedback. It is shown in Fig.



State feedback compensation

The compensation schemes shown in Figs above have one degree of freedom, since there is only one controller in each system. The demerit with one degree of freedom controllers is that the performance criteria that can be realized are limited.

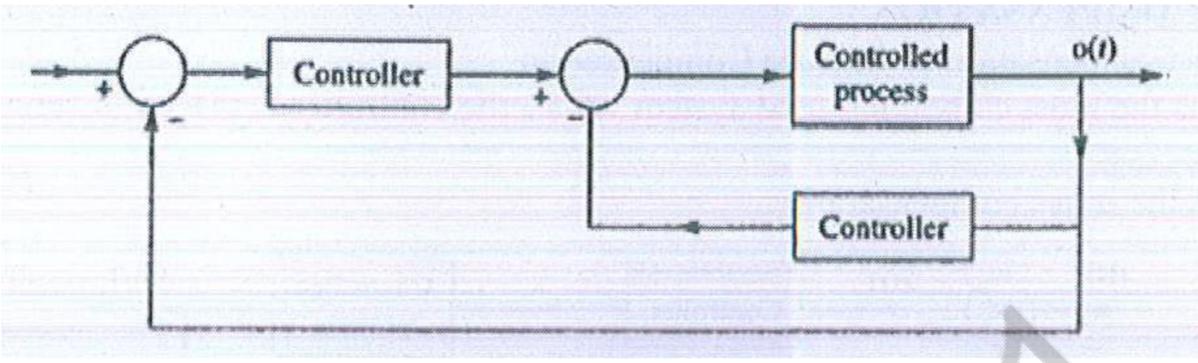
That is why there are compensation schemes which have two degree freedoms, such as:

a) Series – feedback compensation

b) Feed forward compensation

Series- Feedback Compensation

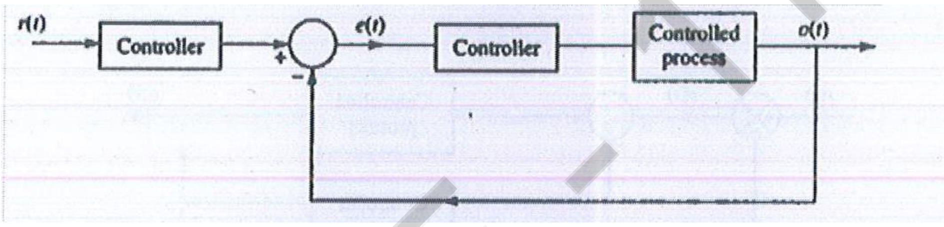
Series-feedback compensation is the scheme for which is series controller and a feedback controller are used. Figure shows the series-feedback compensation scheme.



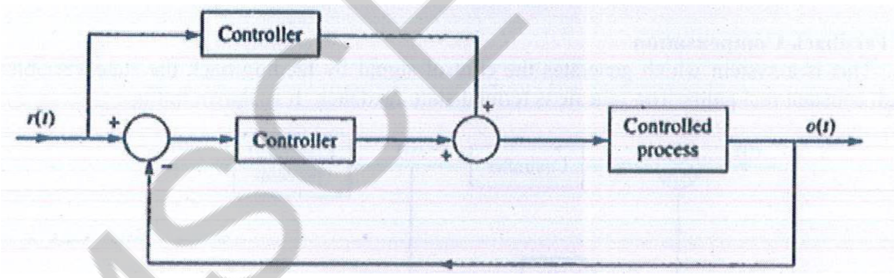
Series-feedback compensation.

Feed forward Compensation

The feed forward controller is placed in series with the closed-loop system which has a controller in the forward path. In Fig. Feed forward the is placed in parallel with the controller in the forward path. The commonly used controller in the above-mentioned compensation schemes are now described in the section below.



Feed forward controller in series with the closed-loop system.



Feed forward controller in parallel with the controller in the forward path.

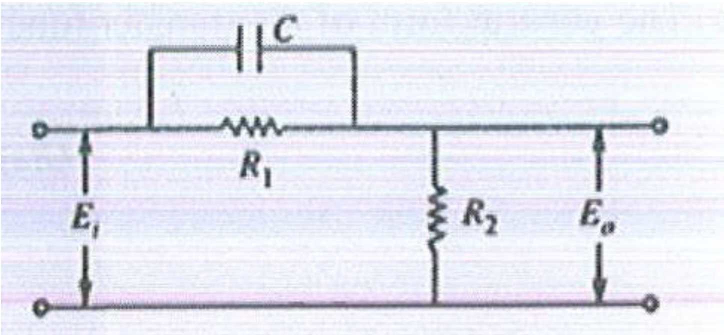
2. Realize the lead compensator using electrical network and obtain the transfer function

Lead Compensator

It has a zero and a pole with zero closer to the origin. The general form of the transfer function of the load compensator is

$$G(s) = \frac{s + \frac{1}{\tau}}{s + \frac{1}{\beta\tau}}$$

$$G(j\omega) = \beta \frac{(\tau j\omega + 1)}{\beta\tau j\omega + 1}$$



$$\frac{E_0(s)}{E_i(s)} = \frac{R_2}{\frac{R_1 \times \frac{1}{Cs} + R_2\left(R_1 + \frac{1}{Cs}\right)}{R_1 + \frac{1}{Cs}}} = \frac{R_2 R_1 + \frac{R_2}{Cs}}{R_1 R_2 + \frac{1}{Cs}(R_1 + R_2)}$$

$$= \frac{CsR_1R_2 + R_2}{CsR_1R_2 + R_1 + R_2}$$

$$= \frac{R_2(CsR_1 + 1)}{(R_1 + R_2)\left(\frac{CsR_1R_2}{R_1 + R_2} + 1\right)}$$

$$= \left(\frac{R_2}{R_1 + R_2}\right)\frac{CR_1 s + 1}{\left(\frac{CR_1R_2 s}{R_1 + R_2} + 1\right)}$$

Subsisting

$\tau = CR_1; \beta\tau = \frac{CR_1R_2}{R_1 + R_2}$ $(\because\ \tau = CR_1)$

Transfer function

$G(s) = \beta\ \frac{\tau s + 1}{\beta\tau s + 1}$

3. Realize the lag compensator using electrical network and obtain the transfer function

Lag Compensator

It has a zero and a pole with the zero situated on the left of the pole on the negative real axis. The general form of the transfer function of the lag compensator is

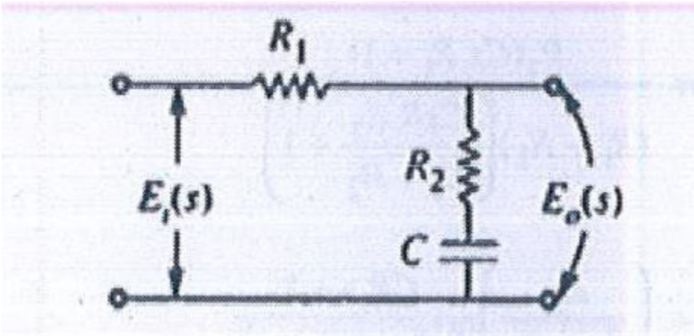
$$G(s) = \frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}} = \frac{\alpha(\tau s + 1)}{\alpha\tau s + 1}$$

Where $\alpha > 1$, $\tau > 0$.

Therefore, the frequency response of the above transfer function will be

$$G(j\omega) = \frac{\alpha(\tau j\omega + 1)}{\alpha\tau j\omega + 1}$$

$$E_0(s) = \frac{E_i(s)}{R_1 + R_2 + \frac{1}{Cs}}\left(R_2 + \frac{1}{Cs}\right)$$



Lag compensator

$$\frac{E_0(s)}{E_i(s)} = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}}$$

$$= \frac{R_2Cs + 1}{(R_1 + R_2)Cs + 1}$$

$$= \frac{R_2C\left(s + \frac{1}{R_2C}\right)}{(R_1 + R_2)C\left(s + \frac{1}{(R_1 + R_2)C}\right)}$$

$$= \frac{R_2}{(R_1 + R_2)} \frac{s + \frac{1}{R_2C}}{\left(s + \frac{1}{(R_1 + R_2)C}\right)} = \frac{R_2}{(R_1 + R_2)} \frac{\left(s + \frac{1}{R_2C}\right)}{\left(s + \frac{R_2}{(R_1 + R_2)R_2C}\right)}$$

Now comparing with

$$G(s) = \frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}}$$

$$\frac{1}{\tau} = \frac{1}{R_2C}; \qquad \frac{1}{\alpha\tau} = \frac{R_2}{(R_1 + R_2)R_2C};$$

$$\frac{1}{a\tau} = \frac{R_2}{(R_1 + R_2)} \frac{1}{\tau} \qquad \left(\because \frac{1}{\tau} = \frac{1}{R_2C}\right)$$

$$\alpha = \frac{R_1 + R_2}{R_2}$$

Therefore

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{\alpha} \frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}}$$

4. Realize the lag-lead compensator using electrical network and obtain the transfer function

Lag-Lead Compensator

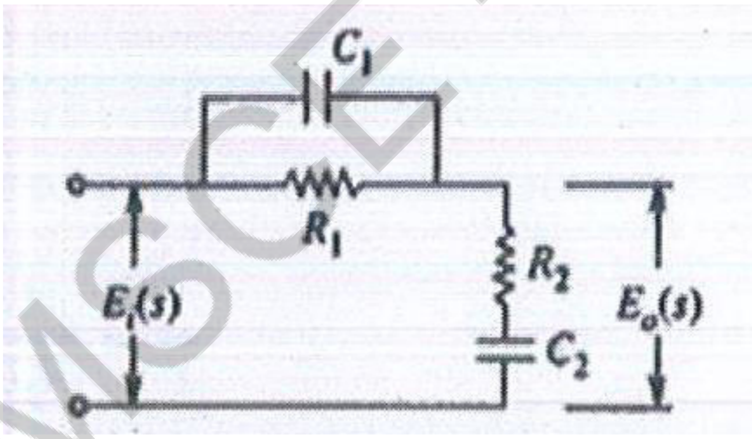
The lag-lead compensator is the combination of a lag compensator and a lead compensator. The lag-section is provided with one real pole and one real zero, the pole being to the right of zero, whereas the lead section has one real pole and one real came with the zero being to the right of the pole.

The transfer function of the lag-lead compensator will be

$$G(s)=\left(\frac{s+\frac{1}{\tau_1}}{s+\frac{1}{\alpha\tau_1}}\right)\left(\frac{s+\frac{1}{\tau_2}}{s+\frac{1}{\beta\tau_2}}\right)$$

The figure shows lag lead compensator

$$E_0(s)=\frac{E_i(s)}{\frac{R_1\times\frac{1}{sC_1}}{R_1+\frac{1}{sC_1}}+R_2+\frac{1}{sC_2}}\left(R_2+\frac{1}{sC_2}\right)$$



Where $\alpha > 1$, $\beta < 1$.

$$\frac{E_0(s)}{E_i(s)}=\frac{\left(R_1\times\frac{1}{sC_1}\right)\left(R_2+\frac{1}{sC_2}\right)}{R_1\times\frac{1}{sC_1}+\left(R_2+\frac{1}{sC_2}\right)\left(R_1+\frac{1}{sC_1}\right)}$$

$$=\frac{\frac{(sC_1R_1+1)}{sC_1}\frac{(sC_2R_2+1)}{sC_2}}{\frac{R_1}{sC_1}+\frac{(R_2sC_2+1)}{sC_2}\frac{(R_1sC_1+1)}{sC_1}}$$

$$=\frac{\dfrac{(1+sC_1R_1)(1+sC_2R_2)}{s^2C_1C_2}}{\dfrac{R_1sC_2+R_2sC_2+1+R_1R_2s^2C_1C_2+R_1sC_1}{s^2C_1C_2}}$$

$$=\frac{(1+sC_1R_1)(1+sC_2R_2)}{s^2R_1R_2C_1C_2+s(R_1C_1+R_2C_2)+1+R_1sC_2}$$

$$=\frac{(1+sC_1R_1)(1+sC_2R_2)}{s^2R_1R_2C_1C_2+s(R_1C_1+R_2C_2)+1+R_1sC_2}$$

$$=\frac{C_1R_1C_2R_2\left(s+\dfrac{1}{C_1R_1}\right)\left(s+\dfrac{1}{C_2R_2}\right)}{R_1R_2C_1C_2\left[s^2+\left\{\dfrac{1}{R_2C_2}+\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_1}\right\}s+\dfrac{1}{R_1R_2C_1C_2}\right]}$$

$$=\frac{\left(s+\dfrac{1}{C_1R_1}\right)\left(s+\dfrac{1}{C_2R_2}\right)}{s^2+\left(\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_1}+\dfrac{1}{R_2C_2}\right)s+\dfrac{1}{R_1R_2C_1C_2}}$$

The above transfer functions are comparing with

$$G(s)=\frac{\left(s+\dfrac{1}{\tau_1}\right)\left(s+\dfrac{1}{\tau_2}\right)}{\left(s+\dfrac{1}{\alpha\tau_1}\right)\left(s+\dfrac{1}{\beta\tau_2}\right)}$$

Then $\dfrac{1}{\tau_1}=\dfrac{1}{C_1R_1}$ $\qquad \dfrac{1}{\tau_2}=\dfrac{1}{C_2R_2}$

$$\frac{1}{\alpha\tau_1}+\frac{1}{\beta\tau_2}=\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}$$

$$\frac{1}{\alpha\beta\tau_1\tau_2}=\frac{1}{R_1R_2C_1C_2}$$

$\tau_1 = C_1 R_1$

$\tau_2 = C_2 R_2$

$\alpha\beta\tau_1 \tau_2 = R_1 R_2 C_1 C_2$

$\alpha\beta = 1$ or $\beta = \dfrac{1}{\alpha}$

Therefore

$$G(s)=\frac{\left(s+\frac{1}{\tau_1}\right)\left(s+\frac{1}{\tau_2}\right)}{\left(s+\frac{1}{\alpha\tau_1}\right)\left(s+\frac{\alpha}{\tau_2}\right)}$$ where $\alpha > 1$

$$\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}=\frac{1}{\alpha\tau_1}+\frac{\alpha}{\tau_2}$$

M&N circles

1. Prove that the loci of the constant magnitude of closed loop transfer function is a circle

Constant M circles

Consider the polar plot of the open loop transfer function of a unity feedback system. A point on the polar plot is given by:

$G(j\omega) = x + jy$

The closed loop frequency response is given by

$$T(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+G(j\omega)}=\frac{x+jy}{1+x+jy}$$

$$\therefore \qquad |T(j\omega)|^2=\frac{x^2+y^2}{(1+x)^2+y^2}$$

Let $\quad |T(j\omega)| = M$

$$\therefore \qquad M^2=\frac{x^2+y^2}{(1+x)^2+y^2}$$

$M^2 (1+ x)^2 + M^2 y^2 = x^2 + y^2$

Rearranging, we have

$$x^2 (M^2 - 1) + 2xM^2 + y^2 (M^2 - 1) = - M^2 \qquad \text{--------------------(a)}$$

$$x^2+\frac{2M^2}{M^2-1}x+y^2=-\frac{M^2}{M^2-1}$$

Making a perfect square of the terms, we have,

$$\left(x^2+\frac{M^2}{M^2-1}\right)^2+y^2=-\frac{M^2}{M^2-1}+\frac{M^4}{(M^2-1)^2}$$

$$=\frac{-M^2(M^2-1)+M^4}{(M^2-1)^2}$$

$$= \left(\frac{M}{M^2 - 1} \right)^2$$

Represents a circle with a radius of $\frac{M^2}{M^2 - 1}$ and centre at $\left(-\frac{M^2}{M^2 - 1}, 0 \right)$.

For various assumed values of M, a family of circles can be drawn which represent the above equation.

These circles are called constant M-circles.

Properties of M-circles:

1. For M =1, the centre of the circle is at $\left(\underset{M \to 1}{Lt} \frac{-M^2}{M^2 - 1}, 0 \right)$. ie., (-∞, 0).

The radius is also infinity

Substituting M = 1 in equation (a), we have

2x = -1

Or x = $-\frac{1}{2}$

This M = 1 is a straight line parallel to y axis at $x = -\frac{1}{2}$.

2. For M > 1, centre of the circle is on the negative real axis and as M → ∞, the centre approaches

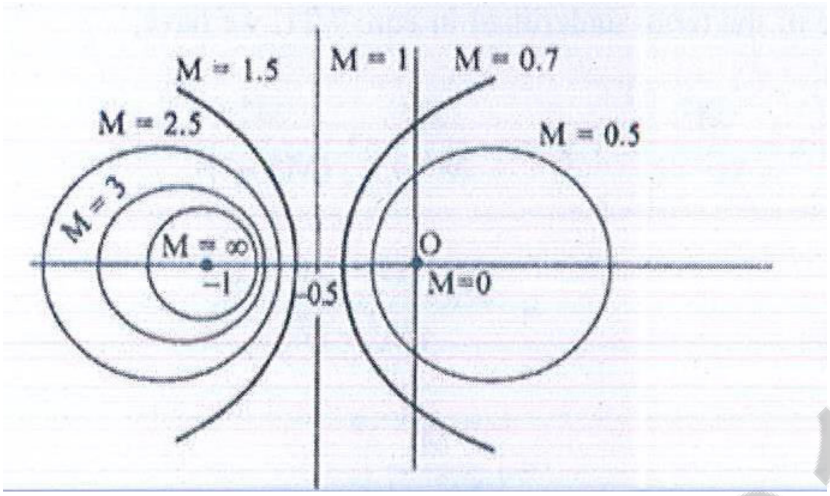
(-1, j0) point and the radius approaches zero; ie (-1, j0) point represents a circle of M = ∞.

3. For 0 < M < 1, $-\frac{M^2}{M^2 - 1}$ is positive and hence the centre is on the positive real axis.

4. For M = 0, the centre is at (0, 0) and radius is 0; ie., origin represents the circle for M = 0.

5. As M is made smaller and smaller than unity, the centre moves from +∞ towards the origin on the positive real axis.

The M circles are sketched in Fig. below



2. Prove that the loci of the constant phase angle of closed loop transfer function is a circle

Constant N circles

Constant N circles are obtained for the points on the open loop polar plot which result in constant phase angle for the closed loop system. Consider the phase angle of the closed loop transfer function

$$\angle T(j\omega) = \theta = \angle \frac{x + jy}{1 + x + jy}$$
$$= \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{y}{1+x}$$

Taking tangent of the angles on both sides of equation. 7.23, we have

$$\tan\theta = \frac{\frac{y}{x} - \frac{y}{1+x}}{1 + \frac{y^2}{x(1+x)}} = \frac{y}{x^2 + y^2 + x}$$

Let tan θ = N

Then $\frac{y}{x^2 + y^2 + x} = N$

Rearranging, we get,

$$N(x^2 + x) + Ny^2 - y = 0$$

$$N\left(x + \frac{1}{2}\right)^2 + N\left(y - \frac{1}{2N}\right)^2 = \frac{N}{4} + \frac{1}{4N}$$

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{4}\left(\frac{N^2 + 1}{N^2}\right)$$

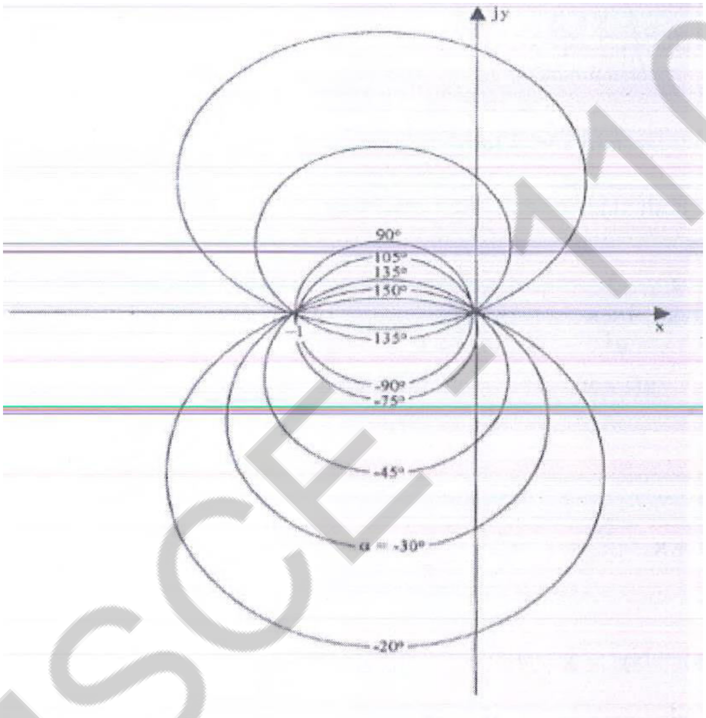
Represents the equation of a family of circles for different values of N with centre at

$$\left(-\frac{1}{2}, \frac{1}{2N}\right)$$

And Radius $$= \frac{\sqrt{N^2+1}}{2N}$$

These circles are known as constant N circles.

The constant N-circles are shown in Figure. Instead of marking the values of N on the various circles, value of $\alpha = \tan^{-1} N$ are marked so that the phase angle can be read from the curves.



UNIT – 4

STABILITY AND COMPENSATOR DESIGN

PART –A

1. Defined stability

A linear relaxed system is said to have BIBIO stability if every bounded input result in a bounded output.

2. **What is nyquist contour?**

The contour that enclosed entire right half of s plane is called nyquist contour.

3. **State Nyquist stability criterion.** (April/May 2019, Nov/Dec 2019, NOV/DEC 2015 & MAY/JUNE 2013)

If the nyquist plot of the open loop transfer function G(s) corresponding to the nyquist contour in the s plane encircle the critical point -1+j0 in the contour in clockwise direction as many time as the number of right half s plane poles of G(s), the closed loop system is stable.

4. Defined relative stability.

Relative stability is degree of closeness of the system; it is an indication of strength or degree of stability.

5. What will be the nature of impulse response when the roots of characteristic equation are lying on imaginary axis?

If the root of characteristic equation lies on imaginary axis the nature of impulse response is oscillatory.

6. What is the relationship between stability and coefficient of characteristic polynomial?

If the coefficients of characteristic polynomial are negative or zero, then some of the roots lies on the negative half of the s plane. Hence the system is unstable. If the coefficients of the characteristic polynomial are positive and if no coefficient is zero then there is possibility of the system to be stable provided all the roots are lying on the left half of the s-plane.

7. What is Routh stability criterion? (APIRL/MAY 2010)

Routh criterion states that the necessary and sufficient condition for stability is that all of the element in the first column of the routh array is positive. If this condition is not met, the system is unstable and the number of sign changes in the element of the first column of the routh array corresponds to the number of roots of characteristic equation in the right half of s plane.

8. What is limitedly stable system?

For a bounded input signal if the output has constant amplitude oscillation , then the system may be stable or unstable under some limited constraints. Such a system is called limitedly stable system.

9. In the routh array what conclusion you can make when there is row of all zero?

All zero rows in the routh array indicate the existence of an even polynomial as a factor of the given characteristic equation. The even polynomial may have roots on imaginary axis.

10. What is the principle of argument?

The principle of argument states that let F(s) are analytic function and if an arbitrary closed contour in a clockwise direction is chosen in the s plane so that F(s) is analytic at every point of the contour. Then the corresponding F(s) plane contour mapped in the F(s) plane will encircle the origin N times in the anti clockwise direction, where N is the difference between number of poles and zeroes of F(s) that are encircle by the chosen closed contour in the s plane.

11. What are the two segment of Nyquist contour?

i. An finite line segment C1 along the imaginary axis.

ii. An arc C2 of infinite radius.

12. What are the root loci?

The path taken by the root of the open loop transfer function when the loop gain is varied from 0 to infinity are called root loci.

13. What is the dominant pole? (NOV/DEC 2015, 2016), APRIL/MAY 2017

The dominant pole is a pair of conjugate pole which decides the transient response of the system. In higher order system the dominant poles are very close to origin and all other poles of the system are widely separated and so they have less effect on transient response of the system.

14. What are the main significance of root locus?

i. The root locus technique is used for stability analysis.
ii. Using root locus techniques the range of value of K, for as stable system can be determined.

15. What are the breakaway point are break in points?

At break away point the root locus break from the real axis to enter into the complex plane. At break in point the root locus enters the real axis from then complex plane. To find the breakaway or break in points, from a equation for K from the characteristic equation and differentiate the equation of K with respect to s. Then find the roots of the equations dK/dS = 0. The root of dK/dS = 0 are breakaway or break in points provided for this value of root the gain K should be positive and real.

16. What are asymptotes? How will you find angle of asymptotes?

Asymptotes are the straight line which are parallel to root locus going to infinity and meet the root locus at infinity.

Angle of asymptotes = $\pm\frac{180°(2q+1)}{n-m}$ $\quad\quad q = 0,1,2,3,....n-m$

N= number of poles

M = number of zeroes.

17. What is the centriod?

The meeting point of the asymptotes with the real axis is called centroid. The centroid is given by Centroid = (sum of the poles-sum of the zeros)/n-m

N= number of poles

M = number of zeroes.

18. What is magnitude criterion?

The magnitude criterion states that s = s_a will be a point on root locus if for that value of s, magnitude of G(s)H(s) is equal to 1.

$$|G(s)H(s)| = K\frac{(\text{product length of vector from open loop zeros to the point } s = s_a)}{(\text{product length of vector from open loop poles to the point } s = s_a)} = 1$$

19. What is angle criterion?

The angle criterion states that s = s_a will be a point on root locus if for that value of s, the argument or phase of G(s)H(s) is equal to an odd multiple 180° .

$$(\text{sum of the angle of vectors from zeros to the point } s = s_a) - (\text{sum of the angle of vectors from poles to the point } s = s_a) = \pm180°(2q+1)$$

20. How will you find the root locus on real axis? (MAY/JUNE 2016)

To find the root locus on real axis choose the test point on real axis to the right of this test point is odd number then the test point lie on the root locus. If it is even the test point does not lie on the root locus.

21. What is characteristic equation? May/June 2016

The denominator polynomial of C(s)/R(s) is the characteristic equation of the system.

22. How the roots of characteristic are related to stability? Nov/Dec 2015

If the root of characteristic equation has positive real part then the impulse response of the system not bounded. Hence the system will be unstable. If the root has negative real part then the impulse response is bounded. Hence the system will be stable.

23. What is necessary condition for stability? (MAY/JUNE 2013, 2016 ,APRIL/MAY 2017, Nov/Dec 2017)

- The necessary condition for stability is that all the coefficient of the characteristic polynomial be positive
- The necessary and sufficient condition for stability is that all of the element in the first column of the routh array should be positive.

24. What are the requirements of BIBO stability? (Nov/DEC 2016)

The requirement of BIBO stability is that the absolute integer of the impulse response of the system should take only the finite value.

PART- B

ROUTH HURWITZ CRITERION

1. Construct Routh array and determine the stability of the system whose characteristic equation is $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$

Solution:

The characteristic equation of the system is , $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$

The order of the equation is 4, so it has 4 roots

s^4 :	1	18	5	… Row 1
s^3 :	8	16		… Row 2

	Column 1			
s^4 :	1	18	5	… Row 1
s^3 :	1	2		… Row 2
s^2 :	16	5		… Row 3
s^1 :	1.7			… Row 4
s^0 :	5			… Row 5

$$s^2; \frac{1\times18-1\times2}{1} \quad \frac{1\times5-1\times0}{1}$$

$$s^2; \quad 16 \qquad 5$$

$$s^1; \frac{16\times2-5\times1}{16}$$

$$s^1: \quad 1.7$$

$$s^0; \frac{1.7\times5-16\times0}{1.7}$$

$$s^0; \quad 5$$

On examining the first column of routh array it is observed that all the elements are positive and there is no sign change. Hence all the roots are lying on left half of the s plane and the system is stable.

2. By routh stability criterion, determine the stability of the system represented by the characteristic equation $s^7+5s^6+2s^5+4s^4+3s^3+8s^2+2s+8=0$

Solution: Routh array

s^7 :	1	2	3	2	…… Row1
s^6 :	5	4	8	8	…… Row2
s^5 :	1.2	1.4	0.4		…… Row3
s^4 :	-1.8	6.3	8		…… Row4
s^3 :	5.6	5.7			…… Row5
s^2 :	8.1	8			…… Row6
s^1 :	0.17	0			…… Row7
s^0 :	8				…… Row8
	Column 1				

Explanation

$$s^5; \frac{5\times2-1\times4}{5} \quad \frac{5\times3-1\times8}{5} \quad \frac{5\times2-1\times8}{5}$$

$$s^5; \quad 1.2 \qquad 1.4 \qquad 0.4$$

$$s^4; \frac{1.2\times4-5\times1.4}{1.2} \quad \frac{1.2\times8-5\times0.4}{1.2} \quad \frac{1.2\times8}{1.2}$$

$$s^4; \quad -1.8 \qquad 6.3 \qquad 8$$

$s^3; \dfrac{-1.8\times1.4-1.2\times6.3}{-1.8} \quad \dfrac{-1.8\times0.4-1.2\times8}{-1.8}$

$s^3; \quad 5.6 \quad\quad 5.7$

$s^2; \dfrac{5.6\times6.3-(-1.8)\times5.7}{5.6} \quad \dfrac{5.7\times8-0}{5.7}$

$s^2; \quad 8.1 \quad\quad 8$

$s^1; \dfrac{8.1\times5.7-5.6\times8}{8.1}$

$s^1: \quad 0.17$

$s^0; \dfrac{0.17\times8-8.1\times0}{8}$

$s^0; \quad 8$

Result

There are two sign changes in the first column of the routh array. Therefore 2 roots are located on the right half of the s p lane & remaining five roots are located on the left half of the plane. Therefore System is unstable

3. By using routh criterion, determine the stability of the system represented by the following characteristic equation $s^5+s^4+2s^3+2s^2+11s+10=0$

Solution:

The characteristic equation is $s^5+s^4+2s^3+2s^2+11s+10=0$, the order of the equation is 5& so it has 5 roots

Routh array;

s^5 :	1	2	11	... Row 1
s^4 :	1	2	10	... Row 2
s^3 :	$0 \to \in$	-2		... Row 3
s^3 :	$\in$	-2		... Row 3
s^2 :	$\dfrac{+2\in+2}{\in}$	10		... Row 4
s^1 :	$\dfrac{-(10\in^2+4\in+4)}{(2\in+2)}$			... Row 5
s^0 :	5			... Row 6

$$s^3;\ \frac{1\times2-1\times2}{1}\quad \frac{1\times11-1\times10}{1}$$

$$s^3:\quad 0\quad -2$$

 replace 0 by $\in$

$$s^2;\ \frac{2\in+2}{\in}\quad \frac{10\in-0}{10}$$

$$s^2;\ \frac{+2\in+2}{\in}\quad 10$$

$$s^1;\ \frac{\frac{(2\in+2)(-2)-10\in}{\in}}{\frac{2\in+2}{\in}}$$

$$s^1;\ \frac{\frac{-4\in-4-10\in^2}{\in}}{\frac{2\in+2}{\in}}=\frac{-10\in^2-4\in-4}{2\in-2}$$

$$s^0;\ \frac{\frac{-10\in^2-4\in-4}{2\in+2}}{\frac{-10\in^2-4\in-4}{2\in+2}}\times10=10$$

On letting $\in\to0$ we get

s^5 :	1	2	11	…… Row1
s^4 :	1	2	10	…… Row2
s^3 :	0	-2		…… Row3
s^2 :	∞	10		…… Row4
s^1 :	-2			…… Row5
s^0 :	5			…… Row6

Column 1

On observing the first column, there are two sign changes. Therefore two roots are located on the right half of the s plane & remaining 3 roots are located on the left half of the s plane. System is unstable.

4. The characteristic polynomial of a system is $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$ **Determine the location of roots on s plane & hence the stability of the system**

Solution:

The characteristic equation is $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$

Routh array

s^7:	1	24	24	23	ROW 1
s^6:	9	24	24	15	ROW 2

Divide s^6 ROW by 3

	1st column				
s^7 :	1	24	24	23	…… Row1
s^6 :	3	8	8	5	…… Row2
s^5 :	1	1	1		…… Row3
s^4 :	1	1	1		…… Row4
s^3 :	0	0			…… Row5
s^3 :	2	1			…… Row5
s^2 :	0.5	1			…… Row6
s^1 :	-3				…… Row7
s^0 :	1				…… Row8

$$s^5;\ \frac{3\times 24 - 8\times 1}{3} \quad \frac{3\times 24 - 8\times 1}{3} \quad \frac{3\times 23 - 5\times 1}{5}$$

$$s^5;\ \frac{21.3}{1} \quad \frac{21.3}{1} \quad \frac{21.3}{1}$$

$$s^4;\ \frac{1\times 8 - 1\times 3}{1} \quad \frac{1\times 8 - 1\times 3}{1} \quad \frac{1\times 5 - 0\times 3}{1}$$

$$s^4;\ \frac{5}{1} \quad \frac{5}{1} \quad \frac{5}{1}$$

$$s^3;\ \frac{1\times 1 - 1\times 1}{1} \quad \frac{1\times 1 - 1\times 1}{1}$$

$$s^3;\ 0 \quad 0$$

The auxiliary polynomial

$$A = s^4 + s^2 + 1$$

$$\frac{dA}{ds} = 4s^3 + 2s$$

$$s^3 = \quad 4 \quad 2$$

$$s^2 = \quad 2 \quad 1$$

$$s^2; \frac{2\times1-1\times1}{2} \quad \frac{2\times1-0\times1}{2}$$

$$s^2; \quad 0.5 \quad 1$$

$$s^1; \quad \frac{0.5\times1-1\times2}{0.5}$$

$$s^1; \quad -3$$

$$s^0; \quad \frac{-3\times1-0.5\times0}{-3}$$

$$s^0; \quad 1$$

On examining the first column element routh array it is found that there are two sign changes. Hence two roots are lying on the right half of s plane and the system is unstable.

The rows of all zeros indicate the possibility of complex roots

The auxiliary equation is $s^4 + s^2 + 1 = 0$

Put s^2= x in auxiliary equation,

$$s^4 + s^2 + 1 = 0 \Rightarrow \quad x^2 + x + 1 = 0$$

The roots of the quadrant equation

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$
$$= 1\angle 120°, 1\angle -120°$$

But s^2 = x, $\therefore$ s = $\pm\sqrt{x} = \pm\sqrt{\angle 120°}, \pm\sqrt{\angle -120°}$

$$= \pm\sqrt{1\angle 120°/2}, \pm\sqrt{\angle -120°/2}$$
$$= \pm\angle 60°, \angle -60°$$
$$= \pm(0.5 + j0.866), \pm(0.5 - j0.866)$$

The two roots of auxiliary polynomial all lying on the right half of s plane & the remaining two on the left half of s plane. The roots of the auxiliary equation are also roots of the characteristic equation. No roots are lying on the imaginary axis.

$\rightarrow$ The system is unstable

$\rightarrow$ Two roots are lying on right half of s plane & five roots are lying on left half of s plane.

5. Use the routh stability criterion to determine the location of roots on the s plane and hence the stability for the system represented by the characteristic equation $s^5+4s^4+8s^3+8s^2+7s+4=0$

Solution:

The characteristic equation is $s^5+4s^4+8s^3+8s^2+7s+4=0$

The order of the characteristic equation is 5& it has 5 roots

Routh array

s^5 :	1	8	7	...... Row1
s^4 :	4	8	4	...... Row2

Divided s^4 row by 4

s^5 :	1	8	7	...... Row1
s^4 :	1	2	1	...... Row2
s^3 :	1	1		...... Row3
s^2 :	1	1		...... Row4
s^1 :	$0 \rightarrow \in$			...... Row5
s^0 :	1			...... Row6

1^{ST} column

$$s^3; \frac{1\times8-2\times1}{1} \quad \frac{1\times7-1\times1}{1}$$

$$s^3; \quad 6 \quad 6$$

$$s^3; \quad 1 \quad 1$$

$s^2; \frac{1\times 2-1\times 1}{1} \quad \frac{1\times 1-0\times 1}{1}$

$s^2; \quad 1 \quad 1$

$s^1; \quad 1\times 1-1\times 1$

$s^1; \quad 0$

Let $0 \rightarrow \in$

$s^1; \in$

$s^0; \frac{\in \times 1-0\times 1}{\in}$

$s^0; \ 1$

When $\in \rightarrow 0$ there is no sign change in the first column of routh array. But we have a row of all zeros (s^1 row).

So there is a possibility of roots on imaginary axis this can be found from the roots of auxiliary equation.

Auxiliary equation is $s^2+1 = 0$

$$s^2 = -1$$

$$s = \pm\sqrt{-1} = \pm j1$$

The roots of auxiliary equation are +j,-j lying on imaginary axis.

- Two roots are lying on imaginary axis and there are no sign changes in the first column of routh array , remaining three roots are located in the left half of s plane
- Hence the system is limitedly or marginally stable.

6. For each of the characteristic equations of feedback control system given, determine the range of K for stability. Determine the value of K so that the system is marginally stable & find the frequency of sustained oscillations.

i) $s^4 + 25s^3 + 15s^2 + 20s + K$

ii) $s^4 + Ks^3 + s^2 + s + 1 = 0$

iii) $s^3 + 3Ks^2 + (K+2)s + 4 = 0$

iv) $s^4 + Ks^3 + 5s^2 + 10s + 10K = 0$

Solution:

i) Given $s^4 + 25s^3 + 15s^2 + 20s + K$

s^4 :	1	15	K	… Row 1
s^3 :	25	20		… Row 2
s^2 :	$\frac{71}{5}$	K		… Row 3
s^1 :	$\frac{284-25K}{71/5}$	0		… Row 4
s^0 :	K			… Row 5
	Column 1			

$$s^2;\ \frac{25\times 15-20\times 1}{25} \quad \frac{25\times K-1\times 0}{25}$$

$$s^2;\ \frac{71}{5} \quad K$$

$$s^1;\ \frac{\frac{71}{5}\times 20-25K}{\frac{71}{5}} \quad 0$$

$$s^1;\ \frac{284-25K}{\frac{71}{5}}$$

$$s^0;\ \frac{\left(\frac{284-25K}{\frac{71}{5}}\right)K-0}{\frac{284-25K}{\frac{71}{5}}}$$

$$s^0;\ K$$

For the system to be stable all the element in the first column must be positive

Therefore for s^1 row, to be positive,

$$\frac{284-25K}{71/5} > 0 \Rightarrow 284-25K > 0$$

$$K < \frac{284}{25};\ K < 11.36 \rightarrow (1)$$

From s^0 row, $K > 0 \rightarrow (2)$

Combining 1 and 2

The range of K for stability is $0 < K < 11.36$. When K=11.36 the elements in s^1 row , becomes zero & the root are on the $j\omega$ axis . Hence the system is under sustained oscillations.

To find the frequency oscillation:

$$\frac{71}{5}s^2 + K = 0\ ;\ \ K = 11.36$$

$$\frac{71}{5}s^2 + 11.36 = 0 \Rightarrow s = \pm j0.894$$

$$\omega = 0.894\,\text{rad/sec}$$

ii) Given $s^4 + Ks^3 + s^2 + s + 1 = 0$

Routh array

s^4 :	1	1	1	… Row 1
s^3 :	K	1		… Row 2
s^2 :	$\frac{K-1}{K}$			… Row 3
s^1 :	$\frac{\frac{K-1}{K}-K}{\frac{K-1}{K}}$			… Row 4
s^0 :	1			… Row 5

For the system to be stable , all the elements in the first column must be positive

From s^3 row, $K > 1$

From s^2 row , $\frac{K-1}{K} > 0 \Rightarrow K > 1$

From s^3 row, $\frac{K-1}{K} - K > 0$

$$\Rightarrow \frac{K-1}{K} > K$$
$$\Rightarrow K-1 > K^2$$

If both condition,K > 1 &K-1 > K^2 are satisfied, then the system is stable. But when K>1, K -1 > K^2 is not satisfied therefore for all values of K, the system is unstable.

iii) Given $s^3 + 3Ks^2 + (K+2)s + 4 = 0$

s^3 :	1	K+2	… Row 1
s^2 :	3K	4	… Row 2
s^1 :	$\frac{3K(K+2)}{3K}$	0	… Row 3
s^0 :	4		… Row 4

For the system to be stable,

$$3K > 0 \quad \Rightarrow K > 0 \left(\text{from } s^2 \text{ row}\right)$$
$$3K(K+2) - 4 > 0 \ \left(\text{from } s^1 \text{ row}\right)$$

$$3K^2 + 6K - 4 > 0$$
$$K^2 - 2K - 1.33 > 0$$
$$(K+2.2527)(K-0.527) > 0$$

From which K > 0.527

For the system to be stable ,

$$\Rightarrow K > 0.527$$

To find auxiliary equation & frequency of sustained oscillation.

When K= 0.527, s^1 row become zero.

$\therefore$ The auxiliary equation

$$3K\ s^2+4=0 \quad put k=0.527$$
$$1.581s^2+4=0.$$
$$\Rightarrow \omega=1.59 \quad rad/sec$$

$$iv) s^4+Ks^3+5s^2+10s+10K=0$$

s^4 :	1	5	10K	… Row 1
s^3 :	K	10		… Row 2
s^2 :	$\frac{5K-10}{K}$	10k		… Row 3
s^1 :	$\frac{\left(\frac{5K-10}{K}\right)10-10K^2}{\frac{5K-10}{K}}$			… Row 4
s^0 :	10K			… Row 5

$$s^0=10K$$

For the system to be stable, $\frac{5K-10}{K}$ >0$\Rightarrow$ K>2 (From s^2 row)

($\frac{5K-10}{K}$)10-10K^2 >0 (from s^1row)

$$\Rightarrow 10K^3-50K^2+100<0$$
$$K^3-5K+10<0$$
$$(K+2.9055)(K^2-2.9055K+3.442)<0$$

K is real when K < -2.9055

Therefore the condition for stability are K > 2, &K< -2.9055.the condition are contradicting to each other. So unstable for all value of k.

7. Using routh Hurwitz criterion determine the stability of a system representing the characteristic equation $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$ **and comment on the location of root of the characteristic equation.**

Solution:

The characteristic equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$

The order of the equation is 6 & number of roots are 6.

Routh array

	Column 1				
s^6 :	1	8	20	16	... Row 1
s^5 :	2	12	16	0	... Row 2
s^4 :	2	12	16		... Row 3
s^4 :	0	0			... Row 4
s^3 :	4	12			... Row 4
s^2 :	6	16			... Row 5
s^1 :	1.33				... Row 6
s^0 :	16				... Row 7

There is no sign change in first column. But row of zeros indicates the presence of complex roots.

$$s^4;\ \frac{2\times8-1\times12}{2}\quad\frac{2\times20-1\times16}{2}\quad\frac{2\times16-0}{2}$$
$$s^4;\quad 2\qquad 12\qquad 16$$

$$s^3;\ \frac{2\times12-2\times12}{2}\quad\frac{2\times16-2\times16}{2}$$
$$s^3;\quad 0\qquad 0$$

Row of zeros;

Auxiliary equation is

$$2s^4+12s^2+16=0$$
$$s^4+6s^2+8=0$$
$$\frac{dA}{ds}=4s^3+12s$$

$s^2;\ \frac{4\times12-2\times12}{4} \quad \frac{4\times16-0\times1}{4}$

$s^2;\quad 6 \quad\quad 16$

$s^1;\quad \frac{6\times12-4\times16}{6}$

$s^1;\quad 1.33$

$s^0;\ \frac{1.33\times16-0}{1.33}$

$s^0;\ 16$

From auxiliary equation

$$
\begin{aligned}
& s^4+6s^2+8=0 \\
& \text{put } s^2=x. \\
& \therefore x^2+6x+8=0 \\
& (x+4)(x+2)=0 \\
& x=-4,-2. \\
& s^2=-4,-2 \\
& s=\pm\sqrt{-4},\pm\sqrt{-2} \\
& s=\pm j\,2,\quad \pm j1.414
\end{aligned}
$$

The roots are +j2, -j2,+j1.414,-j1.414. The four roots are lying on imaginary axis. Remaining 2 roots are located on the left half of the s plane. Hence the system is limitedly or marginally stable.

8. Using Routh Hurwitz criterion, determine the stability of the system represented the characteristic equation $s^5+s^4+2s^3+2s^2+3s+5=0$**. Comment on location of roots of the characteristic equation**

Solution:

The characteristic equation is $s^5+s^4+2s^3+2s^2+3s+5=0$, the order of the equation is 5 & the number of the roots is 5.

Routh array

s^5 :	1	2	3	… Row 1
s^4 :	1	2	5	… Row 2
s^3 :	0	-2		… Row 3
s^3 :	$\in$	-2		… Row 3

s^2 :	$\frac{2\in+2}{\in}$	5	... Row 4
s^1 :	$\frac{-\left(\in^2+4\in+4\right)}{2\in+2}$		... Row 5
s^0 :	5		... Row 6

$$s^3;\frac{1\times2-1\times2}{1}\quad\frac{1\times3-1\times5}{1}$$

$$s^3;\quad 0\qquad -2$$

⎕ replace 0 by $\in$

$$s^3;\quad \in\qquad -2$$

$$s^2;\frac{2\in-(-2)}{\in}\quad\frac{5\in-0}{\in}$$

$$s^2;\quad\frac{2\in+2}{\in}\quad 5$$

$$s^1;\quad\frac{\left(\frac{2\in+2}{\in}\right)(-2)-\in}{\frac{2\in+2}{\in}}$$

$$s^1;\quad\frac{-\left(4\in+4+\in^2\right)}{\frac{2\in+2}{\in}}$$

$$s^0;\frac{\frac{-\left(4\in+4+\in^2\right)}{\frac{2\in+2}{\in}}\times5}{\frac{-\left(4\in+4+\in^2\right)}{\frac{2\in+2}{\in}}}$$

$$s^0;\ 5$$

On letting $\in\to0$

s^5 :	1	2	3	... Row 1
s^4 :	1	2	5	... Row 2
s^3 :	0	-2		... Row 3
s^2 :	∞	5		... Row 4

s^1 :	-2	… Row 5
s^0 :	5	… Row 6
	Column 1	

There are two sign change in the first column of routh array. Therefore two roots are located on the right half of the s plane and remaining three roots are located on the left half of the s plane. Hence the system is unstable.

9. The open loop transfer function of a unity feedback control system is given by $G(s)=\frac{K}{(s+2)(s+4)(s^2+6s+25)}$ **. By applying the routh criterion, discuss the stability of the closed l loop system as a function of K. Determine the value of K which will cause sustained oscillation in the closed loop system what are the corresponding oscillation frequency.**

Solution:

The closed loop transfer function $\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}$

$$\frac{C(s)}{R(s)}=\frac{\frac{k}{(s+2)(s+4)(s^2+6s+25)}}{1+\frac{k}{(s+2)(s+4)(s^2+6s+25)}}$$

The characteristic equation is given by the denominator

polynomial of closed loop transfer function

The characteristic equation is

$$(s+2)(s+4)(s^2+6s+25)+K=0$$
$$(s^2+6s+8)(s^2+6s+25)+K=0$$
$$s^4+12s^3+69s^2+198s+200+K=0$$

The routh array is constructed as below

s^4 :	1	69	200+k	… Row 1
s^3 :	12	198		… Row 2

Divided s^3 by 12

	Column 1			
s^4 :	1	69	200+K	… Row 1
s^3 :	1	16.5		… Row 2
s^2 :	52.5	200+K		… Row 3
s^1 :	$\frac{666.25-K}{52.5}$			… Row 4
s^0 :	200+K			… Row 5

$$s^2; \frac{1\times 69-16.5\times 1}{1} \quad \frac{4\times(200+K)}{1}$$
$$s^2; \quad 52.5 \quad 200+K$$

$$s^1; \frac{52.5\times 16.5-(200+K)\times 1}{52.5}$$
$$s^1; \quad \frac{666.25-K}{52.5}$$

$$s^0; \frac{\frac{666.25-K}{52.5}\times(200+K)}{\frac{666.25-K}{52.5}}$$
$$s^0; \quad 200+K$$

For the system to be stable, all the element in the first column is positive

$$\begin{aligned} &666.25-K>0 \\ &\Rightarrow K<666.25 \quad \ldots 1 \\ &200+K>0 \\ &K>-200 \\ &\text{but practical value of } K \text{ starts from } 0 \\ &\therefore K>0 \quad \ldots 2 \end{aligned}$$

Combining 1& 2

The change K for stability 0 <K < 666.25

To find the sustained oscillation frequency:

When K=666.25, s^1 row will become zero

The auxiliary equation is

$$52.5s^2 + 200 + K = 0 \quad \text{from } s^2 \text{ row}$$
$$\text{put} \quad K = 666.25$$
$$52.5s^2 + 200 + 666.25 = 0$$
$$s^2 = \frac{-200 - 666.25}{52.5} = -16.5$$
$$s = \pm\sqrt{-16.5} = \pm j\sqrt{-16.5} = \pm j4.06$$

When K = 666.25, the system has roots on imaginary axis & so it oscillates.

The frequency of the oscillation is given by the value of root on imaginary axis.

The frequency of oscillation $\omega = 4.06 \ \text{rad/sec}$

NYQUIST STABILITY

1. A unity feedback control system has $G(s) = \frac{10}{s(s+1)(s+2)}$**. Draw the Nyquist plot and determine the closed loop stability.**

Solution:-

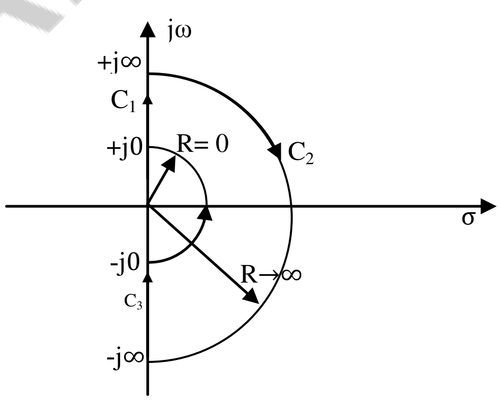
Given that $G(s).H(s) = \frac{10}{s(s+1)(s+2)}$ as $H(s) = 1$

i. Number of poles in the right half of the s – plane P = 0

ii. For stability no of encirclements N = - P =0

The nyquist plot should not encircle (-1+j0) point for absolute stability of this system.

iii. As there is one pole at origin, the Nyquist contour is as shown in figure which contains section C1, C2, C3 & C4



iv. Mapping of section C1:

In section C1 ,ω→0 to ∞, that is mapping of section C1 gives the pole of G(jω) H(jω) in (u-v) plane.

Put s= jω in G(s) H(s)

$$G(j\omega)H(j\omega) = \frac{10}{j\omega(1+j\omega)(2+j\omega)}$$

$$H = |G(j\omega)H(j\omega)| = \frac{10}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}}$$

$$\phi = \angle G(j\omega)H(j\omega) = -90° - \tan^{-1}\omega - \tan^{-1}\omega/2$$

ω	M	ϕ
0	∞	-90^0
∞	0	-270^0

$$G(j\omega)H(j\omega) = \frac{10}{j\omega(1+j\omega)(2+j\omega)}$$

$$= \frac{10}{(-\omega^2+j\omega)(2+j\omega)}$$

$$= \frac{-10}{(\omega^2-j\omega)(2+j\omega)} \times \frac{(\omega^2+j\omega)(2-j\omega)}{(\omega^2+j\omega)(2-j\omega)}$$

$$= \frac{-10(3\omega^2+j)(2\omega+\omega^3)}{(\omega^4+\omega^2)(4+\omega^2)}$$

To find crossing point on –ve axis, equate imaginary of G(jω) H(jω) = 0

$$\frac{-10(2\omega-\omega^2)}{(\omega^4+\omega^2)(4+\omega^2)} = 0$$

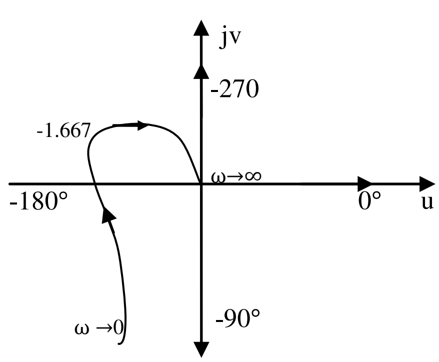
$$\omega^2 - 2 = 0 \Rightarrow \omega = \sqrt{2}$$

$$G(j\omega)H(j\omega) = \frac{-30\times 2}{(4+2)(4+2)} = \frac{-60}{36} = -1.667$$

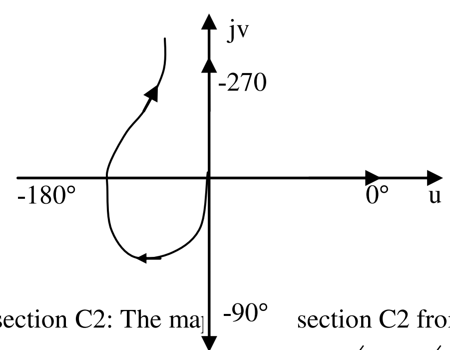
Equate real part of G(jω) H(jω) = 0

$$\frac{-30\omega^2}{(\omega^4+\omega^2)(4+\omega^2)} = 0 \Rightarrow \omega = \infty$$

Thus mapping of section C1 in (u-v) planes is as follows



5. Mapping of section C3: In section C3, ω is varying from $-\infty$ to 0. The mapping of section C3 is given by the locus of $G(j\omega)\,H(j\omega)$ where ω is varying from $-\infty$ to 0. The inverse polar plot is given by the minor image of polarplot with respect to real axis as shown in fig.



6. Mapping of section C2: The ma[illegible] section C2 from s – plane to (u-v) planes is obtained by putting $s = \lim_{R\to\infty} Re^{j\theta}$ in $G(s)H(s)$ and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to Re^{j\theta}$ and $R \to \infty$,

G(s) H(s) can be approximate as $(1+sT \simeq sT)$

$$G(s).H(s) = \frac{10}{s(s+1)(s+2)}$$

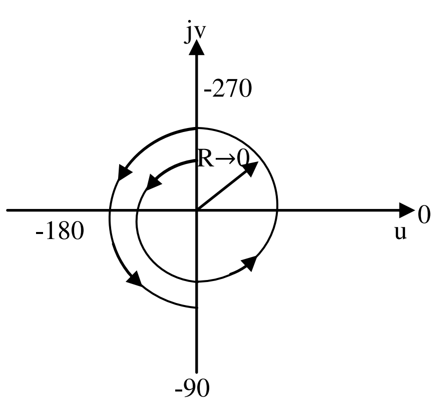
$$= \frac{10}{2s(1+0.5s)(1+s)} = \frac{10}{2\times s\times 0.5s\times s} = \frac{10}{s^3}$$

$$G(s)H(s)\Big|_{s=\underset{R\to\infty}{Lt} Re^{j\theta}} = \frac{10}{\underset{R\to\infty}{Lt} R^3 e^{j3\theta}} = 0e^{-j3\theta}$$

When $\theta = \frac{+\pi}{2}, G(s)H(s) = 0e^{-j\frac{3\pi}{2}}$

When $\theta = \frac{-\pi}{2}, G(s)H(s) = 0e^{+j\frac{3\pi}{2}}$

Therefore in (u-v) plane, θ varies from $-\frac{3\pi}{2}$ to $\frac{3\pi}{2}$ and magnitude of radius R reduces to 0.



7. Mapping section C4: The mapping of section C4 from s – plane to (u-v) plane (ie.G(s)H(s) plane) is obtained by putting $s = \underset{R\to 0}{Lt}\, Re^{j\theta}$ in G(s) H(s) & varying θ from $\frac{-\pi}{2}$ to $\frac{+\pi}{2}$ since $s \to Re^{j0}, \& R \to 0, G(s)H(s)$ can be approximated as 1+sT ≃ T.

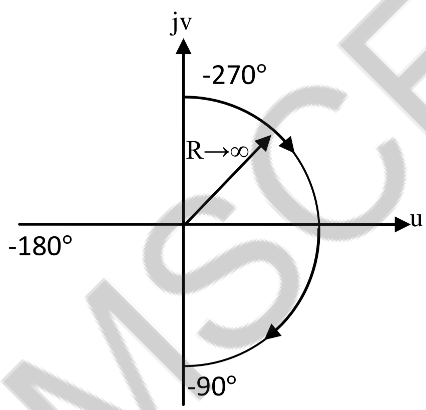
$$G(s)H(s) = \frac{5}{s(1+0.5)(1+s)} = \frac{5}{s\times 1\times 1} = \frac{5}{s}$$

$$\left|G(s)H(s)\right|_{s=\underset{R\to 0}{Lt} Re^{j\theta}} = \frac{5}{\underset{R\to 0}{Lt}\, Re^{j\theta}} = \infty e^{-j\theta}$$

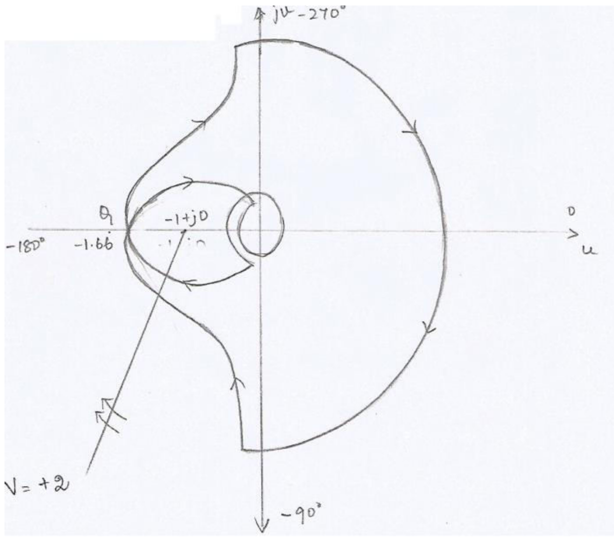
When $\theta = \frac{-\pi}{2}, G(s)H(s) = \infty e^{+j\pi/2}$

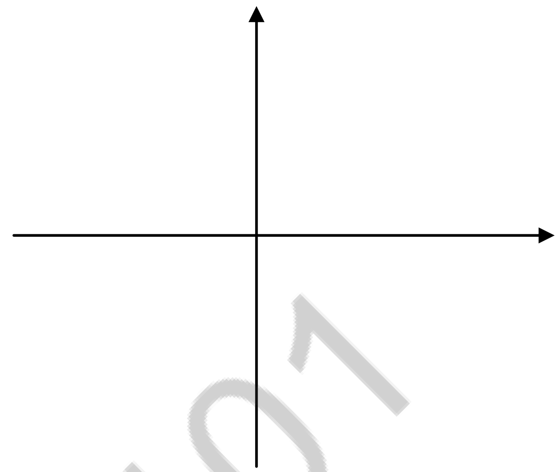
When $\theta = \frac{\pi}{2}, G(s)H(s) = \infty e^{-j\pi/2}$

Therefore section C4 in the s- plane is mapped as a circle of infinite radius with arguments varying from +90°to -90^0



8. Complete Nyquist plot: The complex Nyquist plot is G(s)H(s) plane can be obtained by combining the mappings of individual sections





9. The number of encirclements of (-1+j0) are N = +2

However, for stability, N = 0, the given system is unstable

According to mapping theorem, N = Z – P

$$2 = Z - 0 \rightarrow Z = 2$$

There are two zeros of 1+ G(s) H(s) encircled by Nyquist path, i.e two closed loop poles are there in the right half of the s – plane due to which the closed loop system is unstable.

2. The open loop transfer function of a system is given as $G(s).H(s) = \frac{1+4s}{s^2(1+s)(1+2s)}$ **. Determine the stability of closed loop system by using Nyquist criterian. If the closed loop system is not stable, then find the number of closed loop poles bying on the right half of the S – plane.**

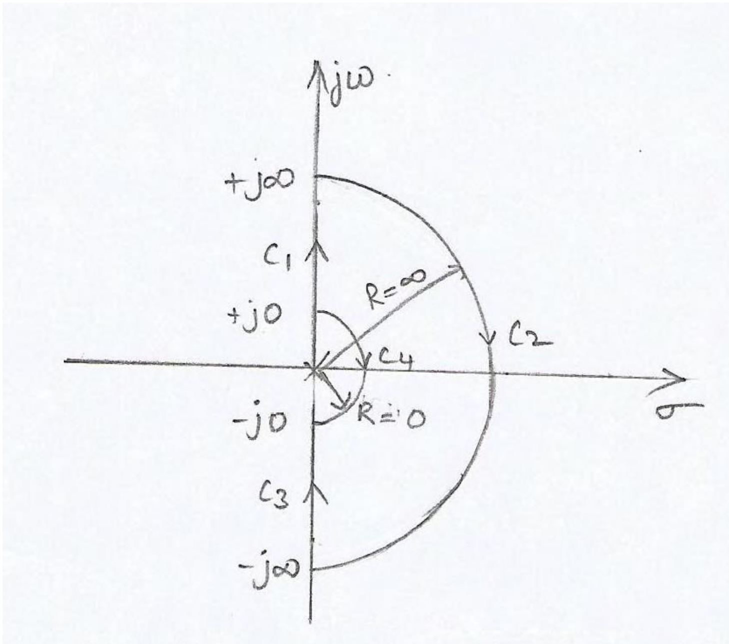
Solution:

Given that $G(s)H(s) = \frac{1+4s}{s^2(1+s)(1+2s)}$

1. Number of poles in the right half of the s – plane P = 0

2. For stability, N = -P = 0

3. As there are two poles at origin, the nyquist contour is as shown in fig which contains the section C1, C2, C3 & C4



4. Mapping of section C1: in section C1, $\omega \rightarrow 0 \text{ to} \infty$ i.e, the mapping of section C1 gives the poles plot of

G(jω) H(jω) in (u-v) plane

Put s = jω in G(s)H(s),

$$G(j\omega)H(j\omega) = \frac{1 + j4\omega}{(j\omega)^2 (1 + j\omega)(1 + j2\omega)}$$

$$M = |G(j\omega)H(j\omega)| = \frac{\sqrt{1 + 16\omega^2}}{\omega^2 \sqrt{(1 + \omega^2)(1 + 4\omega^2)}}$$

$$\phi = \angle G(j\omega)H(j\omega) = -180° - \tan^{-1}\omega - \tan^{-1}2\omega + \tan^{-1}4\omega$$

ω	M	ϕ
0	1.5	-180^0
∞	0	-270^0

$$G(j\omega)H(j\omega) = \frac{1 + j4\omega}{-\omega^2 (1 - 2\omega^2 + j3\omega)}$$

$$= \frac{-(1 + j4\omega)}{\omega^2 (1 - 2\omega^2 + j3\omega)} \times \frac{(1 - 2\omega^2 - j3\omega)}{(1 - 2\omega^2 - j3\omega)}$$

$$= -\frac{\left[1 + 10\omega^2 + j(\omega - 8\omega^3)\right]}{\omega^2 \left[(1 - 2\omega^2)^2 + 9\omega^2\right]}$$

$$= \frac{-(1 + 10\omega^2)}{\omega^2 \left[(1 - 2\omega^2)^2 + 9\omega^2\right]} - \frac{j(\omega - 8\omega^3)}{\omega^2 \left[(1 - 2\omega^2)^2 + 9\omega^2\right]}$$

To find crossing point on –ve real axis,

Equate imaginary part to zero

$$\frac{\omega-8\omega^3}{\omega^2\left[\left(1-2\omega^2\right)^2+9\omega^2\right]}=0$$

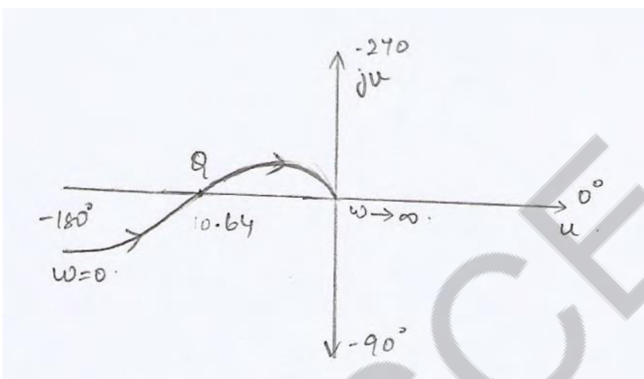
$$\omega\left(1-8\omega^2\right)=0$$

$$\Rightarrow \omega=0,\omega=\frac{1}{2\sqrt{2}}$$

$$\left|G(j\omega)H(j\omega)\right|_{\omega=\frac{1}{8}}=\frac{-\left(1+10\times\frac{1}{8}\right)}{\frac{1}{8}\left[\left(1-2\times\frac{1}{8}\right)^2+\frac{9}{8}\right]}$$

$$OQ=-10.64$$

Thus the mapping of section C1 in (u-v) plane is as shown in figures.



5. Mapping of section C2: Mapping of section C2 from s – plane to (u-v) plane is obtained by taking $s=\underset{R\to\infty}{Lt}Re^{j\theta}$ is $G(s)H(s)$ & varying θ from $\frac{+\pi}{2}$ to $\frac{-\pi}{2}$.

Since $s\to Re^{j\theta}, R\to\infty, 1+sT\approx sT$

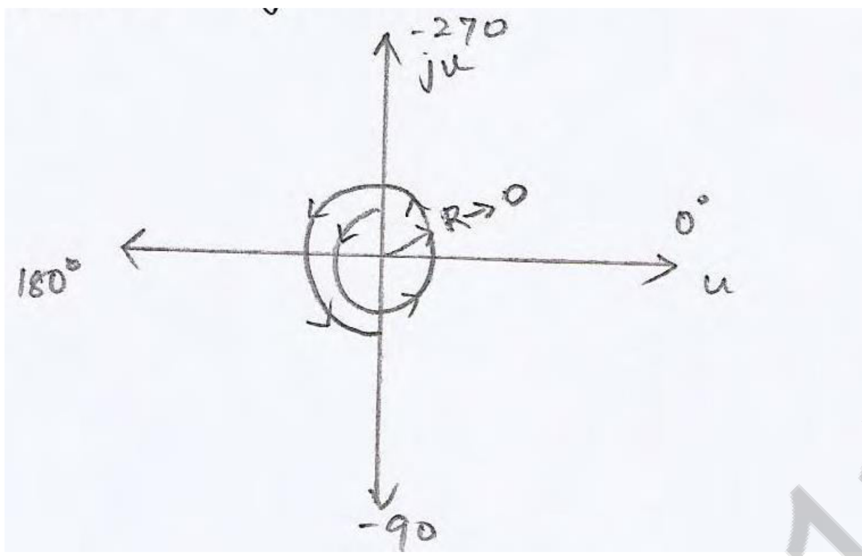
$$G(s)H(s)=\frac{1+4s}{s^2(1+s)(1+2s)}=\frac{4s}{s^2\times s\times 2s}=\frac{2}{s^3}$$

$$G(s)H(s)\Big|_{s=\underset{R\to\infty}{Lt}Re^{j\theta}}=\frac{2}{\underset{R\to\infty}{Lt}R^3e^{j3\theta}}=0e^{-j3\theta}$$

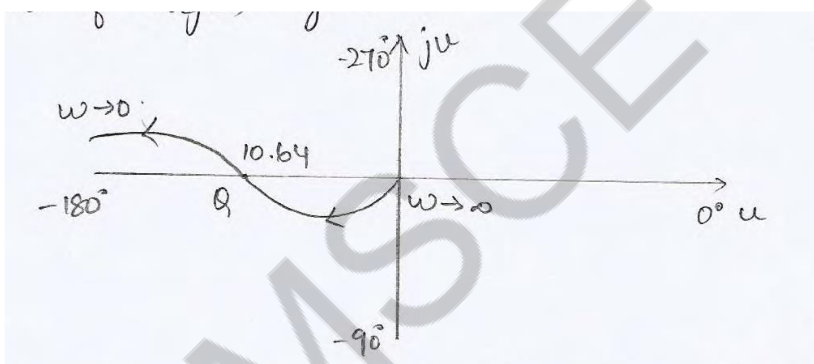
When $\theta=\frac{\pi}{2}, G(s)H(s)=0e^{-j3\pi/2}$

When $\theta=\frac{-\pi}{2}, G(s)H(s)=0e^{j3\pi/2}$

In (u-v) plane, θ varies from $\frac{-3\pi}{2}$ to $\frac{+3\pi}{2}$ of magnitude of radius R reduces to 0.



6. Mapping of section C3: In section C3, $\omega \to \infty \text{ to } 0$ i.e the mapping of section C3 gives the inverse polar plotof G(jω)H(jω) as shown in fig.



7. Mapping of section C4: mapping of section C4 from s plane to (u-v) plane can be obtained by substituting

$s = \underset{R\to 0}{\text{Lt}}\ Re^{j\theta}\ \&\ R \to 0, 1+sT \approx 1$

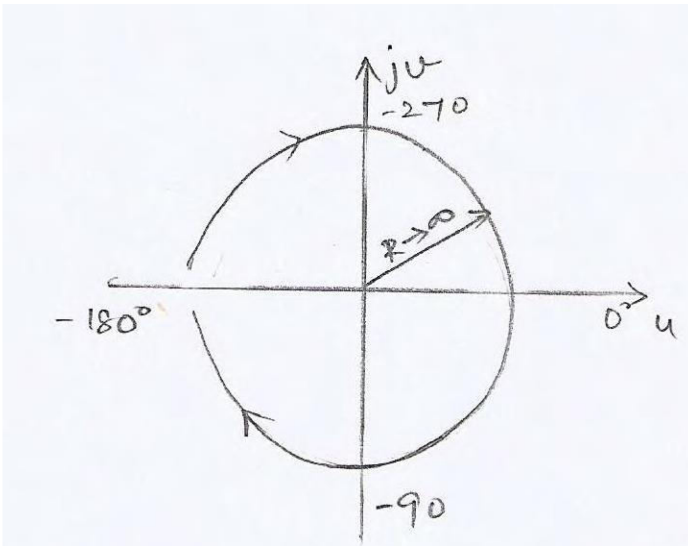
$$\therefore G(s)H(s) = \frac{1+4s}{s^2(1+s)(1+2s)} = \frac{1}{s^2 \times 1 \times 1} = \frac{1}{s^2}$$

$$G(s)H(s)\Big|_{s=\underset{R\to\theta}{\text{Lt}} Re^{j\theta}} = \frac{1}{\underset{R\to\theta}{\text{Lt}} Re^{j2\theta}} = \infty e^{-j2\theta}$$

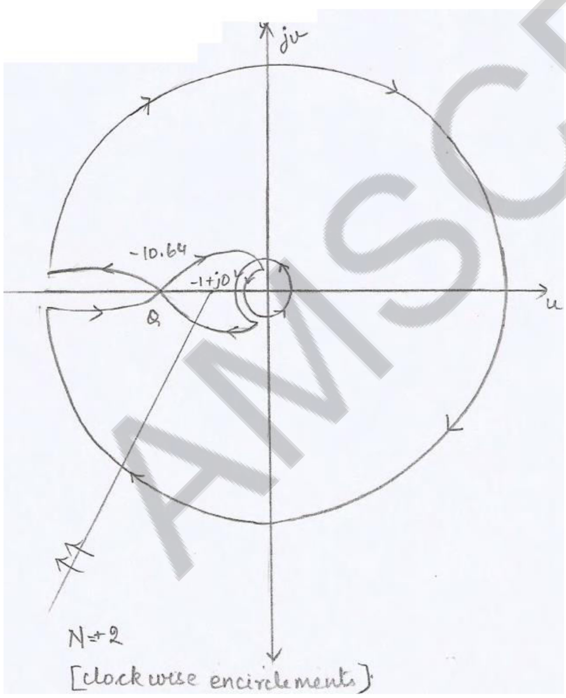
When $\theta = \frac{-\pi}{2}, G(s)H(S) = \infty e^{+j\pi}$

When $\theta = \frac{\pi}{2}, G(s)H(s) = \infty e^{-j\pi}$

Section C4 in the s – plane is mapped into a circle of infinite radius with arguments varying from $+\pi$ to $-\pi$



8. Complete Nyquist plot: The complete Nyquist plot in G(s)H(s) (or) (u-v) plane can be obtained by combining the mapping of the individual sections as shown in fig.



9. The number of encirclement of (-1+jθ) are

N= +2 (clockwise encirclements)

However, for stability, N = 0

The closed loop system is unstable

According to the mapping theorem, we have

$N = Z - P$

$2 = Z - 0 \rightarrow Z = 2$

There are 2 zeros of 1+G(s)H(s) encircled by Nyquist path, that is 2 closed loop poles are there in the right half of the s – plane, due to which the closed loop system is unstable.

3. Draw the Nyquist plot for the system whose open loop transfer function is $G(s)H(s) = \dfrac{K}{s(s+2)(s+10)}$

Determine the range of K for which the closed loop system is stable.

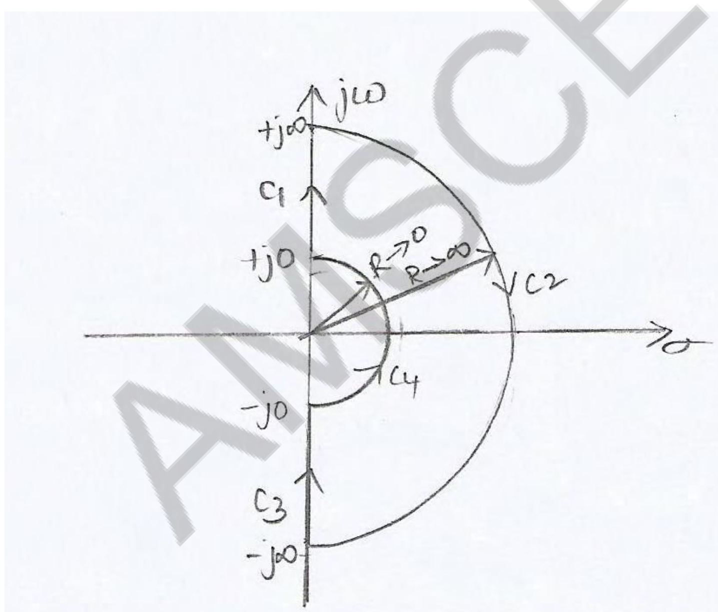
Solution:-

Given that $G(s)H(s) = \dfrac{K}{s(s+2)(s+10)}$

1. Number of poles in the right of half of s – plane P = 0

2. For stability N= -P = 0

3. As there is one pole at origin, the Nyquist contour is chosen as shown in fig. which contains sections C1, C2, C3, & C4.



4. Mapping of section C1: In section C1, $\omega \rightarrow 0 \text{ to} \infty$ that is mapping of section C1 gives the polar plot of G(jω)H(jω) in (u-v) plane

Put s = jω in G(s)H(s)

$$\therefore G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega+2)(j\omega+10)}$$

$$M = |G(j\omega)H(j\omega)| = \frac{K}{\omega\sqrt{\omega^2+4}\sqrt{\omega^2+100}}$$

$$\phi = \underline{|G}(j\omega)H(j\omega) = -90 - \tan^{-1} 0.5\omega - \tan^{-1} 0.1\omega$$

ω	M	ϕ
0	∞	-90^0
∞	0	-270^0

$$G(s)H(s) = \frac{K}{s\times 2(1+0.5s)10\times(1+0.1s)}$$

$$= \frac{0.05K}{s(1+0.5s)(1+0.1s)}$$

$$G(j\omega)H(j\omega) = \frac{0.05K}{-0.6\omega^2 + j\omega(1-0.05\omega^2)}$$

$$= \frac{0.05K\left[-0.6\omega^2 - j\omega(1-0.05\omega^2)\right]}{\left[-0.6\omega^2 + j\omega(1-0.05\omega^2)\right]\left[-0.6\omega^2 - j\omega(1-0.05\omega^2)\right]}$$

$$= \frac{-0.05K\times 0.6\omega^2}{0.36\omega^4 + \omega^2(1-0.05\omega^2)^2} - \frac{j0.05K\omega(1-0.05\omega^2)}{0.36\omega^4 + \omega^2(1-0.05\omega^2)^2}$$

To find crossing point on –ve real axis equate imaginary part to zero

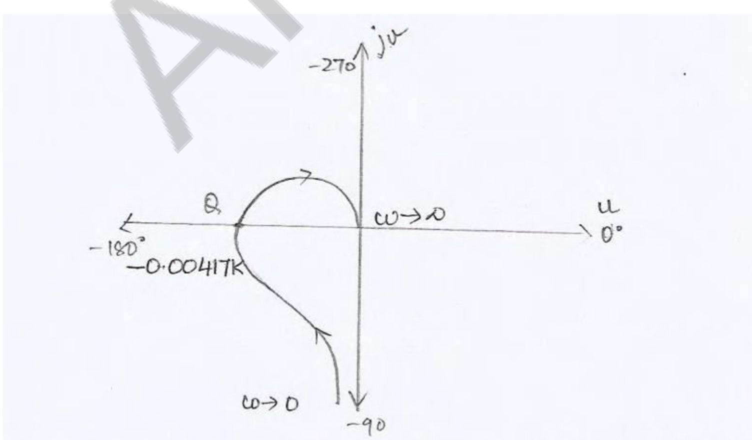
$$1-0.05\omega^2 = 0 \Rightarrow \omega = \frac{1}{\sqrt{0.05}} = 4.47$$

$$\Rightarrow \omega_{pc} = 4.47 \text{ rad/sec}$$

$$OQ = |G(j\omega)H(j\omega)| = \frac{-0.05K\times 0.6\times 4.47^2}{0.36\times 4.47^2 + 4.47^2(1-0.05\times 4.47^2)^2}$$

$$|OQ| = -0.00417K$$

Thus the mapping of section C1 in the (u-v) plane gives the following figure.



5. Mapping of section C2: the mapping of section C2 from S – plane to (u-v) plane is obtained by putting $s = \underset{R\to\infty}{Lt} Re^{j\theta}$ in $G(s)H(s)$ and varying θ from $\frac{+\pi}{2}$ to $\frac{-\pi}{2}$. Since $s \to Re^{j\theta}$ & $R \to \infty$ G(s)H(s) can be approximates as $(1+sT \approx sT)$

$$G(s)\,H(s) = \frac{K}{s(s+2)(s+10)}$$

$$= \frac{0.05K}{s(1+0.5s)(1+0.1s)} \approx \frac{0.05K}{3\times 0.5s\times 0.1s}$$

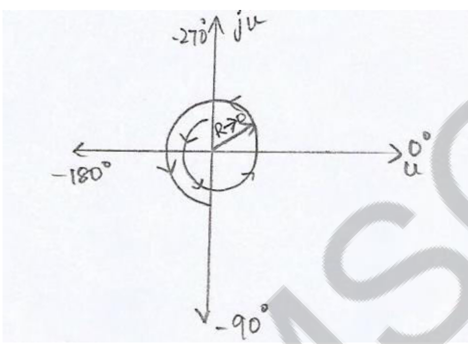
$$= \frac{K}{s^3}$$

$$G(s)H(s)\big|_{s\to \underset{R\to\infty}{Lt} Re^{j\theta}} = \frac{K}{\underset{R\to\infty}{Lt} R^3 e^{j3\theta}} = 0e^{-j3\theta}$$

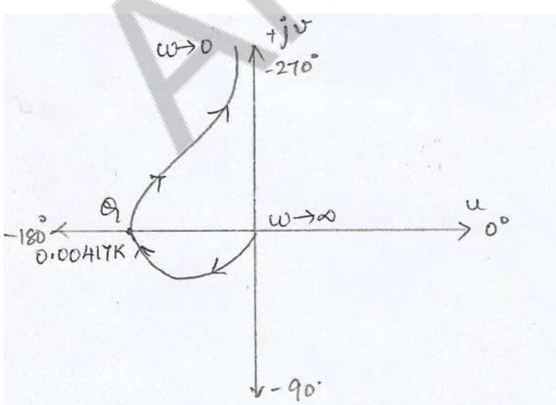
When $\theta = \frac{+\pi}{2}, G(s)H(s) = 0e^{-j3\pi/2}$

When $\theta = \frac{-\pi}{2}, G(s)H(s) = 0e^{j3\pi/2}$

In (u-v) plane, θ varies from $\frac{-3\pi}{2}$ to $\frac{+3\pi}{2}$ and magnitude of radius R reduces to 0.



6. Mapping of section C3: In section C3, $\omega \to \infty$ to 0 that is mapping of section C3 gives the inverse polar plot of G(jω)H(jω) as shown in fig.



7. Mapping of section C4: The rapping of section C4 from S – plane to (u-v) plane can be obtained by substituting $s = \underset{R\to 0}{Lt} Re^{j\theta}$ in $G(s)H(s)$ & varying θ from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$ since $s = Re^{j\theta}$ & $R \to 0$, $1+sT \approx 1$

$$G(s)H(s) = \frac{K}{s(s+2)(s+10)}$$

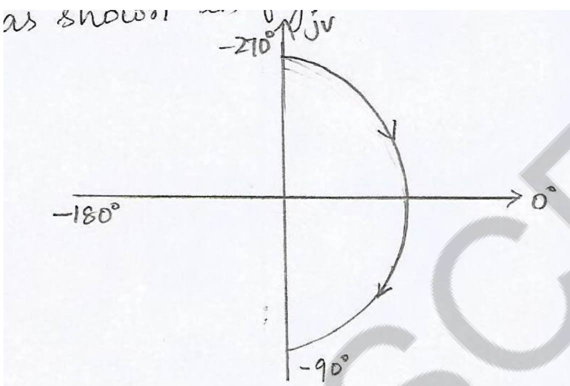
$$= \frac{0.05K}{s(1+0.5s)(1+0.1s)} = \frac{0.05K}{s}$$

$$G(s)H(s)\Big|_{s=\underset{R\to 0}{Lt} Re^{j\theta}} = \frac{0.05K}{\underset{R\to 0}{Lt} Re^{j\theta}} = \infty e^{-j\theta}$$

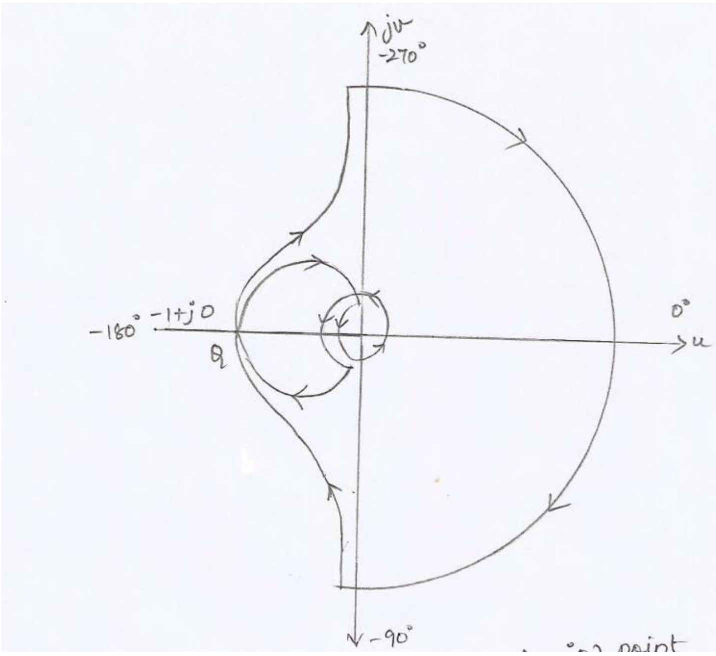
When $\theta = \frac{-\pi}{2}, G(s)H(s) = \theta e^{+j\pi/2}$

When $\theta = \frac{+\pi}{2}, G(s)H(s) = \theta e^{-j\pi/2}$

Section C4 is the S – plane mapped into a circle of infinite radius with arguments varying from $+90^0$ to -90^0 as shown in fig



8. Complete Nyquist plot: The complete Nyquist plot in (u-v) plane can be obtained by combining the mapping of individual section as shown in fig.



9. For absolute stability N = 0 i.e [-1+j0] point should be located on the left side of point Q.

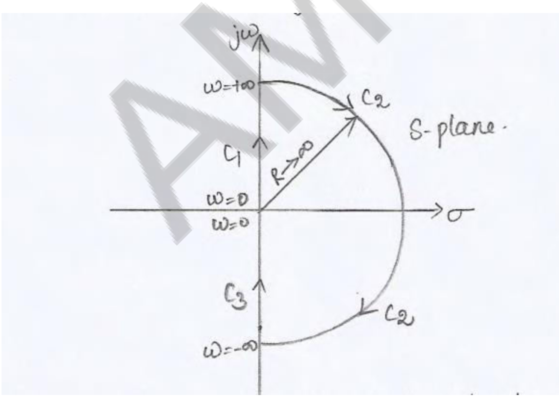
$$|OQ| < 1$$
$$|0.00417K| < 1$$
$$K < \frac{1}{0.00417K} < 240$$

The range of value of K for stability is

$$0 < K < 240$$

4. Describe about Nyquist contour & its various segments. [MAY/JUNE 2016]

In order to investigate the presence of poles of G(s)H(s) on the right half of s – plane a contour C is chosen such that it enclose the entire right half of s – plane. Such a contour is called Nyquist contour.



Nyquist contour is directed clockwise and comprises of three segments.

1. An infinite line segment C1 along the positive imaginary axis

2. An are C2 of infinite radius, enclosing entire right half of s – plane

3. An infinite line segment C3 along the negative imaginary axis.

Along C1, s = jω with ω varying from 0 to ∞

Along C2, $s = \underset{R\to\infty}{Lt} Re^{j\theta}$ with θ varying from $\frac{+\pi}{2}$ to $\frac{\pi}{2}$

Along C3, s = jω with ω varying from $-\infty$ to 0

5. State Nyquist stability criterion and explain the situations while examining the stability of linear control system. [NOV/DEC 2016]

Nyquist stability criterion can be stated as follows.

If the G(s)H(s) contour in the G(s)H(s) plane corresponding to Nyquist contour in the s – plane encircles the point -1+j0 in the number of right half of s plane poles of G(s)H(s), then the closed loop system is stable".

1. No encirclement of -1+j0 point:

This impiles that the system is stable if there are no poles of G(s)H(s) in the right half of s – plane. If there are poles on right half of s – plane then the system is unstable.

2. Anti clockwise encirclements of -1+j0 point:

In this case the system is stable if the number of anticlockwise encirclements is same as the number of poles of G(s)H(s) in the right half of s – plane. If the number of anticlockwise encirclements is not equal to number of poles on right half of s – plane, then the system is unstable.

3. Clockwise encirclements of the -1+j0 point:

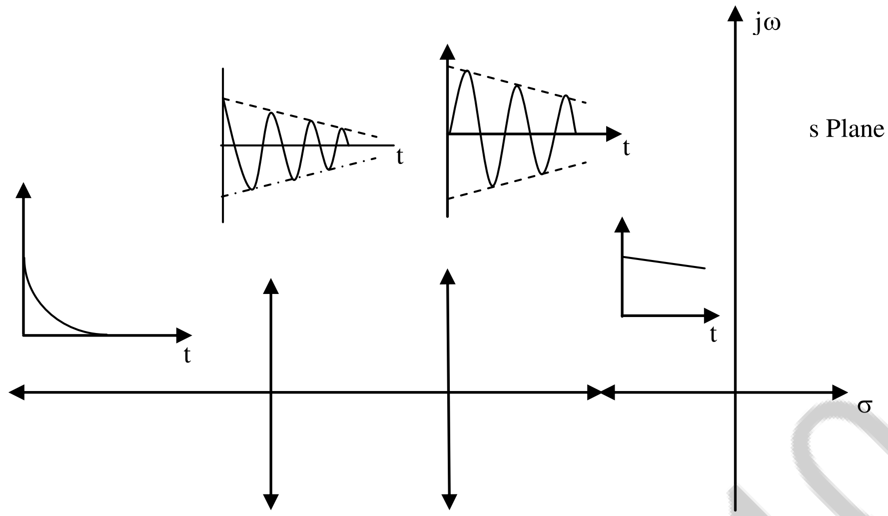
In this case, the system is always unstable. Also in this case if no poles of G(s)H(s) in the right of s – plane, then the number of clockwise encirclements is equal to number of poles of closed loop system on right half of s – plane.,

Relative Stability

1. Write detailed notes on relative stability with its roots of S – plane AU NOV/DEC 2015

The relative stability indicates the closeness of the system to stable region. It is an indication of the strength or degree of stability.

In time domain, the relative stability may be measured by relative settling times of each root (or) pairs of roots. The settling time is inversely proportional to the location of roots of characteristic equation. If the root is located far away the imaginary axis then the transient dies out faster and so the relative stability of the system will improve. The transient response and so the relative stability for various locations of roots in s – plane are shown in fig



INCLUDE THIS

1. Write down the procedure for designing lag compensator using bode plot

The steps to design the lag Compensator are

1. Determine K in uncompensated system to meet the steady state error requirement

2. Sketch the bode plot of the uncompensated system

3. Determine phase margin of the uncompensated system from the bode plot. If the phase margin does not satisfy the requirement then lag compensation is required.

4. Choose a suitable value for the phase margin of the compensated system.

Let γ_d = Desired phase margin of the compensated system

γ_n = Phase Margin of the compensated system

Now $\gamma_n = \gamma_d + \epsilon$

Where ∈ = additional phase lag to compensate for shift in gain crossover frequency.

Choose an initial value of $\epsilon = 5°$

5. Determine the new gain crossover frequency, ω_{gcn}. The new ω_{gcn} is the frequency corresponding to a phase margin of γ_n on the bode plot of uncompensated system.

Let, ϕ_{gcn} = Phase of the $G(j\omega)$ at new gain crossover frequency, ω_{gcn}

Now, $\gamma_n = 180° + \phi_{gcn}$; $\phi_{gcn} = \gamma_n - 180°$

The new gain crossover frequency, ω_{gcn} is given by the frequency at which the phase of $G(j\omega)$ is ϕ_{gcn}

6. Determine the parameter, β of the compensator. The value of β is given by the magnitude of $G(j\omega)$ at new gain crossover frequency, ω_{gcn}. Find the db gain (A_{gcn}) at new gain crossover frequency, ω_{gcn}

Now, $A_{gcn} = 20 \log \beta$ or $\frac{A_{gcn}}{20} = \log \beta$, $\therefore \beta = 10A_{gcn} / 20$

7. Determine the transfer function of the lag compensator

Place the zero of the compensator arbitrarily at 1/10th of the new gain crossover frequency, ω_{gcn}

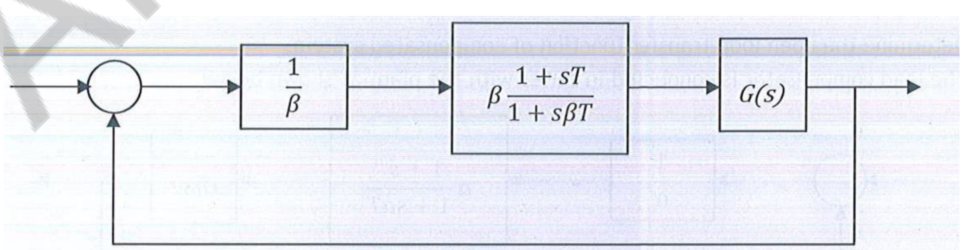
$\therefore$ Zero of the lag compensator, Zc = $\frac{1}{T} = \frac{\omega_{gcn}}{10}$

Now, T = $\frac{10}{\omega_{gcn}}$

Pole of the lag compensator, p_c = $\frac{1}{\beta_T}$

Transfer function of lag compensator $G_c(S)$ = $\frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = \beta\left(\frac{1 + sT}{1 + s\beta T}\right)$

8. Determine the open loop transfer function of compensated system. The lag compensator is connected in series with the plant as shown below



When the lag compensator is inserted in series with plant, the open loop gain of the system is amplified by the factor β. If the gain produced is not required then attenuator with gain $\frac{1}{\beta}$ can be introduced in series with the lag compensator to nullify the gain produced lag compensator. The open loop transfer function of the compensated system,

$$G_0(s) = \frac{1}{\beta} \cdot G_c(s) G(s) = \frac{1}{\beta} \beta \frac{(1+sT)}{(1+s\beta T)} \cdot G(s) = \frac{(1+sT)}{(1+s\beta T)} \cdot G(s)$$

9. Determine the actual phase margin of compensated system. Calculate actual phase angle of the compensated system using the compensated transfer function at new gain crossover frequency, ω_{gcn}

Let, ϕ_{gco} = phase of G_0 ($j\omega$) at $\omega = \omega_{gcn}$

Actual phase margin of the compensated system, $\gamma_0 = 180° + \phi_{gc0}$;

If the actual phase margin satisfies the given specification then the design is accepted.

Otherwise repeat the procedure from step 4 to 9 by taking ∈ as 5° more than the previous design.

2. Write down the procedure for designing lead compensator using bode plot

The steps to design the Lead Compensator are

1. Determine K in uncompensated system to meet the steady state error requirement

2. Sketch the bode plot of the uncompensated system

3. Determine phase margin of the uncompensated system from the bode plot.

4. Determine the amount of phase angle to be contributed by lead network by using the formula

$$\phi_m = \gamma_d - \gamma + \in$$

Where, ϕ_m = maximum phase lead angle of the lead compensator

γ_d = desired phase margin

γ = Phase margin of the uncompensated system

$\in$ = Additional phase lead to compensate for shift in gain crossover frequency. Choose an initial choice of $\in$ as 5°

5. Determine the transfer function of the lead compensator.

Calculate a using the equation $\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m}$

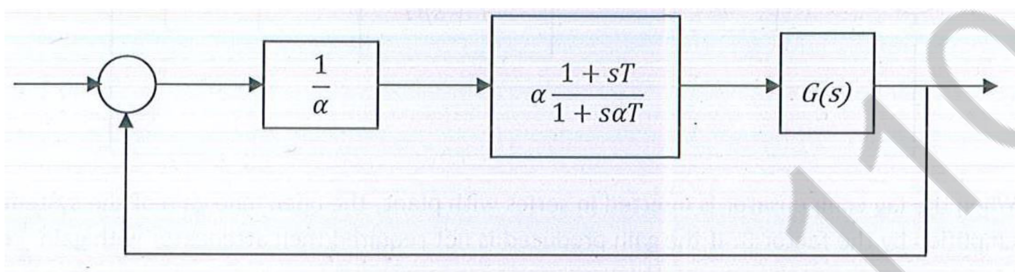
From the bode plot, determine the frequency at which the magnitude of G($j\omega$) is – 20log $\frac{1}{\sqrt{\alpha}}$ db. This frequency is ω_m.

Calculate T from the relation, $\omega_m = \frac{1}{T\sqrt{\alpha}}$ $\therefore T = \frac{1}{\omega_m \sqrt{\alpha}}$

Transfer function of lead compensator $G_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = \frac{\alpha(1+sT)}{(1+\alpha sT)}$

6. Determine the open loop transfer function of compensated system.

The lead compensator is connected in series with the plant as shown below



When the lead compensator is inserted in series with plant, the open loop gain of the system is attenuated by the factor α, so an amplifier of gain $\frac{1}{\alpha}$ can be introduced in series with the lead compensator to nullify the gain produced lead compensator.

The open loop transfer function of the compensated system,

$$G_0(s) = \frac{1}{\alpha} \cdot G_c(s) G(s) = \frac{1}{\alpha} \alpha \frac{(1+sT)}{(1+s\alpha T)} \cdot G(s) = \frac{(1+sT)}{(1+s\alpha T)} \cdot G(s)$$

7. Verify the design.

Finally the bode plot of the compensated system is drawn and verify whether it satisfies the given specifications. If the phase margin of the compensated system is less than the required phase margin then repeat step 4 to 7 by taking ∈ as 5° more than the previous design.

8. Write down the procedure for designing lag – lead compensator using bode plot

The steps to design the Lag – Lead compensator are

1. Determine K in uncompensated system to meet the steady state error requirement

2. Sketch the bode plot of the uncompensated system

3. Determine phase margin of the uncompensated system from the bode plot. If the phase margin does not satisfy the requirement then lag compensation is required.

4. Choose a suitable value for the phase margin of the compensated system.

Let γ_d = Desired phase margin of the compensated system

γ_n = Phase Margin of the compensated system.

Now, $\gamma_n = \gamma_d + \in$

Where $\in$ = additional phase lag to compensate for shift in gain crossover frequency.

5. Determine the new gain crossover frequency, ω_{gcn}. The new ω_{gcn} is the frequency corresponding to a phase margin of γ_n on the bode plot of uncompensated system.

Let, ϕ_{gcn} = Phase of the $G(j\omega)$ at new gain crossover frequency, ω_{gcn}

Now, $\gamma_n = 180° + \phi_{gcn}$; $\phi_{gcn} = \gamma_n - 180°$

The new gain crossover frequency, ω_{gcn} is given by the frequency at which the phase of $G(j\omega)$ is ϕ_{gcn}

Choose the gain crossover frequency of the lag compensator, ω_{gct} greater than ω_{gcn}

6. Calculate β of the lag compensator.

Let $A_{gcl} = |G(j\omega)|$ in db at $\omega = \omega_{gcl}$

From the bode plot find A_{gcl}

Now, $Agcl = 20 \log \beta$ or $\frac{A_{gcl}}{20} = \log \beta$, $\therefore \beta = 10\ A_{gcl} / 20$

7. Determine the transfer function of the lag section

Place the zero of the compensator arbitrarily at 1 / 10th of the new gain crossover frequency , ω_{gcl}

$\therefore$ Zero of the lag compensator, $Z_{c1} = \frac{1}{T_l} = \frac{\omega_{gcl}}{10}$

Now, $T_1 = \frac{10}{\omega_{gcl}}$

Pole of the lag compensator, $p_{c1} = \frac{1}{\beta T_l}$

Transfer function of lag section $G_1(s) = \dfrac{s+\frac{1}{T_1}}{s+\frac{1}{\beta T_1}} = \beta\left(\dfrac{1+sT_1}{1+s\beta T_1}\right)$

8. Determine the transfer function of the lead compensator.

Calculated α using the equation $\alpha = \dfrac{1}{\beta}$

From the bode plot, determine the frequency ω_m. At which the db gain is $-20\log\dfrac{1}{\sqrt{\alpha}}$ db

Calculate T_2 from the relation, $T_2 = \dfrac{1}{\omega_m\sqrt{\alpha}}$ $\quad\therefore T = \dfrac{1}{\omega_m\sqrt{\alpha}}$

Transfer function of lead section $G_2(s) = \dfrac{s+\frac{1}{T_2}}{s+\frac{1}{\alpha T_2}} = \dfrac{\alpha(1+sT_2)}{(1+\alpha sT_2)}$

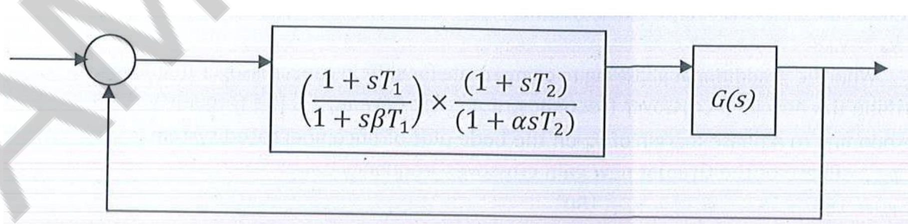
9. Determine the transfer function of lag – lead compensator.

Transfer function of lag – lead compensator $G_C(s) = G_1(s) \times G_2(s) = \beta\left(\dfrac{1+sT_1}{1+s\beta T_1}\right)\times\dfrac{\alpha(1+sT_2)}{(1+\alpha sT_2)}$

Since $= \dfrac{1}{\beta}$, $\quad G_c(s) = \left(\dfrac{1+sT_1}{1+s\beta T_1}\right)\times\dfrac{(1+sT_2)}{(1+\alpha sT_2)}$

10. Determine the open loop transfer function of compensation system.

The lag-lead compensator is connected in series with G(s) as shown below



The open loop transfer function of compensated system

$$G_0(s) = \left(\frac{1+sT_1}{1+s\beta T_1}\right)\times\frac{(1+sT_2)}{(1+\alpha sT_2)}\times G(s)$$

11. Draw the bode plot of compensated system and verify whether the specifications are satisfied or not. If the specifications are not satisfied then choose another choice of a such that $\alpha < \frac{1}{\beta}$ and repeat the steps 8 to 11.

3. Design a phase lag compensator for the given transfer function G(s) = $\frac{K}{s(s+1)(s+4)}$ with the unity feedback has specifications a) Phase Margin is 40° b) Steady state error $e_{ss} \leq 0.2$

Solution

Step 1: Find the value of K for the uncompensated system.

G(s) = $\frac{K}{s(s+1)(s+4)}$; H(s) = 1

$$K_v = \underset{s\to 0}{\text{Lt}}\, s\cdot G(s)\cdot H(s) = \underset{s\to 0}{\text{Lt}}\, s\cdot\frac{K}{s(s+1)(s+4)} = \frac{K}{4}$$

The steady state error e_{ss} = 0.2

$\therefore$ Kv = $\frac{1}{e_{ss}} = \frac{1}{0.2} = 5$

Also K_v = $\frac{k}{4} \Rightarrow$ K = 4 K_v = K = 4 × 5 = 20

Step 2 Construct the Bode plot for uncompensated system & find the value of phase margin (γ)

$$G(s) = \frac{20}{s(s+1)(s+4)} = \frac{5}{s(1+s)(1+0.25s)}$$

Put s = $j\omega$

G($j\omega$) = $\frac{5}{j\omega(1+j\omega)(1+j0.25\omega)}$

Corner frequencies are ω_{c1} = $\frac{1}{1}$ = 1 rad/sec

ω_{c2} = $\frac{1}{0.25}$ = 4 rad/sec

Term	Corner frequency rad/sec	Slope dB/dec	Change in slope (dB/dec)

$\frac{5}{j\omega}$	-	-20	
$\frac{1}{1+j\omega}$	$\omega_{c1} = \frac{1}{1} = 1$	-20	-20 -20 = -40
$\frac{1}{1+j0.25\omega}$	$\omega_{c2} = \frac{1}{0.25} = 4$	-20	-40 -20 = -60

Choose ω_l = 0.1 rad/sec & ω_h = 10 rad/sec

Calculation of Gain A.

(i) when $\omega = \omega_l = 0.1$, $A = 20 \log \left|\frac{5}{j\omega}\right| = 20 \log \left|\frac{5}{0.1}\right| = 33.97 \text{ dB} \approx 34\text{dB}$

(ii) when $\omega = \omega_{c1} = 1$, $A = 20 \log \left|\frac{5}{j\omega}\right| = 20 \log \left|\frac{5}{1}\right| = 33.97 \text{ dB} \approx 14\text{dB}$

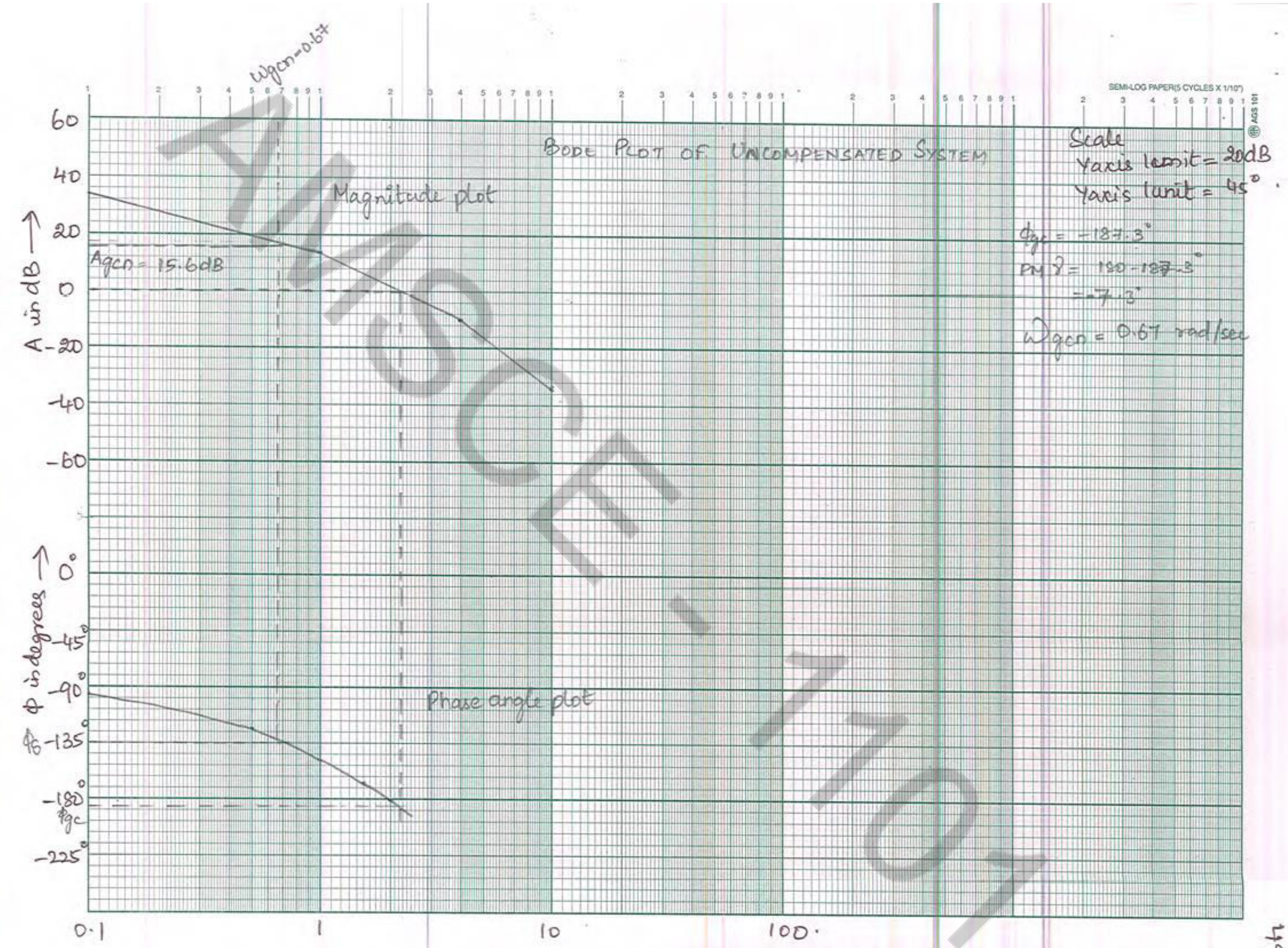
(iii) when $\omega = \omega_{c2} = 4$, A =[change in slope from ω_{c1} to ω_{c2}] $\times \log \frac{\omega_{c2}}{\omega_{c1}}$ + A at $\omega = \omega_{c1}$

$= -40 \log \left(\frac{4}{1}\right) + 13.9 = -10.102\text{dB} \approx -10\text{dB}$

(iv) when $\omega = \omega_h = 10$, A = [change in slope from ω_{c2} to ω_h] $\times \log \frac{\omega_h}{\omega_{c2}}$ + A at $\omega = \omega_{c2}$

$= -60 \log \left(\frac{10}{4}\right) - 10.102 = -33.978\text{dB} \approx 34\text{dB}$

ω rad/sec	A dB
0.1	34
1	14
4	-10
10	-34


BODE PLOT OF UNCOMPENSATED SYSTEM
Scale
Y axis 1 unit = 20dB
Y axis 1 unit = 45°
φgc = -187.3°
PM γ = 180 - 187.3°
= -7.3°
ωgcn = 0.67 rad/sec
ωgcn = 0.67
Magnitude plot
Agcn = 15.6dB
Phase angle plot
A in dB
φ in degrees
60
40
20
0
-20
-40
-60
0°
-45°
-90°
-135°
-180°
φgc
-225°
0.1
1
10
100
SEMI-LOG PAPER (5 CYCLES X 1/10")

Phase Angle Plot

$$\phi = \angle G(j\omega) = -90° - \tan^{-1}\omega - \tan^{-1}0.25\omega$$

ω rad/sec	0.1	0.5	1	1.5	2
ϕ Degrees	-97.14°	-123.7°	-149°	-167°	-180°

From graph, Phase Margin is -7.3°

Step 3 Select the suitable Phase Margin for compensated system

Desired Phase Margin γ_d = 40°

The small value of correction ∈ = 5°

$\gamma_n = \gamma_d + \in = 40° + 5° = 45°$

Step 4 Find the new gain crossover frequency (ω_{gcn}) corresponding to new phase margin

$\phi_{gcn} = \gamma_n - 180°$

ϕ_{gcn} =45° - 180° = -135° & corresponding

ω_{gcn} = 0.67 rad/sec (graph)

Step 5 Obtain β corresponding to the magnitude G(jω) at ω_{gcn}

A_{gcn} = 15.6 dB corresponding to ω_{gcn} = 0.67 rad/sec (from graph)

$A_{gcn} = 20 \log \beta \Rightarrow \beta = 10^{Agcn/20}$

$\Rightarrow \beta = 10^{15.6/20} = 6.025$

Step 6: Obtain the transfer function of lag compensator

Zero of lag compensator Zc = $\frac{1}{T} = \frac{\omega_{gcn}}{8}$

$\therefore$ T = $\frac{8}{\omega_{gcn}} = \frac{8}{0.67} = 11.94$

Poles of lag compensator $P_c = \frac{1}{\beta T}$

The compensated transfer function,

$$Gc(s) = \frac{(1+sT)}{(1+s\beta T)} = \frac{(1+11.94s)}{(1+71.94s)}$$

Step 7 Determine the open loop transfer function

$G_0(s) = Gc(s) \cdot G(s)$

$$= \frac{(1+11.94s)}{(1+71.94s)} \times \frac{20}{s(s+1)(s+4)}$$

Step 8: Determine the phase angle of the compensated system at new frequency (ω_{gcn})

$\phi_{gcn} = \angle G(j\omega)\ G(j\omega)\ |\omega = \omega_{gcn}$

$\phi_{gcn} = -90° - \tan^{-1}\omega - \tan^{-1} 25\,\omega + \tan^{-1}(11.94\omega) - \tan^{-1}(71.94\omega)$

Put $\omega = \omega_{gcn} = 0.67$ rad/sec

$\phi_{gcn} = -90 - \tan^{-1}(0.67) - \tan^{-1}(0.25 \times 0.67) + \tan^{-1}(11.94 \times 0.67) - \tan^{-1}(71.94 \times 0.67)$

$= -139.26°$

The phase margin of compensated system

$\gamma_{new} = 180° - 139.26° = 40.74°$

The given phase margin is 40° & the obtained value of phase margin is 40.74°. Hence the design is acceptable.

4) Design a phase lead compensator for the given transfer function G(s) = $\frac{K}{s(s+2)}$ with a unity feedback system has the specifications a) Phase Margin ≥ 55° b) The steady state error for unit ramp input ≤ 0.40

Solution

Step 1 Find the value of K from the steady state error e_{ss} (or) velocity error constant Kv

For Ramp input

$e_{ss} = \frac{1}{Kv} \leq 0.4 \quad Kv = \frac{1}{0.40} = 2.5$

Given G(s) = $\frac{K}{s(s+1)}$,   H(s) = 1

Kv = $\underset{s\to 0}{Lt}\, s\cdot G(s)\cdot H(s) = \underset{s\to 0}{Lt}\, s\cdot \frac{k}{s(s+2)} = \frac{k}{2}$

K = 2Kv = 2 × 2.5 = 5

Step 2 Draw the bode plot for uncompensated system and find the value of Phase Margin (γ)

The uncompensated transfer function G(s) = $\frac{K}{s(s+2)}$

= $\frac{5}{s(s+2)}$

Put s = jω

$\therefore$ G(jω) = $\frac{5}{j\omega(j\omega+2)} = \frac{2.5}{j\omega(1+j0.5\omega)}$

The corner frequencies $\omega_{c1} = \frac{1}{0.5} = 2\text{rad/sec}$

Term	Corner Frequency rad/sec	Slope dB/dec	Change in slop dB/dec
$\frac{2.5}{j\omega}$		-20	
$\frac{1}{1+j0.5\omega}$	$\omega_{c1} = \frac{1}{0.5} = 2$	-20	-20 -20 = -40

Assume ω_l = 0.1 rad/sec & ω_h = 10 rad/sec

Calculation of Gain A

When $\omega = \omega_l$ = 0.1,   A = 20 log $\left|\frac{2.5}{j\omega}\right| = 20\log\left|\frac{2.5}{1}\right|$

= 27.96 dB

$\cong$ 28dB

When $\omega = \omega_{c1} = 2$, A = 20 log $\left|\frac{2.5}{j\omega}\right| = 20\log\left|\frac{2.5}{2}\right|$

= 1.94 dB

$\cong$ 2dB

When $\omega = \omega_h = 10$, A = [change in slope from $\omega c1$ to ωh] $\times \log\left(\frac{\omega_h}{\omega_{c1}}\right) +$

A at $\omega = \omega_{c1}$

$$= -40\log\left(\frac{10}{2}\right) + 1.94$$

= -26.09 dB

ω rad/sec	A dB
0.1	28
2	2
10	-26

Phase angle plot

Phase angle $\phi = \angle G(j\omega) = -90° - \tan^{-1} 0.5\omega$

ω rad/sec	0.1	0.5	1	5	8	20	40
ϕ Degrees	-92.86°	-104.03°	-116.56°	-158.19°	-165.96°	-174.28°	-177.13°

Phase Margin $\gamma = 180° + \phi_{gc} = 180° - 132° = 48°$

Step 3: maximum phase lead angle of the lead compensator ϕ_m is given by

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = \frac{1-\sin 12°}{1+\sin 12°} = \frac{0.792}{1.2079} = 0.6557 \in$$

The desired phase margin $\gamma_d \geq 55°$

Uncompensated phase margin $\gamma_d = 48°$ (from graph)

Small correction value = $5°$

$$\phi_m = 55° - 48° + 5° = 12°$$

Step-4 Obtain the transfer function of lead compensator

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = \frac{1-\sin 12^\circ}{1+\sin 12^\circ} = \frac{0.792}{1.2079} = 0.6557$$

The value of magnitude dB corresponding to

$$\omega_m = -20\log\frac{1}{\sqrt{\alpha}}$$

$$= -20\log\frac{1}{\sqrt{0.6557}} = -1.833\text{dB}$$

From uncompensated bode plot for dB value -1.833, corresponding value of frequency ω_m is 2.2 rad/sec

$$T = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{2.2\sqrt{0.6557}} = 0.5613$$

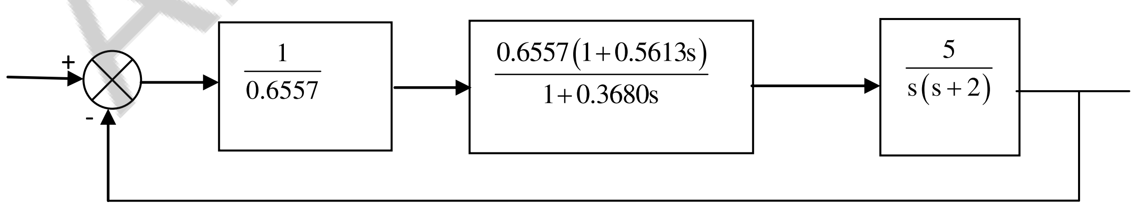
The transfer function of lead compensator

$$G_c(s) = \frac{\left(s+\frac{1}{T}\right)}{\left(s+\frac{1}{\alpha T}\right)} = \frac{\alpha(1+sT)}{(1+s\alpha T)} = \frac{0.6557(1+0.5613s)}{1+(0.6557\times 0.5613)s}$$

$$= \frac{0.6557(1+0.5613s)}{1+0.3680s}$$

Step 5 Open loop transfer function of compensated system

The block diagram of compensated system



The open loop transfer function of lead compensator = $G_0(s) = G_c(s) \cdot G(s)$

$$= \frac{1}{0.6557} \times \frac{0.6557(1+0.5613s)}{1+0.3680s} \times \frac{5}{s(s+2)}$$

$$= \frac{5(1+0.5613s)}{s(s+2)(1+0.3680s)}$$

$$= \frac{2.5(1+0.5613s)}{s(1+0.5s)(1+0.3680s)}$$

Step 6 Draw the Bode plot for the compensated system

$$G_c(s) = \frac{2.5(1+0.5613s)}{s(1+0.5s)(1+0.3680s)}$$

Put s = jω

$$G(j\omega) = \frac{2.5(1+j0.5613\omega)}{j\omega(1+j0.5\omega)(1+j0.3680\omega)}$$

The corner frequencies are

$$\omega_{c1} = \frac{1}{0.5613} = 1.78\,\text{rad/sec} \qquad \omega_{c2} = \frac{1}{0.5} = 2\,\text{rad/sec}$$

$$\omega_{c3} = \frac{1}{0.3680} = 2.7\,\text{rad/sec}$$

Magnitude plot

Term	Corner frequency rad/sec	Slope dB/dec	Change in slope dB/dec
$\frac{2.5}{j\omega}$	-	-20	
1+j0.5613ω	$\omega_{c1} = \frac{1}{0.5613} = 1.78$	20	-20+20=0
$\frac{1}{1+j0.5\omega}$	$\omega_{c2} = \frac{1}{0.5} = 2$	-20	0 - 20 = -20
$\frac{1}{1+j0.368\omega}$	$\omega_{c3} = \frac{1}{0.368} = 2.717$	-20	-20 - 20 = -40

Assume ω_l = 0.1 rad/sec & ω_h =100 rad/sec

Calculation of gain A,

When $\omega = \omega l = 0.1$, $\quad A = 20\log\left|\frac{2.5}{j\omega}\right| = 20\log\left|\frac{2.5}{0.1}\right|$

= 27.95 dB

= 28 dB

When $\omega = \omega c1 = 1.78$, $\quad A = 20\log\left|\frac{2.5}{j\omega}\right| = 20\log\left|\frac{2.5}{1.78}\right|$

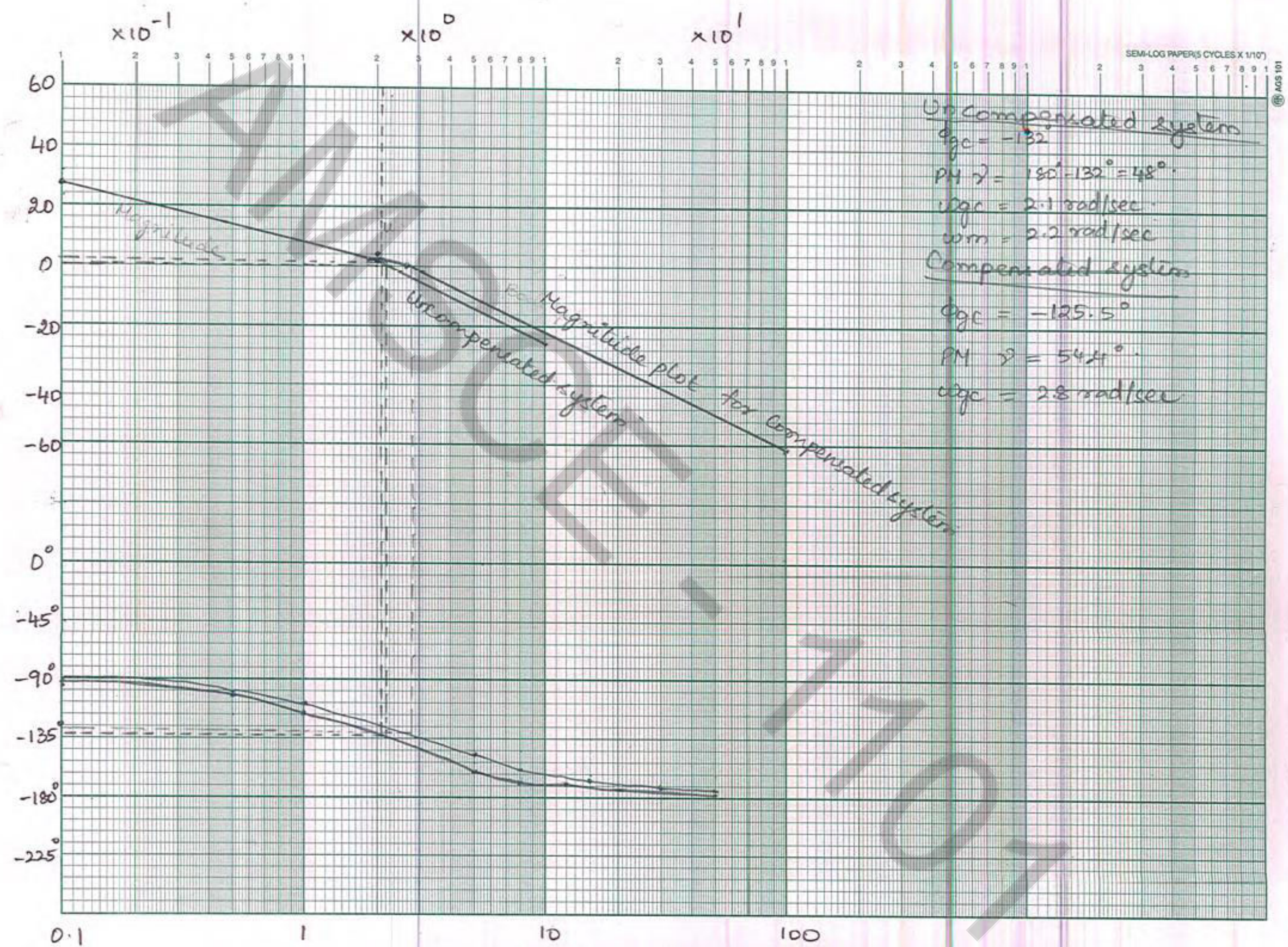
= 2.95 dB

 3dB

When $\omega = \omega_{c2} = 2$, A = change in slope from ω_{c2} to $\omega_{c1} \times \log\left(\frac{\omega_{c_2}}{\omega_{c1}}\right)$

+ A at $\omega = \omega_{c1}$

$$= 0 \times \log\left(\frac{2}{1.78}\right) + 3 = 3\text{dB}$$



x10^-1
x10^0
x10^1
SEMI-LOG PAPER(5 CYCLES X 1/10")
AGS 101
60
40
20
0
-20
-40
-60
0°
-45°
-90°
-135°
-180°
-225°
0.1
1
10
100
Magnitude plot for compensated system
Uncompensated system
Uncompensated system
ϕgc = -132
PM γ = 180°-132° = 48°.
ωgc = 2.1 rad/sec.
ωm = 2.2 rad/sec
Compensated system
ϕgc = -125.5°
PM γ = 54.4°.
ωgc = 2.8 rad/sec

When $\omega = \omega_{c3} = 2.717$, A = (change in slope from ω_{c2} to ω_{c3}) $\times \log\frac{\omega_{c3}}{\omega_{c2}}$

+ A at $\omega = \omega_{c2}$

$= -20 \times \log\left(\frac{2.717}{2}\right) + 3$

$= 0.3388 \approx 0.34\text{dB}$

When $\omega = \omega_h = 100$, A = (change in slope from ω_{c3} to ω_h) $\times \log\frac{\omega_h}{\omega_{c3}}$

+ A at $\omega = \omega_{c3}$

$$= -40 \times \log\left(\frac{100}{2.717}\right) + 0.34$$

$= -62.296$

$\approx -62.3\text{dB}$

ω rad/sec	A dB
0.1	28
1.78	3
2	3
2.717	0.34
100	-62.3

Phase angle plot

$$\phi = \angle G(j\omega) = -90 + \tan^{-1}(0.5613\omega) - \tan^{-1}(0.5\omega) - \tan^{-1}(0.368\omega)$$

ω rad/sec	0.1	0.5	1	5	8	15	30	50
ϕ Degrees	-91.76°	-98.79°	-107.5°	-149.3°	-159.3°	-168.91°	-174.4°	-176.6°

Step 7

Phase Margin of the compensated system

$\gamma = 180° + \phi_{gcn}$

$= 180° + (-125.5)° = 54.5° \approx 55°$

Phase Margin of the compensated system is matching with desired phase margin

Therefore the design is acceptable.

5) Consider the unity feedback system, whose open loop transfer function is $G(s)=\dfrac{K}{s(s+3)(s+6)}$**.**

Design a lag-lead compensator to meet the following specifications. i) Velocity error constant K_v = 80

ii) Phase Margin $\gamma \geq 35°$

Solution

Step 1: Determine K

For unity feedback system

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, S\cdot G(s)$

$$\therefore \underset{s\to 0}{Lt}\, s\cdot G(s) = \underset{s\to 0}{Lt}\, S\cdot\frac{K}{s(s+2)(s+6)} = 80$$

$$\Rightarrow \frac{K}{3\times 6} = 80$$

$$\Rightarrow K = 80\times 3\times 6 = 1440$$

$$\therefore \qquad G(s) = \frac{1440}{s(s+3)(s+6)}$$

$$= \frac{1440}{s\times 3(1+0.33s)\times 6(1+0.167s)}$$

$$= \frac{80}{s(1+0.33s)(1+0.167s)}$$

Step 2: Bode plot of uncompensated system.

In G(s), put s = jω

$$G(j\omega) = \frac{80}{j\omega(1+j0.33\omega)(1+j0.167\omega)}$$

Magnitude Plot

The corner frequencies are

$$\omega_{c1} = \frac{1}{0.33} = 3\text{rad/sec}$$

$$\omega_{c2} = \frac{1}{0.167} = 6\text{rad/sec}$$

Term	Corner frequency rad/sec	Slope dB/dec	Change in slope dB/dec
$\frac{80}{j\omega}$	-	- 20	-
$\frac{1}{1+j0.33\omega}$	$\omega_{c1} = \frac{1}{0.33} = 3$	- 20	- 20 - 20 = - 40
$\frac{1}{1+j0.167\omega}$	$\omega_{c2} = \frac{1}{0.167} = 6$	- 20	- 40 - 20 = - 60

Choose ω_l = 0.5 rad/sec

ω_h = 20 rad/sec

At $\omega = \omega_l$, $A = 20\log\left|\frac{80}{j\omega}\right| = 20\log\frac{80}{0.5} = 44\text{dB}$

At $\omega = \omega_{c1}$, $A = 20\log\left|\frac{80}{j\omega}\right| = 20\log\frac{80}{3} = 28.5\text{dB} \approx 28\text{dB}$

At $\omega = \omega_{c2}$, $A = \left(\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right) + A \text{ at } \omega = \omega_{c1}$

$$= -40 \times \log\frac{6}{3} + 28 = 16\text{dB}$$

At $\omega = \omega_h$, $A = \left(\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right) + A \text{ at } \omega = \omega_{c2}$

$$= -60 \times \log\frac{20}{6} + 16 = -15\text{dB}$$

ω rad/sec	A dB
0.5	44
3	28
6	16
20	-15

Phase angle plot

$$\phi = \angle G(j\omega) = -90^\circ - \tan^{-1} 0.33\omega - \tan^{-1} 0.167\omega$$

ω rad/sec	0.5	1.0	3.0	6	10	20
$\angle G(j\omega)$ Deg	-104	-118	-161	-198	-222	-244.7 ≈ -244

Step 3 Find the phase margin of uncompensated system.

Let ϕ_{gc} = Phase of $G(j\omega)$ at gain cross over frequency

γ = Phase margin of uncompensated system

From bode plot of uncompensated system,

$\phi_{gc} = -226^\circ$

$\gamma = 180^\circ + \phi_{gc} = 180^\circ - 226^\circ = -46^\circ$

Step 4 Choose a new phase margin

The desired phase margin $\gamma_d = 35^\circ$

The phase margin of compensated system

$\gamma_n = \gamma_d + \in$ Let $\in$ be 5°

$\therefore \gamma_n = 35^\circ + 5^\circ = 40^\circ$

Step 5: Determine new gain crossover frequency

Let ω_{gcn} = New gain cross over frequency

ϕ_{gcn} = Phase of $G(j\omega)$ at ω_{gcn}

Now $\gamma_n = 180^\circ + \phi_{gcn}$

$\therefore \phi_{gcn} = \gamma_n - 180^\circ = 40^\circ - 180^\circ = -140^\circ$

From the bode plot, we found that the frequency corresponding to a phase of -140° is 1.8 rad/sec.

Let ω_{gcl} = Gain cross over frequency of lag compensator

Choose ω_{gcl} such that, $\omega_{gcl} > \omega_{gcn}$

Let ω_{gcl} = 4 rad/sec

Step 6: Calculate β of lag compensator

From the bode plot, we found that the dB magnitude at ω_{gcl} is 23 dB.

$\therefore$ $|G(j\omega)|$ in dB at $(\omega = \omega_{gcl}) = A_{gcl} = 23$ dB

Also $A_{gcl} = 20 \log \beta$; $\therefore$ $\beta = A\ 10^{Agcl/20} = 10^{23/20} = 14$

Step 7: Determine the transfer function of lag section

The zero of the lag compensator is placed at a frequency one-tenth of ω_{gcl}

$\therefore$ Zero of lag compensator, $Z_{c1} = \frac{-1}{T_1} = \frac{\omega_{gcl}}{10}$

$$\therefore T_1 = \frac{10}{\omega_{gcl}} = \frac{10}{4} = 2.5$$

Pole of lag compensator, $P_{c1} = \frac{1}{\beta \cdot T_1} = \frac{1}{14 \times 2.5} = \frac{1}{35}$

Transfer function of lag section

$$G_1(s) = \frac{\beta(1+sT_1)}{(1+s\beta T_1)} = \frac{14(1+2.5s)}{(1+35s)}$$

Step 8: Determine the transfer function of lead section

Let $\alpha = \frac{1}{\beta}$ $\therefore$ $\alpha = \frac{1}{14} = 0.07$

The dB gain (Magnitude) corresponding to $\omega_m = -20\log\frac{1}{\sqrt{\alpha}}$

$$= -20\log\frac{1}{\sqrt{0.07}} = -11.5\text{dB} \approx -12\text{dB}$$

From the bode plot of uncompensated system, the frequency ω_m corresponding to a dB gain of -12dB is found to be 17 rad/sec.

$$\therefore \omega_m = 17\text{rad/sec}$$

$$T_2 = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{17\sqrt{0.07}} = 0.22$$

Transfer function of lead section G_2 (s) $= \frac{\alpha(1+sT_2)}{(1+s\alpha T_2)}$

$$= \frac{0.07(1+0.22s)}{(1+0.0154s)}$$

Step 9: Determine the transfer function of lag-lead compensator

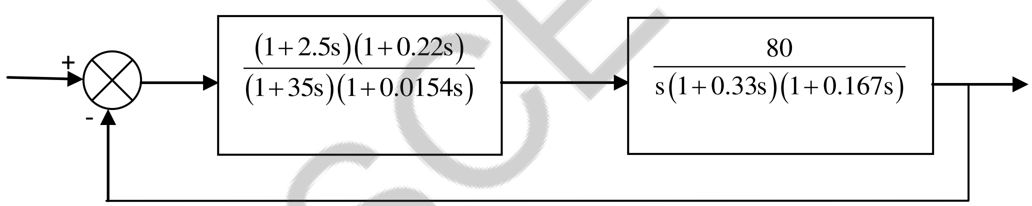
Transfer function of lag-lead compensator $G_c(s) = G_1(s) \times G_2(s)$

$$G_c(s) = \frac{14(1+2.5s)}{(1+35s)} \times 0.07 \frac{(1+0.22s)}{(1+0.0154s)}$$

$$= \frac{(1+2.5s)(1+0.22s)}{(1+35s)(1+0.0154s)}$$

Step 10: Determine open loop transfer function of compensated system.

The lag-lead compensator is connected in series with G(s) as shown in fig.



$$\left.\begin{matrix}\text{Open loop transfer function} \\ \text{of compensated system}\end{matrix}\right\} G_0(s) = \frac{80(1+2.5s)(1+0.22s)}{s(1+35s)(1+0.0154s)(1+0.33s)(1+0.167s)}$$

Step 11 Bode plot of compensated system

Put s = jω in G_0(s)

$$\therefore G_0(j\omega) = \frac{80(1+j2.5\omega)(1+j0.22\omega)}{j\omega(1+j35\omega)(1+j0.0154\omega)(1+j0.33\omega)(1+j0.167\omega)}$$

Magnitude Plot

Corner frequencies:

$\omega_{c1} = \frac{1}{35} = 0.03\text{rad/sec}$ $\omega_{c2} = \frac{1}{2.5} = 0.4\text{rad/sec}$

$\omega_{c3} = \frac{1}{0.33} = 3\text{rad/sec}$ $\omega_{c4} = \frac{1}{0.22} = 4.5\text{rad/sec}$

$\omega_{c5} = \frac{1}{0.167} = 6\text{rad/sec}$ $\omega_{c6} = \frac{1}{0.0154} = 65\text{rad/sec}$

Term	Corner frequency rad/sec	Slope dB/dec	Change in slope dB/dec
$\frac{80}{j\omega}$	-	-20	
$\frac{1}{1+j35\omega}$	$\omega_{c1} = \frac{1}{35} = 0.03$	-20	-20 -20 = -40
1 + j2.5 ω	$\omega_{c2} = \frac{1}{2.5} = 0.4$	+20	-40 + 20 = -20
$\frac{1}{1+j0.33\omega}$	$\omega_{c3} = \frac{1}{0.33} = 3$	-20	-20 -20 = -40
1 + j0.22ω	$\omega_{c4} = \frac{1}{0.22} = 4.5$	+20	-40 +20 = -20
$\frac{1}{1+j0.167\omega}$	$\omega_{c5} = \frac{1}{0.167} = 6$	-20	-20 -20 = -40
$\frac{1}{1+j0.0154\omega}$	$\omega_{c6} = \frac{1}{0.0154} = 65$	-20	-40 -20 = -60

Choose ω_l = 0.01 rad/sec & ω_h = 80 rad/sec

Calculation of Gain:

$A_0 = |G_0(j\omega)|$ in dB

At $\omega = \omega_l$, $A_0 = 20\log\frac{80}{0.01} = 78\text{dB}$

At $\omega = \omega_{c1}$, $A_0 = 20\log\frac{80}{0.03} = 68\text{dB}$

At $\omega = \omega_{c2}$, $A_0 = -40\times\log\frac{0.4}{0.03} + 68 = 23\text{dB}$

At $\omega = \omega_{c3}$, $A_0 = -20\times\log\frac{03}{0.4} + 23 = 5\text{dB}$

At $\omega = \omega_{c4}$, $A_0 = -40 \times \log \frac{4.5}{3} + 5 = -2\text{dB}$

At $\omega = \omega_{c5}$, $A_0 = -20 \times \log \frac{6}{45} + (-2) = -4\text{dB}$

At $\omega = \omega_{c6}$, $A_0 = -40 \times \log \frac{65}{6} + (-4) = -45\text{dB}$

At $\omega = \omega_h$, $A_0 = -60 \times \log \frac{80}{65} + (-45) = -50\text{dB}$

Phase plot

$\phi_0 = \angle G_0(j\omega) = \tan^{-1}2.5\omega + \tan^{-1}0.22\omega - 90° - \tan^{-1}35\omega - \tan^{-1}0.0154\omega - \tan^{-1}0.33\omega - \tan^{-1}0.167\omega$

ω rad/sec	0.01	0.03	0.1	0.4	1	4	10	65	80
ϕ_0 deg	-108	-132	-152	-138	-126	-144	-168	-220	-228

From the Bode plot of the compensated system,

Let ϕ_{gco} = Phase of $G_0(j\omega)$ at gain crossover frequency of compensated system,

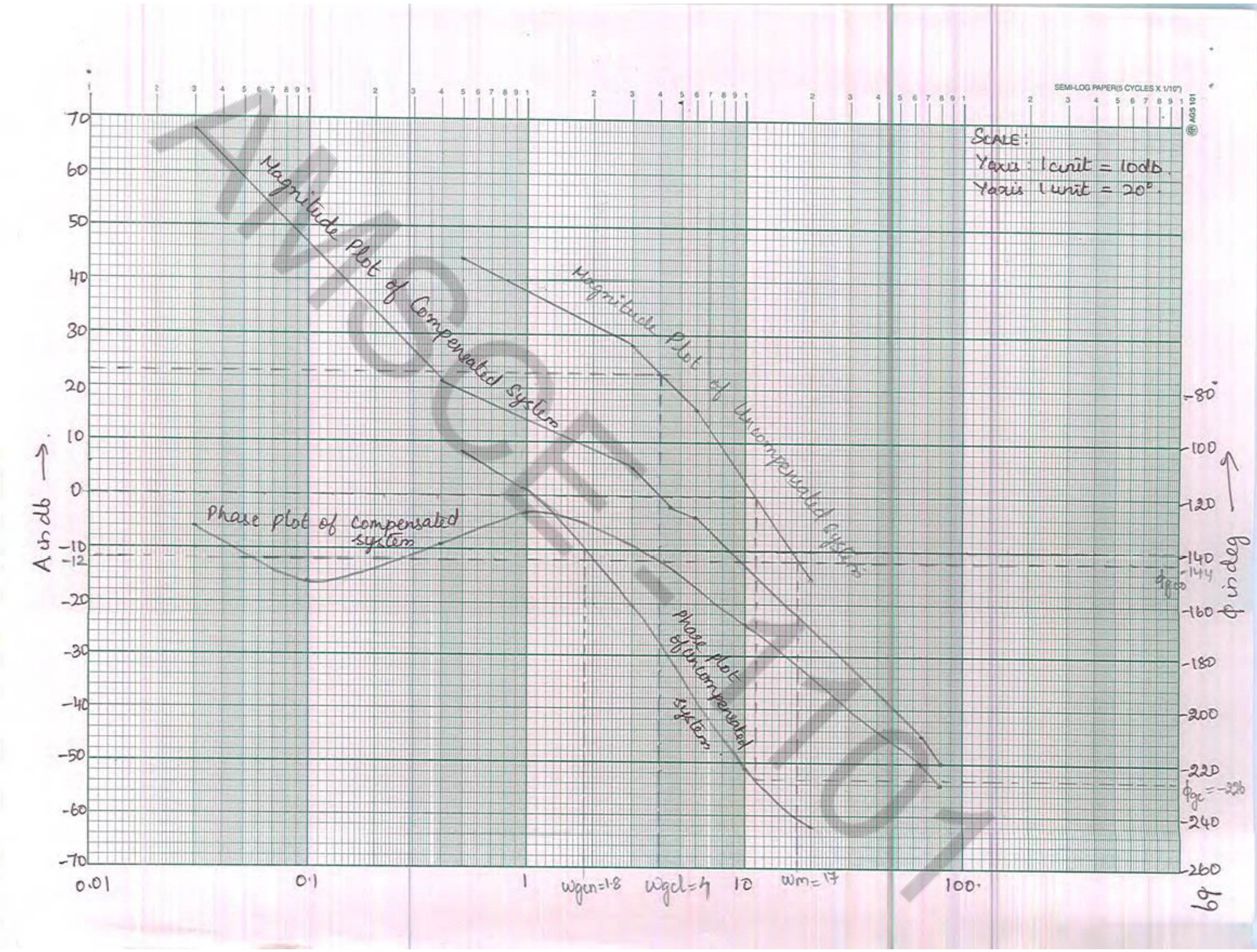
γ_0 = phase margin of compensated system

ϕ_{gco} = -144° (From bode plot)

$\gamma_0 = 180° + \phi_{gc0}° = 180° - 144° = 36°$

Conclusion

The phase margin of the compensated system is satisfactory. Hence the design is acceptable.



SEMI-LOG PAPER(5 CYCLES X 1/10")
AGS 101
SCALE:
Xaxis : 1 unit = 10db.
Yaxis 1 unit = 20°.
Magnitude Plot of Compensated System
Magnitude Plot of Uncompensated System
Phase plot of compensated system
Phase plot of uncompensated system.
A in db →
φ in deg →
-80°
-100
-120
-140
-144
-160
-180
-200
-220
-240
-260
$\phi_{gc} = -226$
ϕ_{gco}
70
60
50
40
30
20
10
0
-10
-12
-20
-30
-40
-50
-60
-70
0.01
0.1
1
$w_{gcn}=1.8$
$w_{gcl}=4$
10
$w_m = 17$
100.

UNIT – 5

STATE VARIABLE ANALYSIS

Part - A

1. Define state and state variable.

The state of a dynamical system is a minimal set of variables (known as state variables) such that we knowledge of these variables at $t - t_0$ together with the knowledge of the input for $t > t_0$, completely determines the behaviour of the system for $t > t_0$.

The state variables are the minimal or the smallest set of variable which determines the dynamic behaviour of the linear system.

2. **Write the general form of state variable matrix.**

The most general state space representation of a linear system with m inputs,p output and n state variable is written in the following form:

$$X = A\dot{X} + BU$$
$$Y = CX + DU$$

Where $\dot{X}$ = state vector of order $n \times 1$,

U = input vector of order $n \times 1$,

A = system matrix of order $n \times n$

B = input matrix of order $n \times m$

C = output matrix of order $p \times n$

D = transmission matrix of order $p \times m$

3. What is the necessary condition to be satisfied for design using state feedback?

The state feedback design requires arbitrary pole to achieve the desire performance. The necessary and sufficient condition to be satisfied for arbitrary pole placement is that the system is completely state controllable.

4. What is controllability? April/May 2017

A system is said to be completely state controllable if it is possible to transfer the system state from any initial stated X(t), in specified finite time by a control vector U(t).

5. What is observability? April/May 2018

A system is said to be completely observable if every state X(t) can be completely identified by measurement of the output Y(t) over a finite time interval.

6. Write the properties of state transition matrix.

The following are the properties of state transition matrix

$$1.\Phi(0) = e^{Ax0} = I(\text{unit matrix})$$

$$2.\Phi(t) = e^{At} = \left(e^{-At}\right)^{-1} = \left[\Phi(-t)\right]^{-1}. \ .$$

$$3.\Phi(t_1 + t_2) = e^{A(t_1+t_2)} = \Phi(t_1)\Phi(t_2)$$

9. What is nyquist rate?

The sampling frequency equal to twice the highest frequency of the signal is called nyquist rate $f_m = 2f_m$

10. What is similarity transformation?

The process of transforming a square matrix A to another similar matrix B by a transformation $P^{-1}AP = B$ is called similarly transformation. The matrix P is called transformation matrix.

11. What is mean by diagonalization?

The process of converting the system matrix A into a diagon al matrix by a similarity transformation using the model matrix M is called diagonalization **.**

12. What is modal matrix?

The modael matrix is a matrix used to diagonalize the system matrix. It is also called diagonalization matrix.

If A = system matrix

M = Modal matrix

And M^{-1}= inverse of modal matrix

Then $M^{-1}AM$ will be a diagonalized system matrix.

13. How the modal matrix is determined?

The modal matrix M can be formed from eigenvectors. Let $m_1, m_2, m_3, \dots m_n$ be the eigenvector of the n^{th} order of the system. Now the modal matrix M is obtained arranging all the eigenvector column wise asshown below.

Modal matrix, M = [$m_1, m_2, m_3, \dots m_n$].

14. What is need for controllability test ?

The controllability test is necessary to find the usefulness of the state variable. If the state variables are controllable then by controlling the state variable the desired output of the system are achieved.

15. What is need for observability test? Nov/Dec 2018, Nov/Dec 2019

` The observability test is necessary to find whether the state variable are measurable or not. If the state variables are measurable then the state of the system can be determined by practical measurement of the state variables.

16. State the condition for controllability by Gilbert's method.

Case (i) when the eigen values are distinct

Consider the canonical form of state model shown below which is obtained by using the transformation

$$X = MZ$$
$$\dot{Z} = \Lambda Z + \tilde{B}U$$
$$Y = \tilde{C}Z + DU$$

Where $\Lambda = M^{-1} AM$: $\tilde{C} = CM$, $\tilde{B} = M^{-1}B$ and M = modal matrix

In this case the necessary and sufficient condition for complex controllability is that, the matrix must have no row with all zeroes. If any row of the matrix is zero then the corresponding state variable is uncontrollable

Case ii) when eigen value have multiplicity

In the case the state modal can be converted to Jordan canonical form shown below

$$Z = JZ + \tilde{B}U$$
$$Y = \tilde{C}Z + DU \quad \text{Where } J = M^{-1}AM$$

In this case the system is completely controllable if the element of any row of that corresponding to the last row of each Jordan block is not all zero.

17. State the condition for observability by Gilbert's method.

Consider the transfer function canonical or Jordan canonical form of the state model shown below which obtained by using the transformation,

$$X = MZ$$
$$\dot{Z} = \Lambda Z + \tilde{B}U$$
$$Y = \tilde{C}Z + DU$$

or

$$Z = JZ + \tilde{B}U$$
$$Y = \tilde{C}Z + DU \quad \text{Where } J = M^{-1}AM$$

Where $\Lambda = M^{-1} A M$: $\tilde{C} = CM$, $\tilde{B} = M^{-1}B$ and M = modal matrix

The necessary and sufficient condition for complete observability is that none of the column of the matrix be zero. If any of column is of all zeroes then corresponding state variable is not observable.

18. State the duality between controllability and observability.

The concept of controllability and observability are dual concept and it is proposed by Kalman as principle of duality. The principle of duality states that a system is completely state controllable if and only if its dual system is completely state controllable if its dual system is completely observable or vice versa.

19. Enumerate the advantages of state space analysis. April/May 2018, April/May 2019

It can be applied to non linear systems, time variant systems and multiple input multiple output systems

20. When a System is said to be completely observable? Nov/Dec 2015, May/ June 2016

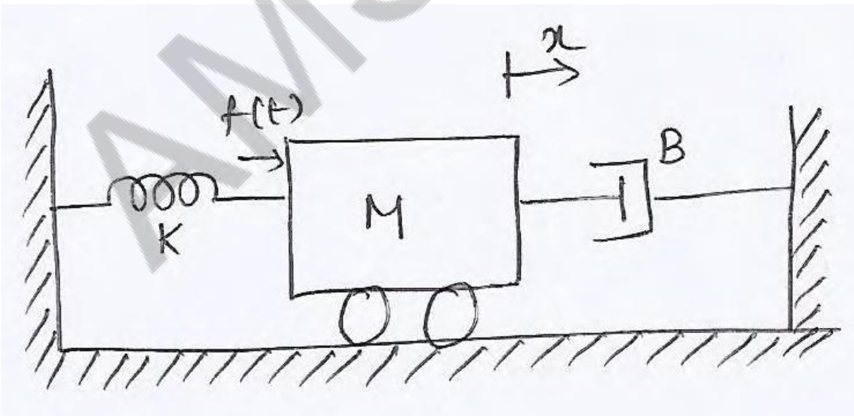
A System is said to be completely observable if all the possible initial states of the system can be observed. Systems that fails this criteria are said to be non observable

21. When a System is said to be completely controllable? Nov/Dec 2015

A System is said to be completely controllable if it is possible to transfer the system state from any initial state $X(t_0)$ at any other desired state X(t), in specified finite time by a control vector v(t).

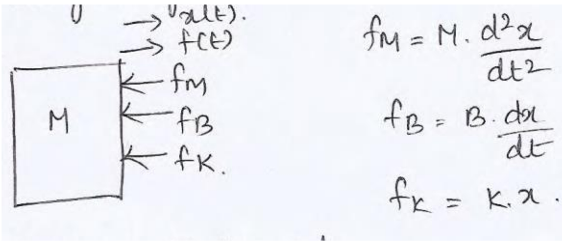
Part – B

1. Obtain the state model of the given mechanical system .



Solution:

Free body diagram



By D'Alembert's principle,

$$\sum \text{applied forces} = \sum \text{opposing forces}$$

$$f(t) = f_M + f_B + f_K$$

$$f(t) = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx \qquad ...1$$

Equation 1 represents the differential equation covering the system. Let position and velocity be chosen as state variables then stable variable $x_1$and x_2 & input variable be u(t).

$$x_1 = x(t) \qquad .....2$$
$$x_2 = \dot{x}(t) \qquad .....3$$
$$u(t) = f(t) \qquad .....3a$$

Therefore $\dot{x}_1 = \dot{x}(t) = x_2$ 4

$$\dot{x}_2 = \ddot{x}(t) \qquad .....5$$

From the equation 1,

$$\frac{d^2x(t)}{dt^2} = \frac{f(t)}{M} - \frac{B}{M}\frac{dx(t)}{dt} - \frac{K}{M}x(t) \quad ....6$$

$$\ddot{x}(t) = \frac{1}{M}f(t) - \frac{B}{M}\dot{x}(t) - \frac{K}{M}x(t) \quad ...7$$

Substituting the state variable,

$$\dot{x}_2 = \frac{1}{M}u(t) - \frac{B}{M}x_2 - \frac{K}{M}x_1 \qquad ...8$$

$\therefore$ the state equation are

$$\dot{x}_1 = x_2 \qquad ...9$$

$$\dot{x}_2 = \frac{-K}{M}x_1 - \frac{B}{M}x_2 + \frac{1}{M}u(t) \qquad ...10$$

Equation 9&10, forms the state equation

Let the displacement x(t) be the output of the system

$$\therefore\ y = x_1 \quad ....11$$

Equation 11 is the output equation.

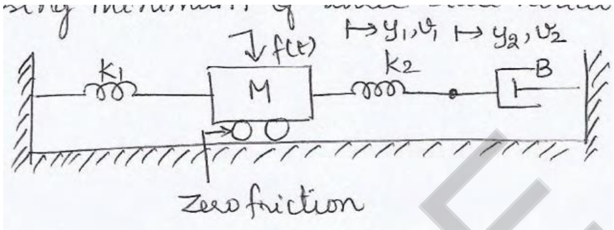
State & output equation in matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \qquad ...12$$

$$[y] = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad ....13$$

Equation 12 & 13 forms the state model.

2. Obtain the state model of the mechanical system by choosing minimum of these state variables .

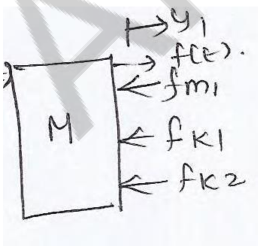


Solution:

Let the the stable variable be x_1, x_2, x_3 input variable is u(t) they are related to the physical variables,

$$x_1 = y_1,\ x_2 = y_2,\ x_3 = \frac{dy_1}{dt} = v_1,\ u(t) = f(t)$$

Free body diagram of mass M is shown in fig



$$f_m = M\frac{d^2y_1}{dt^2};\ f_{K1} = K_1 y_1;\ f_{k2} = K_2(y_1 - y_2)$$

DAlemberts principle

$$\sum \text{applied forces} = \sum \text{opposing forces}$$

$$f(t) = f_M + f_{k1} + f_{k2}$$

$$M\frac{d^2y_1}{dt^2} + K_1y_1 + K_2(y_1 - y_2) = f(t)$$

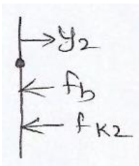
$$M\frac{d^2y_1}{dt^2} + (K_1 + K_2)y_1 - K_2y_2 = f(t) \quad ..1$$

Put $\frac{d^2y_1}{dt^2} = \dot{x}_3;\ y_1 = x_1;\ y_2 = x_2$ & f(t) = u(t) in equ 1.

$$M\dot{x}_3 + (k_1 + k2)x_1 - k_2x_2 = u(t)$$

$$\dot{x}_3 = \frac{-k_1 + k_2}{M}x_1 + \frac{k_2}{M}x_2 + \frac{1}{M}u(t) \quad ...2$$

The free body diagram of node 2



$$f_B = B\frac{d^2y_2}{dt^2};\ f_{K2} = K_2(y_2 - y_1)$$

Writing force balance equation at the meeting point of K_2 & B, we get

$$f_B + f_{K2} = 0$$

$$B\frac{d^2y_2}{dt^2} + K_2(y_2 - y_1) = 0$$

$$\therefore \frac{d^2y_2}{dt^2} = \frac{K_2}{B}y_1 - \frac{K_2}{B}y_2 \quad ....3$$

Put $\frac{d^2y_2}{dt^2} = \dot{x}_2,\ y_1 = x_1,\ y_2 = x_2$ in equ 3

$$\therefore \dot{x}_2 = \frac{K_2}{B}x_1 - \frac{K_2}{B}x_2 \quad ...4$$

State variable $x_1 = y_1$

$$\therefore x_1 = \frac{dy_1}{dt}$$

Let $\frac{dx_1}{dt} = \dot{x}_1,\ \frac{dy_1}{dt} = x_3,\ \dot{x}_1 = x_3 \quad ...5$

The equation 2,4 & 5 are called state equations

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = \frac{K_2}{B}x_1 - \frac{K_2}{B}x_2$$

$$\dot{x}_3 = -\frac{K_1+K_2}{M}x_1 + \frac{K_2}{M}x_2 + \frac{1}{M}u(t)$$

State equation in matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1\\ \frac{K_2}{B} & -\frac{K_2}{B} & 0\\ \frac{-K_1+K_2}{M} & \frac{K_2}{M} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{M}\end{bmatrix} \quad ..a$$

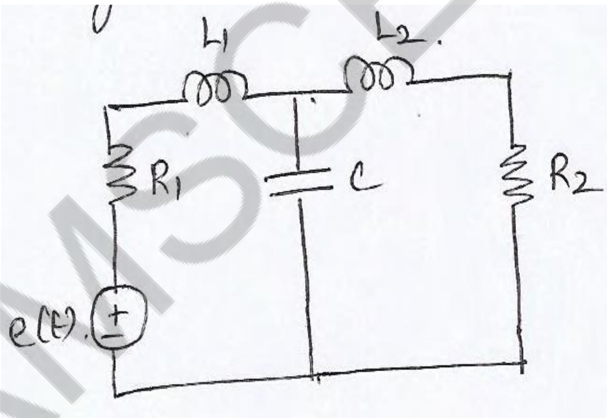
If the desired output are y_1 &y_2 then $y_1 = x_1$, $y_2 = x_2$

The output equation in matrix form

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \quad ..b$$

Equation a& b form the state model of the given mechanical system..

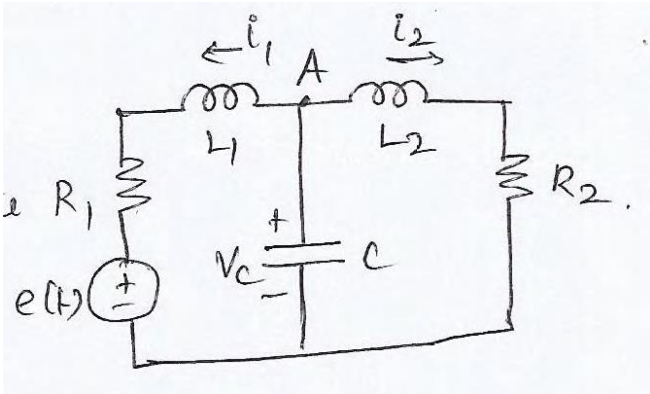
3. Obtained the state model of the electrical network shown in fig choosing minimal number of state variable.



Solution:

Let us chosen the current through the inductance i_1 & i_2 & voltage across the capacitor v_c be the stable variables

Let the stable be



$x_1, x_2 \; \& \, x_3 \; u(t)$ are the input variable

$$x_1 = i_1 \rightarrow \text{current through } L_1$$
$$x_2 = i_2 \rightarrow \text{current through } L_2$$
$$x_3 = v_c \rightarrow \text{voltage across } C;$$
$$u(t) = e(t);$$

At node A, by KCL,

$$i_1 + i_2 + C\frac{dv_c}{dt} = 0 \qquad ..1$$

On substituting the stable variables,

$$x_1 + x_2 + C\dot{x}_3 = 0$$

$$\dot{x}_3 = -\frac{1}{C}x_1 - \frac{1}{C}x_2$$

By krichoff 's voltage law to mesh1

$$e(t) + i_1 R_1 + L_1\frac{di_1}{dt} = v_c$$

On substituting the state variables,

$$u + x_1R_1 + L_1\dot{x}_1 = x_3$$
$$L_1\dot{x}_1 = x_3 - x_1R_1 - u$$
$$\dot{x}_1 = -\frac{R_1}{L_1}x_1 + \frac{1}{L_1}x_3 - \frac{1}{L_1}u \qquad ...4$$

By krichoff's voltage law to mesh 2

$$v_c = L_2\frac{di_2}{dt} + i_2R_2 \quad .....5$$

On substituting the state variable,

$$x_3 = L_2 x_2 + x_2 R_2$$
$$L_2 \dot{x}_2 = x_3 - x_2 R_2$$
$$\dot{x}_2 = \frac{-R_2}{L_2} x_2 + \frac{1}{L_2} x_3 \quad \text{-----6}$$

The equation 2, 4 & 6 are the state equation of the system

$$\dot{x}_1 = -\frac{R_1}{L_1} x_1 + \frac{1}{L_1} x_3 - \frac{1}{L_1} u$$
$$\dot{x}_2 = -\frac{R_2}{L_1} x_2 + \frac{1}{L_1} x_1$$
$$\dot{x}_3 = -\frac{1}{C} x_1 - \frac{1}{C} x_2$$

On arranging state equation in matrix form we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{-1}{L_1} \\ 0 \\ 0 \end{bmatrix} [u]$$

Let uis chosen the voltage across the resistance as output variable are denoted by y_1 &y_2.

$$y_1 = i_1 R_1 \quad ...8$$
$$y_2 = i_2 R_2 \quad ...9$$

On substituting the state variables,

$$y_1 = x_1 + R_1$$
$$y_2 = x_2 + R_2 \quad ...10$$

On arranging output equation in matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad ....11$$

The stable equation 7 & the output equation 11 together constant the state model of the system.

4. Construct a state model for a system characterized by the differential equation, $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0$ **given the block diagram representation of the state model.**

Solution:

Let us choose y and their derivation as state variables the system is governed by third order differential equation & the number of state variable are three

Let the state variable be x_1, x_2, x_3 are related to phase variable as follows

$$x_1 = y$$
$$x_2 = \frac{dy}{dt} = \dot{x}_1$$
$$x_3 = \frac{d^2y}{dt^2} = \dot{x}_2$$

Put $y = x_1, \; \frac{dy}{dt} = x_2, \; \frac{d^2y}{dt^2} = x_3 \; \& \frac{d^3y}{dt^3} = \dot{x}_3$ the given equation

$$\therefore \dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

The state equations are

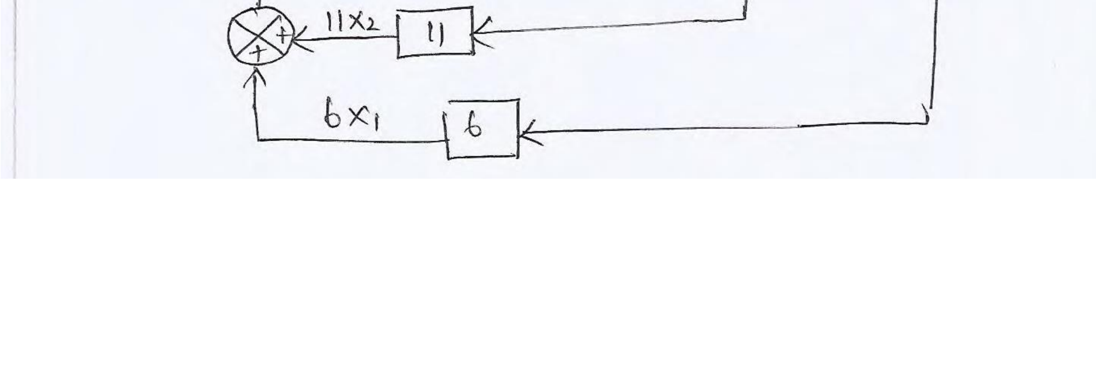
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

On arranging the state equation in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [u]$$

Let $y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The state equation and output equation constitutes the state model of the system . Block diagram of the state model is shown in fig



5. For the transfer function $\frac{Y(S)}{U(S)}=\frac{10(s+4)}{s(s+2)(s+5)}$, obtain the state space representation using 1) controllable canonical form 2) Observe canonical form using mason's gain formula.

Given:1.Controllable canonical form

$$\frac{Y(s)}{U(s)}=\frac{10(s+4)}{s(s+2)(s+5)}=\frac{10s+40}{s^3+7s^2+10s}$$

$$\frac{Y(s)}{U(s)}=\frac{Y(s)}{X_1(s)}.\frac{X_1(s)}{U(s)}=\frac{10s+40}{s^3+7s^2+10s}$$

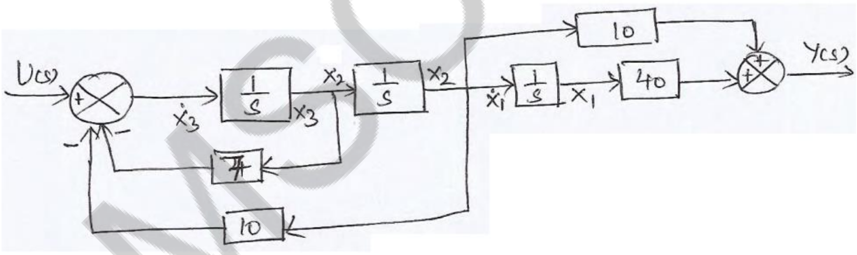
Where $\frac{X_1(s)}{U(s)}=\frac{1}{s^3+7s^2+10s}$ and $\frac{Y(s)}{X_1(s)}=10s+40$

$$\frac{X_1(s)}{U(s)}=\frac{1}{s^3+7s^2+10s}$$

$$\frac{Y(s)}{X_1(s)}=10s+40$$

$$Y(s)=10sX_1(s)+40X_1(s) \qquad ...2$$

Realization of equation 1 & 2 are shown fig



Let the state variable be $x_1, x_2, x_3,$ are marked at the output of integrators.

$$\dot{x}_1=x_2$$
$$\dot{x}_2=x_3$$
$$\dot{x}_3=u-7x_3-10x_2 \qquad ...3$$

The output is given by

$$y=40x_1+10x_2 \qquad ...4$$

The state equation and output equation in matrix form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u] \qquad ...5$$

And $$y = \begin{bmatrix} 40 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad ...6$$

Equation 5& 6 gives the state model in controllable canonical form.

ii) Observable canonical from using mason's gain formula.

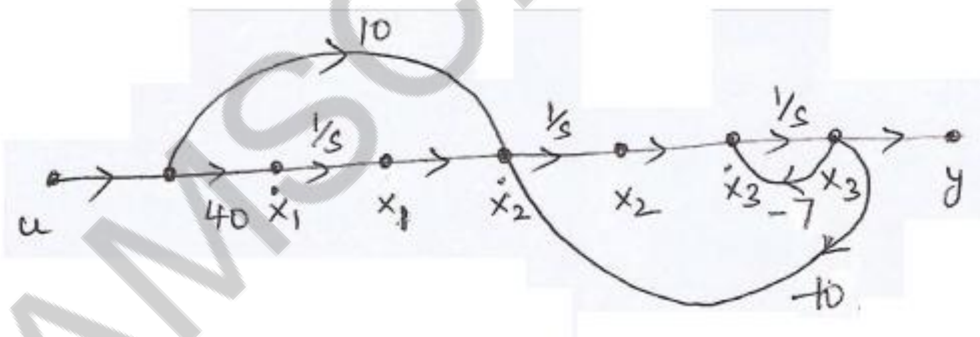
$$G(s) = \frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+2)(s+5)} = \frac{10s+40}{s^3+7s^2+10s} \qquad ...1$$

$$= \frac{\frac{10}{s^2} + \frac{40}{s^3}}{1 - \left(-\frac{7}{s} - \frac{3}{s^2}\right)}$$

Comparing with masons gain formula, there are two forward path with gain $\frac{10}{s^2}, \frac{40}{s^3}$

Two feedback loop with gain $\frac{-7}{s}$ and $\frac{-3}{s^2}$.

Signal flow graph



From fig

$$\dot{x}_1 = 40u$$
$$\dot{x}_2 = x_1 - 10x_3 + 10u$$
$$\dot{x}_3 = x_2 - 7x_3$$
$$y = x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -10 \\ 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} 40 \\ 10 \\ 0 \end{bmatrix} [u]$$

And $\quad [y] = [0 \quad 0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

6. Obtain the state model of the system by drawing the signal flow graph whose t/f is given as $\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1}$ **.**

Solution:

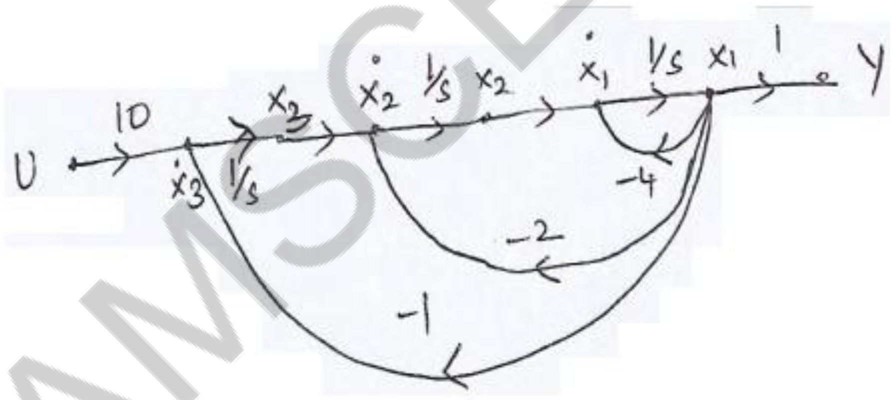
Given

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1} = \frac{10}{s^3\left(1 + \frac{4}{s} + \frac{2}{s^2} + \frac{1}{s^3}\right)}$$

$$= \frac{\frac{10}{s^3}}{1 - \left(-\frac{4}{s} - \frac{2}{s^2} - \frac{1}{s^3}\right)}$$

Comparing with mason gain formula, forward path gain = $\frac{10}{s^3}$

Three individual loop gain = $-\frac{4}{s}, \frac{-2}{s^2}, \frac{-1}{s^3}$



Assign state variable at the output of the integrators

The state equations are

$$\dot{x}_1 = -4x_1 + x_2$$
$$\dot{x}_2 = -2x_1 + x_3$$
$$\dot{x}_3 = -x_1 + 10u$$

The output equation is $y = x_1$

The state model in the matrix form is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-4 & 1 & 0\\ -2 & 0 & 1\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 10\end{bmatrix}[u] \text{ and}$$

$$[y] = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

7. Determine the diagonal canonical state model of the system, whose transfer function is $T(s) = \frac{2(s+5)}{[(s+2)(s+3)(s+4)]}$

Solution:

Given:

Let $\frac{Y(s)}{U(s)} = \frac{2(s+5)}{[(s+2)(s+3)(s+4)]}$

By partial fraction expansion,

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{[(s+2)(s+3)(s+4)]} = \frac{A}{(s+2)} + \frac{B}{(s+3)} + \frac{C}{(s+4)}$$

$$A = \left.\frac{2(s+5)}{(s+2)(s+4)}\right|_{s=-2} = \frac{2(-2+5)}{(-2+3)(-2+4)} = \frac{2\times 3}{1\times 2} = 3$$

$$B = \left.\frac{2(s+5)}{(s+2)(s+4)}\right|_{s=-3} = \frac{2(-3+5)}{(-3+2)(-3+4)} = \frac{2\times 2}{-1\times 1} = -4$$

$$C = \left.\frac{2(s+5)}{(s+2)(s+4)}\right|_{s=-4} = \frac{2(-4+5)}{(-4+2)(-4+3)} = \frac{2\times 1}{-2\times -1} = 1$$

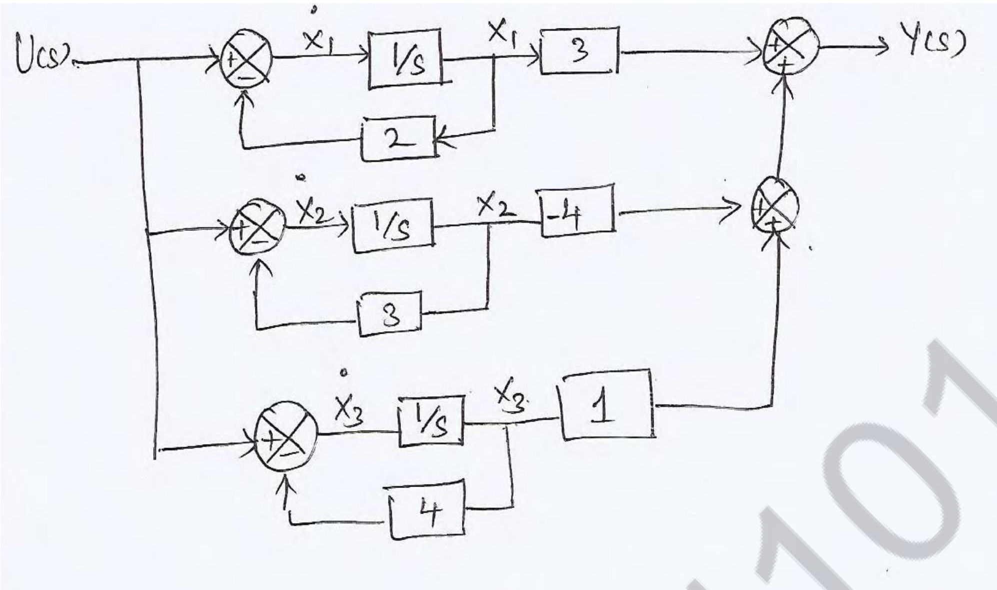
$$\therefore \frac{Y(s)}{U(s)} = \frac{3}{s+2} - \frac{4}{s+3} + \frac{1}{s+4} \qquad ...1$$

Equation 1 can be rearranged a follows

$$\frac{Y(s)}{U(s)} = \frac{3}{s\left(1+\frac{2}{s}\right)} - \frac{4}{s\left(1+\frac{3}{s}\right)} + \frac{1}{s\left(1+\frac{4}{s}\right)}$$

$$\therefore Y(s) = \left[\frac{\frac{1}{s}}{\left(1+\frac{1}{s}.2\right)}\times 3\right]U(s) - \left[\frac{\frac{1}{s}}{\left(1+\frac{1}{s}.3\right)}\times 4\right]U(s) + \left[\frac{\frac{1}{s}}{\left(1+\frac{1}{s}.4\right)}\times 1\right]U(s) \qquad ...2$$

Equation 2 can be represented in block diagram



Assign state variables at the output of the integer as shown in fig. at the input of the integrators the derivation of the stable variable are assigned.

State equation s are

$$\dot{x}_1 = -2x_1 + u$$
$$\dot{x}_2 = -3x_2 + u$$
$$\dot{x}_3 = -4x_3 + u$$

The output equation is y = $3x_1 - 4x_2 + x_3$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [u]$$

$$[y] = [3 \quad -4 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

8. Obtained the state space representation in Jordan canonical form for the given transfer function $\frac{Y(S)}{U(S)} = \frac{2s^2+6s+7}{(s+1)^2(s+2)}$

Solution:

Given

$$\frac{Y(s)}{U(s)} = \frac{2s^2+6s+7}{(s+1)^2(s+2)} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \underset{s\to -1}{Lt}\left[\frac{\left(2s^2+6s+7\right)\left(s+1\right)^2}{\left(s+1\right)^2\left(s+2\right)}\right]$$

$$= \underset{s\to -1}{Lt}\left[\frac{2s^2+6s+7}{s+2}\right] = \frac{2-6+7}{-1+2} = 3$$

$$B = \underset{s\to -1}{Lt}\frac{d}{ds}\left[\frac{\left(2s^2+6s+7\right)\left(s+1\right)^2}{\left(s+1\right)^2\left(s+2\right)}\right]$$

$$= \underset{s\to -1}{Lt}\frac{d}{ds}\left[\frac{2s^2+6s+7}{s+2}\right] = \underset{s\to -1}{Lt}\left[\frac{(s+2)(4s+6)-\left(2s^2+6s+7\right)}{(s+2)^2}\right]$$

$$= \frac{(-4+6)-(2-6+7)}{(-1+2)^2} = \frac{2-3}{1} = 1$$

$$C = \underset{s\to -2}{Lt}\left[\frac{\left(2s^2+6s+7\right)(s+2)}{(s+1)^2(s+2)}\right]$$

$$= \underset{s\to -2}{Lt}\left[\frac{2s^2+6s+7}{(s+1)^2}\right]$$

$$= \frac{8-12+7}{(-2+1)^2} = 3$$

$$\frac{Y(s)}{U(s)} = \frac{3}{(s+1)^2} + \frac{-1}{(s+1)} + \frac{2}{s+2}$$

$$Y(s) = \frac{3U(s)}{(s+1)^2} + \frac{-U(s)}{(s+1)} + \frac{2U(s)}{s+2}$$

Let the state variable be

$$X_1(s) = \frac{U(s)}{(s+1)^2}$$

$$X_2(s) = \frac{U(s)}{(s+1)}$$

$$X_3(s) = \frac{U(s)}{(s+2)}$$

$$\frac{X_1(s)}{X_2(s)} = \frac{1}{(s+1)}$$

$$sX_1(s) = -X_1(s) + X_2(s)$$

$$sX_2(s) = -X_2(s) + U(s)$$

$$sX_3(s) = -2X_3(s) + U(s)$$

$$Y(s) = 3X_1(s) - X_2(s) + 3X_3(s)$$

Taking inverse Laplace transform

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -x_2 + u$$
$$\dot{x}_3 = -2x_3 + u$$
$$y = 3x_1 - x_2 + 3x_3$$

This equation can be represented in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

9. Obtain the transfer function model for the following state space system.

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix},\ B\begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = [0].$$

Solution:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+5)+6} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s(s+5)+6} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + [0]$$

$$= \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s+5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{s^2 + 5s + 6} [s+5]$$

$$\frac{Y(s)}{U(s)} = \frac{s+5}{s^2 + 5s + 6}$$

10. Find the transfer function for the system, which is represented in the state space representation as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$D[0]$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

$$(sI - A) = s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} s+2 & -1 & 0 \\ 0 & s+3 & -1 \\ 3 & 4 & s+5 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\operatorname{adj}(sI - A)}{\det(sI - A)}$$

$$\operatorname{adj}(sI - A) = \begin{bmatrix} s^2 + 8s + 19 & s+5 & 1 \\ -3 & s^2 + 7s + 10 & s+2 \\ -3(s+3) & -4s - 11 & s^2 + 5s + 6 \end{bmatrix}$$

$$\det(sI - A) = [(s+2)((s+3)(s+5) + 4) + 1 \times 3]$$

$$= (s+2)(s^2 + 8s + 19) + 3$$

$$= s^3 + 10s^2 + 35s + 41$$

$$\therefore (S - I - A)^{-1} = \frac{1}{s^3 + 10s^2 + 35s + 41} \begin{bmatrix} s^2 + 8s + 19 & s+5 & 1 \\ -3 & s^2 + 7s + 10 & s+2 \\ -3(s+3) & -4s - 11 & s^2 + 5s + 6 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 10s^2 + 35s + 41} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s^2 + 8s + 19 & s+5 & 1 \\ -3 & s^2 + 7s + 10 & s+2 \\ -3(s+3) & -4s - 11 & s^2 + 5s + 6 \end{bmatrix}$$

$$= \frac{1}{s^3 + 10s^2 + 35s + 41}\begin{bmatrix} -3 & s^2 + 7s + 10 & s + 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^3 + 10s^2 + 35s + 41}. \ s + 2$$

$$\frac{Y(s)}{U(s)} = \frac{s+2}{s^3 + 10s^2 + 35s + 41}$$

11. A linear time invariant system is characterised by the state equation

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[u]$ **when u is a unit step function complete the solution of these equation assuming initial condition** $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **use inverse Laplace technique.**

Solution:

Given $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$X(t) = L^{-1}\left[\phi(s)X(s)\right] + L^{-1}\left[\phi(s)BU(s)\right].$$

$$\phi(s) = (sI - A)^{-1}$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\text{adj } (sI - A)}{\det \ (sI - A)} = \frac{1}{(s-1)^2}\begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$\phi(s)X(0) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{(s-1)^2} \end{bmatrix}$$

$$L^{-1}[\phi(s)X(0)] = L^{-1}\begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{(s-1)^2} \end{bmatrix} = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

$$\phi(S)B = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{s-1} \end{bmatrix}$$

$$\phi(s)B\ U(s) = \begin{bmatrix} 0 \\ \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{s(s-1)} \end{bmatrix}$$

$$L^{-1}\left[\phi(s)B\ U(s)\right] = L^{-1}\begin{bmatrix} 0 \\ \frac{1}{s(s-1)} \end{bmatrix}$$

$$L^{-1}\left[\phi(s)B\ U(s)\right] = \begin{bmatrix} 0 \\ -\frac{1}{s} + \frac{1}{s-1} \end{bmatrix}$$

$$L^{-1}\left[\phi(s)B\ U(s)\right] = \begin{bmatrix} 0 \\ -1 + e^t \end{bmatrix}$$

$$X(t) = \begin{bmatrix} e^t \\ te^t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 + e^t \end{bmatrix}$$

$$\Rightarrow X(t) = \begin{bmatrix} e^t \\ -1 + (t+1)e^t \end{bmatrix}$$

12. Test the controllability & observability of the system by any one method whose state space representation is given as. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u$ **,** $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ **.**

Solution:

Method: Gilberts Method.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

To find Eigen values.

The characteristic equation is $|\lambda I - A| = 0$

$$[\lambda I - A] = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$=\begin{bmatrix}\lambda & 0 & -1\\ 2 & \lambda+3 & 0\\ 0 & -2 & \lambda+3\end{bmatrix}$$

$$`\left|\lambda I-A\right|=\begin{vmatrix}\lambda & 0 & -1\\ 2 & \lambda+3 & 0\\ 0 & -2 & \lambda+3\end{vmatrix}$$

$$=\lambda\left(\lambda+3\right)^{2}-1\left(-4\right)-\lambda\left(\lambda^{2}+6\lambda+9\right)+4-(\lambda^{3}+6\lambda^{2}+9\lambda+4$$
$$=\left(\lambda+1\right)\left(\lambda+1\right)\left(\lambda+4\right)=\left(\lambda+1\right)^{2}\left(\lambda+4\right)$$

The Eigen values are $\lambda=-1,\lambda=-1,\lambda=-4$

To find eigen vectors

$$\left|\lambda_1 I-A\right|=\lambda_1\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}0 & 0 & 1\\ -2 & -3 & 0\\ 0 & 2 & -3\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_1 & 0 & -1\\ 2 & \lambda_1+3 & 0\\ 0 & -2 & \lambda_1+3\end{bmatrix}$$

Let C_{11},C_{12},C_{13} be the cofactore along the 1st row of the matrix $\left[\lambda_1 I-A\right]$

$$C_{11}=\left(+1\right)\begin{vmatrix}\lambda_1+3 & 0\\ -2 & \lambda_1+3\end{vmatrix}=(\lambda_1+3)^{2}=\lambda_1^{2}+6\lambda_1+9$$

$$C_{12}=\left(-1\right)\begin{vmatrix}2 & 0\\ 0 & \lambda_1+3\end{vmatrix}=-\left(2\left(\lambda_1+3\right)\right)=-2\lambda_1-6$$

$$C_{13}=1\begin{vmatrix}2 & \lambda_1+3\\ 0 & -2\end{vmatrix}=-4$$

$$m_1=\begin{bmatrix}C_{11}\\ C_{12}\\ C_{13}\end{bmatrix}=\begin{bmatrix}\lambda_1^{2}+6\lambda_1+9\\ -2\lambda_1-6\\ -4\end{bmatrix}=\begin{bmatrix}1-6+9\\ 2-6\\ -4\end{bmatrix}=\begin{bmatrix}4\\ -4\\ -4\end{bmatrix}$$

$$m_2=\begin{bmatrix}\dfrac{dC_{11}}{d\lambda_1}\\ \dfrac{dC_{12}}{d\lambda_1}\\ \dfrac{dC_{13}}{d\lambda_1}\end{bmatrix}=\begin{bmatrix}2\lambda_1+6\\ -2\\ 0\end{bmatrix}=\begin{bmatrix}-2+6\\ -2\\ 0\end{bmatrix}=\begin{bmatrix}4\\ -2\\ 0\end{bmatrix}$$

$$[\lambda_3 I - A] = -4\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0 & 1\\-2 & -3 & 0\\0 & 2 & -3\end{bmatrix} = \begin{bmatrix}-4 & 0 & -1\\2 & -1 & 0\\0 & -2 & -1\end{bmatrix}$$

Let C_{11}, C_{12}, C_{13} be the cofactor along 1st row of the matrix $[\lambda_3 I - A]$

$$C_{11} = (+1)\begin{vmatrix}-1 & 0\\-2 & -1\end{vmatrix} = 1$$

$$C_{12} = (-1)\begin{vmatrix}2 & 0\\0 & -1\end{vmatrix} = 2$$

$$C_{13} = (+1)\begin{vmatrix}2 & -1\\0 & -2\end{vmatrix} = -4$$

$$m_3 = \begin{bmatrix}C_{11}\\C_{12}\\C_{13}\end{bmatrix} = \begin{bmatrix}1\\2\\-4\end{bmatrix}$$

To find the canonical form of state from of state model

The model matrix, M is given by

$$M = [m_1 \quad m_2 \quad m_3] = \begin{bmatrix}4 & 4 & 1\\-4 & -2 & 2\\-4 & 0 & -4\end{bmatrix}$$

$$M^{-1} = \frac{[\text{Cofactor of } M]^T}{\text{Deter min ation of } M} = \frac{{M_{cof}}^T}{\Delta M}$$

$$\Delta M = \begin{vmatrix}4 & 4 & 1\\-4 & -2 & 2\\-4 & 0 & -4\end{vmatrix} = 4(8) - 4(24) + 1(-8) = -72$$

$${M_{cof}}^T = \begin{bmatrix}8 & 24 & -8\\16 & -12 & -16\\10 & -12 & 8\end{bmatrix}^T = \begin{bmatrix}8 & 16 & 10\\-24 & -12 & -12\\-8 & -16 & 8\end{bmatrix}$$

$$M^{-1} = \frac{1}{-72}\begin{bmatrix}8 & 16 & 10\\-24 & -12 & -12\\-8 & -16 & 8\end{bmatrix} = \frac{1}{18}\begin{bmatrix}-2 & -4 & -25\\6 & 3 & 3\\2 & 4 & -2\end{bmatrix}$$

$$J = M^{-1}AM = \frac{1}{18}\begin{bmatrix}-2 & -4 & -25\\6 & 3 & 3\\2 & 4 & -2\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\-2 & -3 & 0\\0 & 2 & -3\end{bmatrix}\begin{bmatrix}4 & 4 & 1\\-4 & -2 & 2\\-4 & 0 & 4\end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 8 & 7 & 5.5 \\ -6 & -3 & -3 \\ -8 & -16 & 8 \end{bmatrix}\begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} -18 & 18 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -72 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\tilde{B} = M^{-1}B = \frac{1}{18}\begin{bmatrix} -2 & -4 & -2.5 \\ 6 & 3 & 3 \\ 2 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -8/18 \\ 6/18 \\ 8/18 \end{bmatrix} = \begin{bmatrix} -4/9 \\ 3/9 \\ 4/9 \end{bmatrix}$$

$$\tilde{C} = CM = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 1 \end{bmatrix}$$

The Jordan canonical form of state model

$$\dot{Z} = JZ + \tilde{B}U$$

$$Y = \tilde{C}Z + DU$$

$$\begin{bmatrix} \dot{Z}_1 \\ \dot{Z}_2 \\ \dot{Z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} + \begin{bmatrix} -4/9 \\ 3/9 \\ 4/9 \end{bmatrix}\begin{bmatrix} u \end{bmatrix} \text{ and}$$

$$Y = \begin{bmatrix} 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$$

Conclusion

- The elements of the row of $\tilde{B}$ are not all zero. Hence the system is completely controllable.
- The elements of the column of $\tilde{C}$ are not all zero. Hence the system is completely observable

13. Consider the system defined by $\dot{X} = AX + BU, \quad Y = CX$

Where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 10 & 5 & 1 \end{bmatrix}$

Check controllability and observability of the system Nov/Dec 2015

using Kalman's method

i) To check for controllability

$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A.B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -12 \end{bmatrix}$$

$$A^2B = A.(A.B) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 61 \end{bmatrix}$$

The composite matrix for controllability

$$Q_c = \begin{bmatrix} B & AB & AB^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -12 & 61 \end{bmatrix}$$

$$\Delta Q_c = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -12 & 61 \end{vmatrix} = 1(61-144)+1(-1) = -83-1 = -84$$

Since $|Q_c| \neq 0$ the rank of Q_c is 3 hence the system is completely controllable.

To check for observability

$$A^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}, \qquad C^T = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}$$

$$A^T.C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0+6 \\ 10+0-11 \\ 0+5-6 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T.\left(A^T C^T\right) = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0+0+6 \\ -6+0+11 \\ 0-1+6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}$$

The composite matrix for observability

$$Q_o = \left[C^T \quad A^T C^T \quad \left(A^T\right)^2 C^T\right]$$

$$= \begin{bmatrix} 10 & -6 & 6 \\ 5 & -1 & 5 \\ 1 & -1 & 5 \end{bmatrix}$$

$$\Delta Q_o = 10(-5+5)+6(25-5)+6(-5+1)$$
$$= 6(20)+6(-4)$$
$$= 120-24 = 96$$

Since $|Q_o| \neq 0$, the rank of Q_o is 3. Hence the system is completely observable.

18. State the properties of state transition matrix.

$\phi(t) = e^{At}$ = state transition matrix

1. $\phi(0) = e^{A\times 0} = I =$ Identity matrix

2. $\phi(t) = e^{At} = \left(e^{-At}\right)^{-1} = \left[\phi(-t)\right]^{-1}$

i.e., $\phi^{-1}(t) = \phi(-t)$

3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$

$= \phi(t_1)\phi(t_2)$

$= \phi(t_2)\phi(t_1)$

4. $e^{A(t+s)} = e^{At} \cdot e^{As}$

5. $e^{(A+B)t} = e^{At} \cdot e^{Bt}$ only if AB = BA

6. $\left[\phi(t)\right]^n = \left[e^{At}\right]^n = e^{Ant} = \phi(nt)$

7. $\phi(t_2 - t_1) \cdot \phi(t_1 - t_0) = \phi(t_2 - t_0)$

This property states that the process of transition of state can be divided into number of sequential transition. Thus t_0 to t_2 can be divided as t_0 to t_1 & t_1 to t_2, as stated in the property In terms of $\phi(t)$, the solution is expressed as

$$X(t) = \phi(t - t_0) X(t_0) + \int_{t_0}^{t} \phi(t - \tau) B \cdot U(\tau) \cdot d\tau$$

Where $\phi(t - t_0) = e^{A(t - t_0)}$ & $\phi(t - \tau) = e^{A(t - \tau)}$

8. $\phi(t)$ is an non-singular matrix for all values of t.

19. Draw the state model of a linear single input-single output system and obtain its corresponding equations.

The state model of a linear single input single output system can be obtained by putting m = 1 & p = 1 in the state model of a linear multi input – multi- output system as

$$\dot{x}(t) = Ax(t) + B \cdot u(t) \;\rightarrow \text{State Equation}$$

$$y(t) = Cx(t) + D \cdot u(t) \;\rightarrow \text{Output Equation}$$

Where $x(t) = \begin{bmatrix} x_1(t) \\ x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n \times 1} \rightarrow \text{State Vector}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n \times n} \rightarrow \text{System Matrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1} \rightarrow \text{Input Matrix}$$

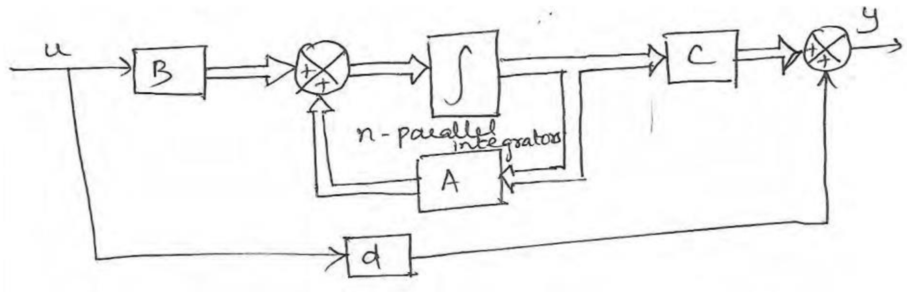
$$C = [C_1\, C_2\, C_3 \dots C_n]_{1 \times n} \rightarrow \text{Output matrix}$$

d = Transmission Constant

u(t) = Input (or) Control Variable (Scalar)

y(t) = Output Variable (Scalar)

The Block diagram representation of the state model of linear single input single output system is shown in fig.



20. Consider the following system with differential equation given by.

$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$ **obtain the state model in diagonal canonical form Nov/Dec 2015**

Solution

Given $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$

Taking Laplace transform on both sides

$$s^3Y(s) + s^2 6Y(s) + 11sY(s) + 6Y(s) = 6U(s)$$

$$\left[s^3 + 6s^2 + 11s + 6\right]Y(s) = U(s)\cdot 6$$

$$\frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6}$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{6}{(s+1)(s+2)(s+3)}$$

$$= \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)}$$ [By partial fraction expansion]

$$A = \left.\frac{6}{(s+2)(s+3)}\right|_{s=-1} = \frac{6}{1\times 2} = 3$$

$$B = \left.\frac{6}{(s+1)(s+3)}\right|_{s=-2} = \frac{6}{(-1)(1)} = -6$$

$$C = \left.\frac{6}{(s+1)(s+2)}\right|_{s=-3} = \frac{6}{(-2)(-1)} = 3$$

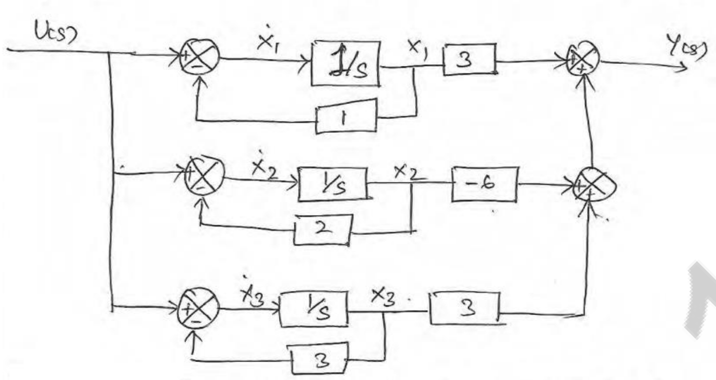
$$\therefore \frac{Y(s)}{U(s)} = \frac{3}{(s+1)} + \frac{-6}{(s+2)} + \frac{3}{(s+3)} \qquad (1)$$

Equation (1) can be rearranged as follows

$$\frac{Y(s)}{U(s)} = \frac{3}{s\left(1+\frac{1}{s}\right)} - \frac{6}{s\left(1+\frac{2}{s}\right)} + \frac{3}{s\left(1+\frac{3}{s}\right)}$$

$$Y(s) = \frac{3/s}{\left(1+\frac{1}{s}\cdot 1\right)}\cdot U(s) - \frac{6/s}{s\left(1+\frac{1}{s}\cdot 2\right)}\cdot U(s) + \frac{3/s}{s\left(1+\frac{1}{s}\cdot 3\right)}\cdot U(s)$$

Equation 2 can be represented in block diagram



Assign state variables at the output of the integrators as shown in fig. At the input of the integrators, the derivatives of the state variables are assigned.

The state equations are

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -2x_2 + u \qquad \rightarrow (3) \\ \dot{x}_3 &= -3x_3 + u \end{aligned}$$

The output equation is

$y = 3x_1 - 6x_2 + 3x_3$ (4)

Equation (3) & (4) forms the state model

State model in Matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$